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A VARIATIONAL PRINCIPLE FOR ROTATING STARS IN GENERAL RELATIVITY*

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ABSTRACT

A variational principle for the structure of differentially rotating stars in general relativity is developed both as an action principle from which the Einstein equations and the equation of hydrostatic equilibrium can be derived and as an energy-extremization principle. We show that the equilibrium configuration of the star and its gravitational field extremizes the gravitational mass of momentarily stationary, axisymmetric configurations which satisfy the initial-value equations of general relativity and in which each ring of matter has a fixed number of baryons, a fixed angular momentum, and a fixed entropy per baryon. A simple positive-definite expression for the gravitational mass of such a momentarily stationary configuration is obtained directly from the initial-value equations. Ways in which the different forms of the variational principle can be used in numerical calculations of the structure of rotating stars in a general relativity are discussed, as well as some analytic results relating to stability against convection.

I. INTRODUCTION

The study of rotating stars in general relativity is important for understanding how effective supermassive-star models can be as energy sources for QSOs (Fowler 1966; Roxburgh 1965) and has more recently received considerable impetus from the identification of pulsars as rotating neutron stars (Gold 1968; Pacini 1968). In both cases the general-relativistic corrections may be rather large, so the post-Newtonian approximation (Chandrasekhar 1965) does not always give an accurate description of the gravitational fields and the structure of the star. Although treating the effects of rotation as a small perturbation on the structure of a spherical star (Hartle 1967; Hartle and Thorne 1968) is adequate in models of the observed pulsars, it is not likely to be adequate in studying the formation of pulsars or the structure of supermassive stars.

In this paper, the calculation of the structure of a rapidly rotating star in the full theory of general relativity is formulated as a variational principle. This formulation may be of use in actual numerical calculations, but perhaps more important is the insight it gives into the general properties of equilibrium models and into the relationships between different models without the labor involved in a detailed calculation. The nonlinear partial differential equations for a rapidly rotating star in a strong gravitational field are difficult to solve numerically, even on a large computer.

The variational principle is related to two earlier variational principles. Taub (1954) shows how to derive the gravitational-field equations and the equations of motion for *any* perfect-fluid configuration from an action principle. On the other hand, Hartle and Sharp (1967) construct a variational principle which gives the Einstein equations and the equation of hydrostatic equilibrium for a uniformly rotating star. The value of the variational integral, which is extremized with the total rest mass (or baryon number) and the total angular momentum of the star kept constant, is equal to the total gravitational mass when evaluated for the equilibrium configuration. Hartle and Sharp assume that

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the pressure and energy density of the perfect fluid are functions of the baryon density alone; in other words, that the entropy per baryon is constant throughout the star.

Our variational principle applies to rotating perfect-fluid stars in which the entropy per baryon and the angular velocity vary with position inside the star. Furthermore, it is shown to be an energy-extremization principle in a fuller sense than Hartle and Sharp prove for their variational principle. What we show is that among all trial configurations for the rotating star which are axisymmetric, which are "momentarily stationary" in a sense to be defined later, which satisfy the initial-value equations on the momentarily stationary spacelike hypersurface, and in which each ring of matter has fixed values of the baryon number, entropy, and angular momentum, the equilibrium configuration extremizes the total gravitational mass. The gravitational mass is determined by the asymptotic behavior of the spatial metric components, which are assumed to correspond asymptotically to some form of the Schwarzschild metric. The word *extremum* is, in this paper, synonymous with "stationary with respect to small perturbations" and does not imply an actual minimum or maximum.

After introduction of the variables used to describe the gravitational field and the structure of the rotating star in § II of this paper, the variational principle is stated as an action principle in § III. The Euler-Lagrange equations for the action principle are derived in § IV. In § V it is shown that as an action principle our variational principle can be considered a special case of Taub's variational principle, adapted to stationary, axisymmetric configurations.

The physical interpretation of the metric-tensor components is discussed in § VI as background for the detailed proof of the energy-extremization properties of our variational principle in § VII. A relatively simple integral for the gravitational mass on a momentarily stationary hypersurface is obtained directly from the initial-value equations in § VIII. The integrand is positive-definite, and its form sheds light on the nature of the energy extremization. Finally, applications of the variational principle, primarily based on an analytic expression for the difference in energy between neighboring equilibrium models, are discussed in § IX.

The orthonormal tetrad for a "locally nonrotating" observer and the corresponding physical Riemann curvature tensor are given in Appendices, as well as additional formulae used in the main part of the paper.

Throughout the paper $G = c = 1$, and the sign conventions follow those in Landau and Lifshitz (1962). A slash denotes covariant differentiation; a comma, ordinary differentiation. Greek indices range from 0 to 3; Latin indices, from 2 to 3.

For background information on perfect-fluid stars in general relativity, both rotating and nonrotating, the reader is referred to reviews by Thorne (1967, 1970).

II. DESCRIPTION OF THE GRAVITATIONAL FIELD AND MATTER DISTRIBUTION

The gravitational field generated by an axisymmetric perfect-fluid star in equilibrium possesses two Killing vectors $\zeta_t^{\alpha} = \delta_0^{\alpha}$ and $\zeta_\phi^{\alpha} = \delta_1^{\alpha}$, such that the time coordinate $x^0 = t$ labels the spacelike hypersurfaces which are invariant under time translations and $x^1 = \phi$ is the axial-angle coordinate around the axis of symmetry. The metric-tensor components can only depend on the two remaining spatial coordinates x^2 and x^3. Furthermore, the energy-momentum tensor is invariant under a simultaneous change of sign of t and ϕ, since each change of sign just reverses the direction of rotation of the star. This additional symmetry implies (Carter 1969) that the metric-tensor components g_{02}, g_{03}, g_{12}, and g_{13} are identically zero.

The line element takes the form

$$ds^2 = g_{00}dt^2 + 2g_{01}dtd\phi + g_{11}d\phi^2 + g_{km}dx^k dx^m . \quad (1)$$

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This form of the metric is left invariant by transformation of x^2 and x^3 among themselves, so the three g_{km} can be expressed in terms of one independent function using these two coordinate transformations. However, if the full complement of Einstein equations is to be obtained from the variational principle, the coordinate conditions must not be applied until after the variation is taken.

The properties of the matter in its local rest frame are described by the energy density (including the rest-mass energy density) ϵ, the pressure p, the baryon density n, the temperature T, and the entropy per baryon s. We assume that the composition is the same everywhere in the star, but this condition could be relaxed by introducing composition variables which would be handled in the same way as the specific entropy s in the variational principle. Thorne (1967) discusses the properties of the more general equation of state. For our purposes the equation of state is the equation $\epsilon = \epsilon(n, s)$, from which the temperature T is

$$T = \frac{1}{n}\left(\frac{\partial\epsilon}{\partial s}\right)_n \tag{2}$$

and the pressure is

$$p = n\left(\frac{\partial\epsilon}{\partial n}\right)_s - \epsilon \tag{3}$$

using the first law of thermodynamics.

The four-velocity of a fluid element, U^a, must be a linear combination of the Killing vectors, so $U^2 = U^3 = 0$ identically. The angular velocity $\Omega = d\phi/dt$ of the fluid element measured by a distant observer in an asymptotically flat space-time is

$$\Omega = U^1/U^0 \,. \tag{4}$$

Since the four-velocity is a unit timelike vector,

$$U^a U_a = -1 \,, \tag{5}$$

the angular velocity Ω plus the metric completely specifies U^a. In particular,

$$U^0 = [-g_{00} - 2\Omega g_{01} - \Omega^2 g_{11}]^{-1/2} \,. \tag{6}$$

The energy-momentum tensor of a perfect fluid is

$$T^{a\beta} = (\epsilon + p)U^a U^\beta + g^{a\beta} p \,. \tag{7}$$

The angular momentum per baryon, something which is conserved in any axisymmetric, but not necessarily stationary, motion as a consequence of $T^{a\beta}{}_{/\beta} = 0$ and the equation for conservation of baryons $(nU^a)_{/a} = 0$ is

$$j = \frac{\epsilon + p}{n} U_1 = \frac{\epsilon + p}{n}(g_{01} + \Omega g_{11})U^0 \,. \tag{8}$$

The location of a particular ring of matter with a particular entropy and angular momentum is not known in terms of the coordinates x^k before the variations are carried out and the equilibrium configuration found, since it is not convenient to make the x^k comoving coordinates when the exterior metric as well as the interior metric is involved. The relative positions of different rings in the star are determined by the relative values of two Lagrangian coordinates ξ and η. Fixing the number of baryons in a ring means fixing the number of baryons having Lagrangian coordinate ξ in the range $d\xi$ and η in the range $d\eta$. The structure of the star is specified by the two functions $\xi(x^k)$ and $\eta(x^k)$ which determine which ring, if any, is present at a particular "Eulerian" coordinate point x^k. The entropy per baryon and the angular momentum per baryon of each ring can be specified by the functions $s(\xi, \eta)$ and $j(\xi, \eta)$, respectively.

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III. THE ACTION PRINCIPLE

In this section the variational principle is formulated as an action principle, without trying to relate the value of the action to the total gravitational mass or energy. The integral

$$I_1 = 2\pi \int\int \left[-\frac{1}{16\pi} R - T_0{}^0 \right] (-g)^{1/2} dx^2 dx^3 , \tag{9}$$

in which R is the scalar curvature constructed from the metric (1) and is considered a functional of the metric-tensor components $g_{\alpha\beta}(x^k)$ specifying the gravitational field and the geometry and of the functions $\xi(x^k)$ and $\eta(x^k)$ specifying the locations of the rings of matter. The entropy per baryon is a known function $s(\xi, \eta)$, and the functions $N(\xi, \eta)$ and $j(\xi, \eta)$ fixing the number of baryons

$$dA = N(\xi, \eta) d\xi d\eta \tag{10}$$

and the angular momentum

$$dJ = j(\xi, \eta) N(\xi, \eta) d\xi d\eta \tag{11}$$

in each ring of the star are known. The baryon density n and the angular velocity Ω are determined for particular functions ξ, η, and $g_{\alpha\beta}(x^k)$ by the constraints (10) and (11). The number of baryons in the coordinate range dx^2dx^3 is

$$dA = 2\pi n U^0 (-g)^{1/2} dx^2 dx^3 = N(\xi, \eta) \left(\frac{\partial \xi}{\partial x^2} \frac{\partial \eta}{\partial x^3} - \frac{\partial \eta}{\partial x^2} \frac{\partial \xi}{\partial x^3} \right) dx^2 dx^3 , \tag{12}$$

and Ω is related to $j(\xi, \eta)$ by equation (8). The Jacobian of the transformation from Lagrangian to Eulerian coordinates in equation (12) is required to be nonsingular to maintain the relative arrangement of the rings of matter. Then the statement of the action principle is that the functions $g_{\alpha\beta}(x^k)$, $\xi(x^k)$, and $\eta(x^k)$ which "extremize" the functional I_1 (about which I_1 is stationary) are the functions which characterize the equilibrium configuration of the rotating star, and vice versa. There is no guarantee that a solution exists for a particular choice of the constraints.

In practice, it is awkward to solve the constraints explicitly for n and Ω. The baryon density n is involved in equation (8) for Ω in a complicated nonlinear way. Therefore, it is useful to introduce an alternative form of the variational principle which replaces the explicit constraints by Lagrange multipliers. Since the baryon number and angular momentum are specified separately for each ring, there are separate Lagrange multipliers for each ring (for each pair of values of ξ and η). The Lagrange multiplier for the baryon number is $\Phi = \Phi(\xi, \eta)$; and the Lagrange multiplier for the angular momentum is $\Lambda = \Lambda(\xi, \eta)$. The new form of the variational principle is that the functional

$$\begin{aligned} I_2 &= I_1 - \int \Phi dA - \int \Lambda dJ \\ &= I_1 - 2\pi \int\int \Phi(\xi, \eta) n U^0 (-g)^{1/2} dx^2 dx^3 \\ &\quad - 2\pi \int\int \Lambda(\xi, \eta)(\epsilon + p) U_1 U^0 (-g)^{1/2} dx^2 dx^3 \end{aligned} \tag{13}$$

is stationary under independent first-order variations in the functions $g_{\alpha\beta}(x^k)$, $\xi(x^k)$, $\eta(x^k)$, $n(x^k)$, and $\Omega(x^k)$ if and only if these functions describe an equilibrium configuration which is a solution to the Einstein equations.

IV. THE EULER-LAGRANGE EQUATIONS

The proof of the variational principle as an action principle consists in showing that the Euler-Lagrange equations obtained by requiring the action to be stationary under arbitrary, independent variations of the appropriate functions are the Einstein equations

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and the equations of hydrostatic equilibrium for a rotating perfect-fluid star in general relativity. The second of the two equivalent forms of the variational principle stated in the last section is easiest to work with. The arbitrary, independent variations of the $g_{\alpha\beta}$, ξ, η, n, and Ω are carried out at constant Eulerian coordinate values x^k. Since the Lagrange multipliers are constant during the variation for a given ring of matter, their values at a particular x^k vary if the functions $\xi(x^k)$ and $\eta(x^k)$ are varied.

The variations with respect to $\Omega(x^k)$ and $n(x^k)$ are considered first, since these allow us to evaluate the Lagrange multipliers Φ and Λ. Note that

$$\frac{\delta U^0}{\delta \Omega} = U_1(U^0)^2. \tag{14}$$

The variation of I_2 with respect to $\Omega(x^k)$ gives

$$\begin{aligned}\delta I_2 = 2\pi \int\int \{ &-(\epsilon + p)(U^0)^2[2U_1U_0 + g_{01}] - \Lambda(\epsilon + p)(U^0)^2[2U_1U_1 + g_{11}] \\ &- \Phi n U_1(U^0)^2\} \delta\Omega(x^k)(-g)^{1/2}\, dx^2 dx^3 .\end{aligned} \tag{15}$$

The variation in I_2 is zero for arbitrary $\delta\Omega(x^k)$ if and only if

$$\begin{aligned}2(\epsilon + p)U_1U_0(U^0)^2 + 2\Lambda(\epsilon + p)(U_1U^0)^2 + (\epsilon + p)(g_{01} + \Lambda g_{11})(U^0)^2 \\ + \Phi n U_1(U^0)^2 = 0 .\end{aligned} \tag{16}$$

In varying the function $n(x^k)$, the independent functions ξ and η, and therefore the specific entropy $s(\xi, \eta)$, are kept constant. Equation (3) can then be used to write the Euler-Lagrange equation as

$$\frac{\epsilon + p}{n}(U_0U^0 + \Lambda U_1U^0) + \left(\frac{\partial p}{\partial n}\right)_s [(U_0U^0 + \Lambda U_1U^0) + 1] + \Phi U^0 = 0 . \tag{17}$$

Equations (16) and (17) are very similar to equations obtained in the proof of the Hartle-Sharp variational principle. The only physically acceptable solution for Φ and Λ is

$$\Lambda = \Omega \tag{18}$$

and

$$\Phi = \frac{\epsilon + p}{n}(U^0)^{-1} . \tag{19}$$

The Lagrange multiplier $\Lambda(\xi, \eta)$ is the angular velocity of the ring in the equilibrium configuration as seen by a distant observer. The quantity $(\epsilon + p)/n$ is the energy required to add one baryon to the ring at constant pressure and constant specific entropy, evaluated by a local comoving observer. The Lagrange multiplier Φ, the "injection energy," is this energy evaluated at infinity, since $(U^0)^{-1}$ is the ratio of energy at infinity to energy in the comoving frame for a freely falling particle with zero angular momentum (see Thorne 1967).

When Φ is constant over the star, equation (19) is the integral of the equation of hydrostatic equilibrium found by Hartle and Sharp (1967). In our more general case the equations of hydrostatic equilibrium are obtained by requiring I_2 to be stationary with respect to variations in the functions $\xi(x^k)$ and $\eta(x^k)$. The immediate result is

$$\begin{aligned}\left[(U_0U^0 + \Lambda U_1U^0)\left(\frac{\partial \epsilon}{\partial s}\right)_n + (U_0U^0 + \Lambda U_1U^0 + 1)\left(\frac{\partial p}{\partial s}\right)_n\right]\frac{\partial s}{\partial \xi} \\ + \frac{\partial \Lambda}{\partial \xi}(\epsilon + p)U_1U^0 + \frac{\partial \Phi}{\partial \xi} n U^0 = 0 ,\end{aligned}$$

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plus a similar equation from the variation with respect to η. By use of equations (18), (2), (4), and (5), these simplify to

$$\frac{\partial\Phi}{\partial\xi}+\frac{\partial\Omega}{\partial\xi}\frac{\epsilon+p}{n}U_1-T(U^0)^{-1}\frac{\partial s}{\partial\xi}=0 \tag{20}$$

and

$$\frac{\partial\Phi}{\partial\eta}+\frac{\partial\Omega}{\partial\eta}\frac{\epsilon+p}{n}U_1-T(U^0)^{-1}\frac{\partial s}{\partial\eta}=0\,. \tag{21}$$

The derivatives with respect to ξ and η can obviously be replaced by derivatives with respect to the x^k.

A more recognizable form of the equations of hydrostatic equilibrium $T_k{}^a{}_{/a}=0$ is obtained by substituting

$$nU^0\frac{\partial\Phi}{\partial x^k}=\left(\frac{\partial p}{\partial n}\right)_s\frac{\partial n}{\partial x^k}+\left[\left(\frac{\partial\epsilon}{\partial s}\right)_n+\left(\frac{\partial p}{\partial s}\right)_n\right]\frac{\partial s}{\partial x^k}-\frac{(\epsilon+p)}{U^0}\frac{\partial U^0}{\partial x^k} \tag{22}$$

into equations (20) and (21). The result is

$$\frac{\partial p}{\partial x^k}-\frac{(\epsilon+p)}{U^0}\frac{\partial U^0}{\partial x^k}+\frac{\partial\Omega}{\partial x^k}(\epsilon+p)U_1U^0=0\,. \tag{23}$$

However, the form (20), (21) is very useful in applying the variational principle and will be analyzed in more detail in § IX.

The variation of the $g_{\alpha\beta}(x^k)$ in I_2 yields the Einstein equations. Standard results (Landau and Lifshitz 1962) are that

$$\frac{\delta(-g)^{1/2}}{\delta g_{\alpha\beta}}=\tfrac{1}{2}(-g)^{1/2}g^{\alpha\beta}\,, \tag{24}$$

and that within a divergence

$$\frac{\delta[R(-g)^{1/2}]}{\delta g_{\alpha\beta}}=[-R^{\alpha\beta}+\tfrac{1}{2}g^{\alpha\beta}R](-g)^{1/2}\,. \tag{25}$$

Also,

$$\frac{\delta U^0}{\delta g_{\alpha\beta}}=\tfrac{1}{2}U^\alpha U^\beta U^0\,. \tag{26}$$

The variation with respect to g_{00} gives

$$\begin{aligned}\frac{1}{16\pi}[R^{00}-\tfrac{1}{2}g^{00}R]-(\epsilon+p)(U^0)^2[U_0U^0+1]-\tfrac{1}{2}g^{00}[(\epsilon+p)U_0U^0+p]\\-\Lambda(\epsilon+p)U_1U^0[(U^0)^2+\tfrac{1}{2}g^{00}]-\tfrac{1}{2}\Phi nU^0[g^{00}+(U^0)^2]=0\,.\end{aligned} \tag{27}$$

This simplifies to

$$\begin{aligned}R^{00}-\tfrac{1}{2}g^{00}R&=8\pi[(\epsilon+p)U^0U^0+pg^{00}]\\&=8\pi T^{00}\,.\end{aligned} \tag{28}$$

Similarly, the variations with respect to g_{01} and g_{11} yield

$$R^{01}-\tfrac{1}{2}g^{01}R=8\pi(\epsilon+p)U^0U^1=8\pi T^{01} \tag{29}$$

and

$$R^{11}-\tfrac{1}{2}g^{11}R=8\pi[(\epsilon+p)U^1U^1+pg^{11}]=8\pi T^{11} \tag{30}$$

when the Lagrange multipliers Λ and Φ are eliminated through equations (18) and (19). The remaining Einstein equations come from the variations with respect to the g_{kl}:

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$$\frac{1}{16\pi}[R^{kl} - \tfrac{1}{2}g^{kl}R] - \tfrac{1}{2}g^{kl}[(\epsilon + p)U_0U^0 + p + \Lambda(\epsilon + p)U_1U^0 + \Phi n U^0] = 0 ; \quad (31)$$

so

$$R^{kl} - \tfrac{1}{2}g^{kl}R = 8\pi p g^{kl} = 8\pi T^{kl} . \quad (32)$$

Nothing is lost by setting the g_{0k} and g_{1k} equal to zero before taking the variation, since the Ricci tensor components R_{0k} and R_{1k} are identically zero as a result of the symmetries. There are only six nontrivial Einstein equations for a stationary, axisymmetric rotating-star metric.

What is now established is that the functional I_2 is stationary under arbitrary independent variations of the $g_{\alpha\beta}$, ξ, η, n, and Ω, for given $\Phi(\xi, \eta)$ and $\Lambda(\xi, \eta)$, if and only if the Einstein equations are satisfied by the metric tensor and the covariant divergence of the energy-momentum tensor is zero for a stationary, axisymmetric, rotating perfect-fluid distribution. Because of the role of the Φ and the Λ as Lagrange multipliers, this implies that the functional I_1 is stationary under arbitrary independent variations of the $g_{\alpha\beta}$, ξ, and η, when the constraints (10) and (11) for given baryon-number and angular-momentum distributions $N(\xi, \eta)$ and $j(\xi, \eta)$ are used to calculate the variations in n and Ω, if and only if the same equations are satisfied.

The fact that coordinate conditions on the x^k can be used to express the g_{km} in terms of one independent function plays a role in the variational principle, along with the fact that the equations $T_k{}^\alpha{}_{/\alpha} = 0$ are a direct consequence of the Einstein equations and the Bianchi indentities and in this sense are redundant. Treating the g_{kl} as independent functions in the variational principle yields all six Einstein equations, and one has a complete set of equations without requiring that the action be stationary under variations of the functions ξ and η. However, if the coordinate conditions are applied *before* taking the variation, so the g_{kl} are not independent, the three Euler-Lagrange equations (32) reduce to *one* equation which is a linear combination of the three. Now the variations with respect to ξ and η are needed to give the two $T_k{}^\alpha{}_{/\alpha} = 0$ equations which, together with the four modified Einstein equations, form a complete set (see Hartle and Sharp 1967).

One could make the coordinate choice $x^2 = \xi$, $x^3 = \eta$ in the interior of the star (in which case the variations of ξ and η would be suppressed) and keep all six metric components as independent. This does not greatly increase the complexity of the calculations in the interior, where there are six independent functions anyway, but it complicates the problem of joining the interior solution to the exterior solution. Also, interpretation of the geometry is difficult unless the metric has a simple form. Coordinate conditions on the form of the metric are probably best applied from the beginning in both interior and exterior in an actual numerical calculation.

V. RELATIONSHIP TO OTHER ACTION PRINCIPLES

As an action principle our variational principle is a straightforward generalization of the Hartle-Sharp variational principle, also considered as an action principle. Hartle and Sharp (1967) constrain only the total baryon number and total angular momentum of the star, instead of the baryon number and angular momentum of each ring, so their Lagrange multipliers are constants over the whole star.

Our variational principle is also related to a variational principle of Taub (1954). Taub's principle derives the Einstein equations, the equations of motion $T_\alpha{}^\beta{}_{/\beta} = 0$, and the constancy of entropy along the fluid-element world lines for *any* perfect fluid configuration, not necessarily time independent or axisymmetric, from variations of the functional

$$I_T = \int \left[\frac{1}{16\pi}R + \epsilon - nTs + n\sigma g_{\alpha\beta}U^\alpha U^\beta\right](-g)^{1/2}d^4x . \quad (33)$$

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This functional is required to be stationary with respect to arbitrary, independent variations of the metric-tensor components $g_{\alpha\beta}$, of the trajectories of the fluid elements as specified by the dependence of their coordinate positions x^α in the space-time as functions of proper time, and of the temperature T along the world line of each fluid element. The constraint that the rest mass (or baryon number) of each fluid element be conserved is applied explicitly and determines the variation in the baryon density n in terms of the variations in the other quantities. The Lagrange multiplier σ is determined for each fluid element by requiring that the four-velocity U^α be a unit timelike vector in the final solution.

The details of Taub's principle are quite different from ours. For instance, Taub's variations are carried out along the fluid-element world lines, as opposed to our Eulerian variations at constant x^k. We explicitly normalize the four-velocity, instead of introducing a Lagrange multiplier. Perhaps most important, we determine the solution more or less uniquely by the symmetry conditions and by specifying the rest mass and angular momentum of each fluid element, whereas there is nothing in Taub's principle to pick out a particular solution.

Still, as an action principle our variational principle can be considered a special case of Taub's. To show this, we first specialize Taub's principle to stationary, axisymmetric configurations. Then the integral over $x^1 = \phi$ can be done. The principle is further modified by dropping the variation of the temperature and the term $-nTs$ in (33), since these are only included to derive the time independence of the entropy, which is true automatically for a stationary, axisymmetric rotating star. Also, the last term in equation (33) can be eliminated by explicitly normalizing the four-velocity to a unit vector.

The variation of the functions $\xi(x^k)$ and $\eta(x^k)$ in our principle is the Eulerian version of the variation of particle paths in Taub's principle as far as the location of the trajectory in the (x^2, x^3)-plane is concerned. The variation of particle paths would also involve variations in the motion in the ϕ-direction, but to specify a particular solution we will fix $\Omega = d\phi/dt$ for each fluid element. This is formally equivalent to applying the angular-momentum constraint through the Lagrange multiplier Λ in our principle, since $\Lambda = \Omega$ after the variation of the angular velocity has been carried out. Therefore, if in our principle $\Lambda(\xi, \eta)$ is specified, Ω set equal to Λ, and baryon number conserved by explicit constraint, the variational integral becomes

$$\begin{aligned} I_3 &= \pi \int\int \left[-\frac{1}{16\pi} R - T_0{}^0 - \Lambda(\epsilon + p) U_1 U^0 \right] (-g)^{1/2} dx^2 dx^3 \\ &= 2\pi \int\int \left[-\frac{1}{16\pi} R + \epsilon \right] (-g)^{1/2} dx^2 dx^3 \,. \end{aligned} \tag{34}$$

The modified version of the two variational principles are now identical except for the technical point of Eulerian versus Lagrangian variations.

As Hartle and Sharp emphasize, the additional interpretation of their and our variational principles as energy-extremization principles makes them much more useful in actual physical applications to rotating stars than the pure action principle of Taub.

VI. PHYSICAL INTERPRETATION OF THE GEOMETRY

Most discussions of the physical effects of rotation on the properties of space-time when the gravitational fields are strong have been based on the properties of slowly rotating configurations, with the effects of rotation treated as perturbing spherically symmetric space-time (see Hartle 1967; Brill and Cohen 1966; Hartle and Thorne 1968; Thorne 1970). Until recently (Bardeen and Wagoner 1969) there have been no solutions to the Einstein equations for a perfect-fluid configuration in an asymptotically flat space-time, either analytic or numerical, involving fast rotation in arbitrarily strong

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gravitational fields and encompassing both the sources of the gravitational field and the vacuum exterior metric. In such cases an analysis based on *static* observers, observers at rest as seen from infinity, is not appropriate because there is no local property of the space-time which singles out these observers and because such observers may not even exist in certain regions of the space-time. In this section the components of the metric tensor are interpreted physically in ways which remain valid no matter how rapid the rotation and how strong the gravitational fields, as long as the space-time is stationary and axisymmetric.

This physical interpretation is facilitated by writing

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 . \tag{35}$$

One degree of freedom in the choice of coordinates has been used to set $g_{23} = 0$. The remaining degree of freedom is left unspecified for the present.

The signs chosen for the exponentials are the signs that must hold if the metric is to be stationary, that is, if the $t =$ constant hypersurfaces are to be spacelike and if the metric is to have the correct signature. On the other hand,

$$g_{00} = -e^{2\nu} + \omega^2 e^{2\psi} \tag{36}$$

can have either sign.

In a region of space-time where $g_{00} > 0$, the "time axis," the line $(\phi, x^2, x^3) =$ constant, is not a timelike direction. In other words, no local observer with $\Omega = d\phi/dt = 0$ and $U^2 = U^3 = 0$, an observer who would be stationary as seen from infinity in an asymptotically flat space-time, can exist in such a region.

A set of local observers identifiable uniquely by local measurements in a stationary, axisymmetric space-time is what might be called the "locally nonrotating" observers. An observer with $U^2 = U^3 = 0$ and an angular velocity $\Omega = d\phi/dt$ as measured by a distant stationary observer can measure his absolute velocity of rotation locally (extending local to all values of ϕ at the same x^2 and x^3) by the following gedankenexperiment. Imagine that mirrors have been set up on the circle $x^2 = x^3 =$ constant containing the observer, so he can send a light signal around this circle in either the forward ($d\phi > 0$) or backward ($d\phi < 0$) direction. In terms of the line element (35) and the angular velocity Ω of the observer, the proper time the light signal takes to make the circuit in the forward direction is

$$\Delta S_1 = 2\pi e^{\psi}\left(\frac{1+v}{1-v}\right)^{1/2} , \tag{37}$$

where

$$v = (\Omega - \omega)e^{\psi-\nu} . \tag{38}$$

The proper time it takes when sent in the backward direction is

$$\Delta S_2 = 2\pi e^{\psi}\left(\frac{1-v}{1+v}\right)^{1/2} . \tag{39}$$

Just as in special relativity, a measure of the proper distance around the circle which is independent of the velocity of rotation of the observer is (with $c = 1$) the geometric mean of the two proper times (37) and (39). This identifies e^{ψ} as the *proper circumferential radius* of the circle around the axis of symmetry. Also, comparing equations (37) and (39) with the corresponding special-relativistic formulae, we identify v as the local *velocity of rotation* of the observer. The *locally nonrotating observer* then has $v = 0$, or $\Omega = d\phi/dt = \omega$, and is the only observer at each (x^2, x^3) for which $\Delta S_1 = \Delta S_2$. It is this observer whose world line is perpendicular to the $t =$ constant spacelike hypersurfaces. The congruence of world lines with $U^1 = \omega U^0$, $U^2 = U^3 = 0$ has an angular velocity four-vector (Synge 1960, p. 172) which is identically zero everywhere. The function $\omega(x^2, x^3)$, which is the angular velocity as seen from infinity of the locally nonrotating

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observer, is sometimes called the angular velocity of the local inertial frame (see Hartle 1967), and is the angular velocity of any particle or photon which has zero angular momentum.

The ratio of frequency observed at infinity to emitted frequency in the locally nonrotating frame is equal to e^{ν} for photons of zero angular momentum. Therefore $(e^{-\nu} - 1)$ is identified as the *gravitational redshift*, and ν can be considered the general-relativistic gravitational potential.

Photons of zero angular momentum emitted in a frame with rotational velocity $v \neq 0$ will be Doppler-shifted. The ratio of observed to emitted frequency is $(U^0)^{-1} = e^{\nu}(1 - v^2)^{1/2}$, where U^0 is the time component of the velocity four-vector of the observer with rotational velocity v. The frequency shift is a combination of gravitational redshift and transverse Doppler shift, since zero angular momentum implies emission in a direction perpendicular to the ϕ-direction in the locally nonrotating frame.

The physical simplicity of the locally nonrotating frame is echoed by the mathematical simplicity of the Riemann and Ricci tensor components measured by the locally nonrotating observer when expressed in terms of the metric functions in equation (35). The orthonormal tetrad for the locally nonrotating observer's frame of reference is defined in Appendix A, and the Riemann tensor components projected onto this tetrad are given in Appendix B.

VII. ENERGY EXTREMIZATION

The Euler-Lagrange equations from the variational integral I_1 (eq. [9]) are unaffected by adding a pure divergence to the integrand. Therefore, the value of the action I_1 can be altered more or less at will. Hartle and Sharp (1967) show how to alter the action so that when evaluated for the final equilibrium configuration it is equal to the total gravitational mass, or energy, of the equilibrium configuration. Here we go further and show that by proper choice of the divergence the action can be made equal to the gravitational mass of all trial, nonequilibrium configurations which are *momentarily stationary* and satisfy the initial-value equations of general relativity. Furthermore, the action is modified to contain at most first derivatives of the metric functions such that the integral (9) converges for all trial configurations in an asymptotically flat space-time.

Because of the importance of the asymptotic behavior of the metric functions at spatial infinity, we will take x^2 and x^3 to be spherical coordinates at infinity with $x^2 = r$ and $x^3 = \theta$. If functions $\alpha(r, \theta)$, $\mu(r, \theta)$, and $\gamma(r, \theta)$ are defined by

$$e^{\psi} = r \sin\theta \, e^{\alpha}, \quad e^{\mu_2} = e^{\mu}, \quad e^{\mu_3} = re^{\gamma}, \tag{40}$$

then asymptotically flat boundary conditions give $\nu, \alpha, \mu, \gamma \sim r^{-1}$ and $\omega \sim r^{-3}$ as $r \to \infty$. The one remaining degree of freedom in the choice of coordinate system can be used to set $\gamma = \alpha$, as appropriate for the study of perturbations of slowly rotating stars (Hartle 1967), or to set $\gamma = \mu$, which gives simpler equations for rapidly rotating stars (Hartle and Sharp 1967; Bardeen and Wagoner 1969).

The functions ν, ω, α, μ, and (γ) will now be the independent functions describing the gravitational field in the variational principle, instead of the metric-tensor components $g_{\alpha\beta}$ themselves. The Euler-Lagrange equations obtained by varying these functions will then be certain combinations of the Einstein equations.

The concept of a momentarily stationary configuration enters in the following way. The variational integral is considered as an integral over the spacelike hypersurface $t = 0$. This hypersurface is embedded in an axisymmetric, but not necessarily time-independent, space-time. Momentarily "stationary" implies that the space-time is symmetric under *simultaneous* change of sign of the time coordinate t and the axial angle ϕ. The four-velocity components U^2 and U^3 of the fluid must then be zero on the $t = 0$ hypersurface. In Appendix D we show that coordinates can always be chosen such that all *first* time

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derivatives of the metric-tensor components are zero at $t = 0$ and such that on the momentarily stationary hypersurface the line element has the form (35) as further specified by equations (40). The rest of the discussion assumes that this choice has been made.

If the metric-tensor components in the (nonstationary) space-time are to obey the Einstein equations, certain consistency conditions must be satisfied on the $t = 0$ hypersurface. These are the initial-value equations (Bruhat 1962). The nontrivial initial-value equations here are

$$R_0{}^0 - \tfrac{1}{2}R = 8\pi T_0{}^0 \quad (41)$$

and

$$R_1{}^0 = 8\pi T_1{}^0 \,. \quad (42)$$

These equations contain at most first time derivatives of the metric, which means that on the momentarily stationary hypersurface they are completely equivalent to the corresponding equations for the equilibrium configuration. They are discussed in detail in § VIII.

The total energy or gravitational mass of a momentarily stationary configuration satisfying the initial-value equations can be calculated if the *spatial* metric on the $t = 0$ hypersurface is asymptotically Schwarzschild. Physically this condition means that gravitational waves are not present in this hypersurface at arbitrarily large distances from the star, and mathematically it means that to order r^{-1} in an expansion of the potentials in powers of $1/r$ at infinity the spatial metric components correspond to some form of the Schwarzschild metric. Independent of the particular choice of the radial coordinate in the Schwarzschild metric, the gravitational mass on the asymptotically time-symmetric hypersurface can be evaluated from (Hernandez and Misner 1966)

$$M = \lim_{r\to\infty} \{r[1 - (re^{\gamma})_{,r}e^{-\mu}] = r[1 - (re^{\alpha})_{,r}e^{-\mu}]\} \,. \quad (43)$$

The asymptotic behavior of ν has no direct bearing on the gravitational mass on an only momentarily stationary hypersurface (see Harrison *et al.* 1965). The function ω has an asymptotic behavior (Papapetrou 1948)

$$\lim_{r\to\infty} (r^3\omega) = 2J \,, \quad (44)$$

where J is the total angular momentum, if the initial-value equation (42) is satisfied.

The scalar curvature constructed from the metric (1) is not the scalar curvature evaluated on a momentarily stationary hypersurface in a nonstationary space-time, since the latter contains second time derivatives which will not be zero. Let R be the scalar curvature of the nonstationary space-time and let R^* be R minus the second time derivatives, evaluated on the momentarily stationary hypersurface. Then R^* has the same form as the scalar curvature for the equilibrium configuration and is what is called R in the variational integrals I_1 and I_2.

Since adding a pure divergence to the integrand of a variational integral does not affect the Euler-Lagrange equations, a variational integral equivalent to I_1 is

$$I'_1 = 2\pi \int\int [-L_G - T_0{}^0(-g)^{1/2}] dr d\theta \,, \quad (45)$$

where the "Lagrangian density" of the gravitational field L_G is

$$8\pi L_G = \tfrac{1}{2}R^*(-g)^{1/2} + W^k{}_{,k} \,. \quad (46)$$

The integration is over the momentarily stationary hypersurface. In terms of the functions ν, ω, α, μ, and γ, the expression (C3) for R^* gives

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$$\tfrac{1}{2}R^*(-g)^{1/2} = r^2 \sin\theta\, e^{\nu+\alpha+\gamma+\mu}\Big\{e^{-2\mu}[\alpha_{,r}\gamma_{,r} + \nu_{,r}(\alpha_{,r} + \gamma_{,r})] + \frac{1}{r^2}\, e^{-2\gamma}[\alpha_{,\theta}\mu_{,\theta} + \nu_{,\theta}(\alpha_{,\theta} + \mu_{,\theta})] + \tfrac{1}{4}r^2 \sin^2\theta\, e^{2\alpha-2\nu}\left(e^{-2\mu}\omega_{,r}\omega_{,r} + \frac{1}{r^2}\, e^{-2\gamma}\omega_{,\theta}\omega_{,\theta}\right) + \frac{1}{r}\, e^{-2\mu}(2\nu_{,r} + \alpha_{,r} + \gamma_{,r}) + \frac{1}{r^2}\, e^{-2\mu} + \frac{\cot\theta}{r^2}\, e^{-2\gamma}(\nu_{,\theta} + \mu_{,\theta})\Big\} - [e^{-\mu}(r^2 \sin\theta\, e^{\nu+\alpha+\gamma})_{,r}]_{,r} - [e^{-\gamma}(\sin\theta\, e^{\nu+\alpha+\mu})_{,\theta}]_{,\theta}\,. \tag{47}$$

Expressions are now chosen for W^r and W^θ so that (1) L_G contains at most first derivatives of the metric functions; (2) the integral I'_1 converges for any choice of the metric functions so long as they have the appropriate asymptotic behavior as $r \rightarrow \infty$, and (3) the value of I'_1 is equal to the total gravitational mass of the momentarily stationary configuration when the initial-value equations (41) and (42) are satisfied and the spatial metric is asymptotically Schwarzschild. One satisfactory choice is

$$W^r = e^{-\mu}(r^2 \sin\theta\, e^{\nu+\alpha+\gamma})_{,r} - 2r \sin\theta\, e^{\nu+\alpha}\,, \tag{48}$$

$$W^\theta = e^{-\gamma}(\sin\theta\, e^{\nu+\alpha+\mu})_{,\theta} - 2\cos\theta\, e^{\nu+\alpha} + \cos\theta\, e^{\nu+\alpha+\gamma+\mu}\,. \tag{49}$$

The result for L_G is

$$8\pi L_G = r^2 \sin\theta\, e^{\nu+\alpha+\gamma+\mu}\Big\{e^{-2\mu}[\alpha_{,r}\gamma_{,r} + \nu_{,r}(\alpha_{,r} + \gamma_{,r})] + \frac{1}{r^2}\, e^{-2\gamma}[\alpha_{,\theta}\mu_{,\theta} + \nu_{,\theta}(\alpha_{,\theta} + \mu_{,\theta})] + \tfrac{1}{4}r^2 \sin^2\theta\, e^{\alpha-\nu}\left(e^{-2\mu}\omega_{,r}\omega_{,r} + \frac{1}{r^2}\, e^{-2\gamma}\omega_{,\theta}\omega_{,\theta}\right) + \frac{1}{r}\,[e^{-2\mu}(2\nu_{,r} + \alpha_{,r} + \gamma_{,r}) - 2e^{-\mu-\gamma}(\nu_{,r} + \alpha_{,r})] + \frac{\cot\theta}{r^2}\left[e^{-2\gamma}(\nu_{,\theta} + \mu_{,\theta}) - 2e^{-\mu-\gamma}(\nu_{,\theta} + \alpha_{,\theta}) + e^{-2\mu}(\nu_{,\theta} + \alpha_{,\theta} + \gamma_{,\theta} - \mu_{,\theta})\right]\Big\}\,, \tag{50}$$

which by inspection satisfies the first two conditions since ν, α, γ, $\mu \sim r^{-1}$ and $\omega \sim r^{-3}$ as $r \rightarrow \infty$. The form of L_G simplifies considerably if the remaining coordinate condition is applied to make $\gamma = \mu$ and becomes equivalent to a form given by Hartle and Sharp (1967).

The initial-value equation (41) is the one used to evaluate I'_1. Since the left-hand side of equation (41) contains no second time derivatives, the second time derivatives in $R_0{}^0$ must be precisely the same as those in $\frac{1}{2}R$. If $R^*{}_0{}^0$ is defined to be $R_0{}^0$ computed from the stationary metric (1) or (35), equation (41) can be rewritten as

$$R^*{}_0{}^0 - \tfrac{1}{2}R^* = 8\pi T_0{}^0 \tag{51}$$

on the momentarily stationary hypersurface. Equation (51) gives

$$I'_1 = \tfrac{1}{4}\int\!\!\int[-R^*{}_0{}^0(-g)^{1/2} - W^k{}_{,k}]drd\theta\,. \tag{52}$$

The expression for $R^*{}_0{}^0$ is obtained from Appendix C. It is

$$R^*{}_0{}^0(-g)^{1/2} = [r^2 \sin\theta\, e^{\nu+\alpha+\gamma-\mu}(-\nu_{,r} + \tfrac{1}{2}r^2 \sin^2\theta\, e^{2\alpha-2\nu}\omega\omega_{,r})]_{,r} + [\sin\theta\, e^{\nu+\alpha+\mu-\gamma}(-\nu_{,\theta} + \tfrac{1}{2}r^2 \sin\theta\, e^{2\alpha-2\nu}\omega\omega_{,\theta})]_{,\theta} = Z^k{}_{,k}\,. \tag{53}$$

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Since the integrand in equation (52) has the form of a simple divergence, the volume integral can be converted to a surface integral to obtain

$$I'_1 = \tfrac{1}{4}\int_0^\pi d\theta[-Z^r - W^r]_{r=\infty} . \quad (54)$$

From equations (53) and (48)

$$-Z^r - W^r = r \sin\theta\, e^{\nu+\alpha}\left[2 - \frac{1}{r}\, e^{-\alpha-\mu}(r^2 e^{\alpha+\gamma})_{,r} - \tfrac{1}{2} r^3 \sin^2\theta\, e^{2\alpha-2\nu+\gamma-\mu}\omega\omega_{,r}\right] . \quad (55)$$

In the limit $r \to \infty$, $\omega \sim r^{-3}$ and asymptotically Schwarzschild implies that $\alpha = \gamma$ to order r^{-1}. Thus equation (43) can be used in equation (55) to give

$$-Z^r - W^r \to 2M \sin\theta \quad (56)$$

as $r \to \infty$, and

$$I'_1 = \tfrac{1}{2} M \int_0^\pi \sin\theta d\theta = M , \quad (57)$$

the total gravitational mass of the momentarily stationary configuration.

While the initial-value equation (42) has not been used explicitly in deriving equation (57), it must be satisfied if the momentarily stationary configuration is to be physically realizable.

Combining the results of this section with those of §§ III and IV, we have finally proved that the equilibrium configuration of a rotating star extremizes the total energy or gravitational mass of momentarily stationary configurations which satisfy the initial-value equations, keeping the baryon number or rest mass, the angular momentum, and the specific entropy of each ring of matter in the star fixed. Since the action I'_1 is stationary with respect to arbitrary variations of the metric functions and the matter functions ξ and η about the equilibrium configuration, it is still stationary when the additional constraints of the initial-value equations are imposed. However, the action I'_1 is not minimized by the equilibrium configuration, so it is not necessarily true that *only* the equilibrium configuration extremizes the total energy. The functions ν and ω, in particular, cannot be completely determined by extremizing the total energy, since they are affected by coordinate transformations which do not alter the form of the metric in the vicinity of the momentarily stationary hypersurface (see Appendix D).

In spherically symmetric, momentarily static configurations ω is identically zero and the total energy is completely independent of ν. The extent to which the energy depends on ν when $\omega \neq 0$ is discussed further in § VIII.

The equilibrium configuration may or may not locally minimize the energy. If the equilibrium configuration is unstable to axisymmetric, adiabatic perturbations, there will be some nearby momentarily stationary configurations of lower energy than the equilibrium configuration. No proof of the connection between stability and the nature of the energy extremum has been carried out for rotating stars in general relativity as it has been for spherically symmetric stars (see Harrison *et al.* 1965). Any such proof must deal with the gravitational radiation emitted by the perturbed configuration, though this may well not be important if the change of stability occurs through a zero-frequency mode. The stability of rotating stars in Newtonian theory is analyzed in detail by Lynden-Bell and Ostriker (1967). See also the review by Lebovitz (1967).

VIII. THE INITIAL-VALUE EQUATIONS

Harrison *et al.* (1965) established an energy-extremization principle for spherically symmetric stars which is based on an expression for the gravitational mass on a momentarily static (time-symmetric) spacelike hypersurface obtained directly from the initial-value equation. When the mass is extremized with the number of baryons in each spheri-

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cal shell explicitly kept constant, the "TOV" equation of hydrostatic equilibrium is obtained as an Euler-Lagrange equation. The metric function ν does not appear in this expression for the total energy and thus is left undetermined, in contrast to its appearance in the action I'_1 with L_G given by equation (50).

In this section we show that an expression for the total energy of a momentarily stationary rotating star can also be obtained from the initial-value equations. This expression for the energy is *positive definite*, giving an explicit proof that the energy of a momentarily stationary rotating star is positive no matter what axisymmetric gravitational waves are present on the hypersurface. We have not been able to obtain Euler-Lagrange equations from this expression because of the difficulty of solving the initial-value equations.

The initial-value equations (C7) and (C9) of Appendix C are rewritten in terms of the two spatial metric functions λ and β defined by

$$e^{\psi} = r \sin\theta\, e^{\lambda+\beta}\,, \qquad e^{\mu_2} = \frac{1}{r} e^{\mu_3} = e^{\lambda-\beta}\,. \tag{58}$$

The coordinate condition $\gamma = \mu$ is implicit in equation (58). The functions λ and β are chosen so that no second derivatives of β appear in the initial-value equation (C7). The new form of the equations is

$$\begin{aligned}\nabla^2\lambda = &-\tfrac{1}{2}\nabla(\lambda+\beta)\cdot\nabla(\lambda+\beta) - \tfrac{1}{8}r^2\sin^2\theta\, e^{2\lambda+2\beta-2\nu}\nabla\omega\cdot\nabla\omega \\ &- \frac{1}{r\sin\theta}\nabla(r\sin\theta)\cdot\nabla\beta - 4\pi e^{2\lambda-2\beta}\frac{\epsilon+pv^2}{1-v^2}\end{aligned} \tag{59}$$

and

$$\nabla\cdot[r^2\sin^2\theta\, e^{3\lambda+3\beta-\nu}\nabla\omega] = -16\pi e^{3\lambda-\beta} r\sin\theta\, e^{\lambda+\beta}\frac{(\epsilon+p)v}{1-v^2}\,. \tag{60}$$

The differential operators ∇, $\nabla\cdot$, and ∇^2 are the flat-space gradient, divergence, and Laplacian, respectively, in the spherical r, θ coordinates.

At spatial infinity the coordinate conditions are such that the metric approaches the Schwarzschild metric as written in isotropic coordinates. Since $\alpha = \lambda$ to order $1/r$, $\beta \sim 1/r^2$ as $r \to \infty$. The expression (43) for the total gravitational mass M on the momentarily stationary hypersurface gives

$$\lambda \simeq M/r \tag{61}$$

as $r \to \infty$.

Local flatness and regularity of the metric at the axis of symmetry require that β go to zero, with

$$\beta \sim r^2 \sin^2\theta\,, \tag{62}$$

as $r\sin\theta \to 0$. Otherwise, the ratio of circumference to radius of an infinitesimal circle around the axis of symmetry would not be 2π.

From equation (59) and the asymptotic behavior (61), the gravitational mass of a momentarily stationary configuration can be written

$$\begin{aligned}M = \int\int r^2\sin\theta\, dr d\theta \Big\{ &2\pi e^{2\lambda-2\beta}\frac{\epsilon+pv^2}{1-v^2} + \tfrac{1}{4}\nabla(\lambda+\beta)\cdot\nabla(\lambda+\beta) \\ &+ \tfrac{1}{16}r^2\sin^2\theta\, e^{2\lambda+2\beta-2\nu}\nabla\omega\cdot\nabla\omega + \tfrac{1}{2}\nabla[\ln(r\sin\theta)]\cdot\nabla\beta \Big\}\,.\end{aligned} \tag{63}$$

In this integral over the whole spacelike hypersurface the functions λ and ω are understood to be solutions of equations (59) and (60), respectively. The only restrictions on

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the function β are the boundary conditions mentioned above, and the matter distribution is only required to be bounded.

The integral (63) can be simplified by integrating the last term by parts,

$$\int\int \nabla[\ln (r \sin \theta)] \cdot \nabla\beta r^2 \sin \theta dr d\theta = - \int\int \beta\nabla^2[\ln (r \sin \theta)] r^2 \sin \theta dr d\theta \,. \quad (64)$$

The boundary terms vanish from the conditions on β at infinity and on the axis of symmetry. Since

$$\nabla^2[\ln (r \sin \theta)] = 0 \,, \quad (65)$$

the integral (64) vanishes. The expression for M reduces to

$$M = \int\int r^2 \sin \theta dr d\theta \Big\{ 2\pi e^{2\lambda - 2\beta} \frac{\epsilon + p v^2}{1 - v^2} + \tfrac{1}{4}\nabla(\lambda + \beta)\cdot\nabla(\lambda + \beta) + \tfrac{1}{16} r^2 \sin^2 \theta \, e^{2\lambda + 2\beta - 2\nu} \nabla\omega \cdot \nabla\omega \Big\} \,, \quad (66)$$

which is positive definite as long as the energy density of the matter in the locally nonrotating frame is nonnegative. Since M includes the energy of any gravitational waves present, this is a direct proof that the presence of gravitational waves on a momentarily stationary hypersurface cannot affect the sign of the energy. Brill and Deser (1968) have shown the positive-definiteness of gravitational-wave energy more generally, but by indirect arguments. Brill (1959) showed earlier that axisymmetric configurations which are time-symmetric have positive energy.

A variational principle based directly on extremizing the integral (66) for the gravitational mass on a momentarily stationary hypersurface, analogous to the variational principle of Harrison *et al.* in the spherically symmetric case, cannot hope to determine completely the functions ν and ω for the stationary equilibrium configuration. There is a certain amount of freedom in the choice of scale of the time coordinate as a function of position on the hypersurface. In Appendix D we show that the most general coordinate transformation which leaves the form of the metric and the Einstein equations on the $t = 0$ hypersurface unchanged has the result of altering the metric functions ν and ω to ν' and ω' with

$$\omega' = \int_0^{\omega} e^{F(\omega)} d\omega \,, \qquad \nu' = \nu + F(\omega) \,. \quad (67)$$

The other metric functions are invariant under the transformation. The function $F(\omega)$ is arbitrary, except for the condition $F(0) = 0$ which keeps the metric asymptotically flat. Any coordinate transformation which does not affect the asymptotic behavior of the spatial metric cannot change the gravitational mass, and consistent with this the transformation (67) leaves invariant the combination $e^{-\nu}\nabla\omega$ which appears in equation (66) and in equation (59) for λ.

An attempt to use equation (66) as the basis for a variational principle would proceed as follows. Vary the functions β, ν, ξ, and η contained either explicitly or implicitly in equation (66). Determine the variations in λ and ω induced by the primary variations by solving the initial-value equations (59) and (60). The matter constraints of fixed angular momentum, baryon number, and entropy for each ring determine the variations in ϵ, p, and the rotational velocity v. The functions β, λ, v, ϵ, and p determined by extremizing the gravitational mass M should be those of the equilibrium configuration. The functions ν and ω should be those of the globally stationary equilibrium configuration within a transformation of the type (67), but the transformation function $F(\omega)$ cannot be determined by the variational principle.

This conjectured variational principle is not likely to be of much use for determining

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ν unless the configuration is both rapidly rotating and highly relativistic, since ν appears in such a minimal way in equation (66). Therefore, it may be more practical to determine ν directly from the equation

$$R_{(0)(0)} = 8\pi[T_{(0)(0)} + \tfrac{1}{2}T^{(a)}{}_{(a)}] = 4\pi\left[(\epsilon + p)\frac{1+v^2}{1-v^2} + 2p\right], \tag{68}$$

where it is assumed that the combination of second time derivatives in $R_{(0)(0)}$ is zero, so $R_{(0)(0)}$ is given by equation (C10). It is the second time derivative of the determinant of the spatial metric which appears in $R_{(0)(0)}$, so the constraint (68) would not seem to limit the gravitational-wave degree of freedom in the trial configurations in an important way. The gravitational waves intrinsically involve only the "transverse, traceless" part of the spatial metric, which does not appear in the determinant (see Arnowitt, Deser, and Misner 1961).

In the Newtonian limit it *is* possible to construct a variational principle directly from equation (66), the variational principle of Lynden-Bell and Ostriker that has been used to calculate the structure of Newtonian white-dwarf models (Ostriker *et al.* 1966). The function λ in equation (66) is related to the Newtonian gravitational potential Φ by $\lambda = -\Phi$, while β is second order in c^{-2} and can be set equal to zero. The Newtonian energy E is M minus the rest-mass energy M_0. If ρ_0 is the rest-mass density, so that $\epsilon = \rho_0(1 + u)$ where u is the internal energy per gram, then

$$M_0 = 2\pi \int\int e^{3\lambda-\beta} \frac{\rho_0}{(1-v^2)^{1/2}} r^2 \sin\theta dr d\theta\,. \tag{69}$$

The expression for E obtained from equations (66) and (69) is

$$E = 2\pi \int\int \left[\rho_0(u + \tfrac{1}{2}v^2 + \Phi) + \frac{1}{8\pi}\nabla\Phi\cdot\nabla\Phi\right]\cdot r^2 \sin\theta dr d\theta\,. \tag{70}$$

The Newtonian version of the initial-value equation (59),

$$\nabla^2\Phi = 4\pi\rho_0\,, \tag{71}$$

gives the usual Newtonian expression for E,

$$E = 2\pi \int\int \rho_0(u + \tfrac{1}{2}v^2 + \tfrac{1}{2}\Phi) r^2 \sin\theta dr d\theta\,, \tag{72}$$

when used in equation (70). The proof of the variational principle of Lynden-Bell and Ostriker is based on this form for E and the explicit solution of the initial-value equation (71),

$$\Phi = -\int \frac{\rho_0(r')}{|r - r'|} dV'\,. \tag{73}$$

IX. APPLICATIONS OF THE VARIATIONAL PRINCIPLE

The variational principle developed in this paper may in the future be used to calculate the structure of rotating stars in general relativity. In doing so, a basic strategic decision is whether to treat the variational principle as an action principle or as an energy-extremization principle. If the full complement of the metric functions is to be obtained from the variational principle, the action-principle formulation is the one to use. The action I'_1 or, if the rest-mass and angular-momentum constraints are not applied explicitly, the action I'_2 which is the result of using equation (50) for L_G in I_2, can be made stationary under independent variations of the full set of four metric functions left after coordinate conditions have been applied. Unless there is some special reason to choose particular angular velocities for the rings of matter, the physically most appropri-

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ate form of the action principle is the one which fixes the rest mass and angular momentum of each ring, for reasons made clear by Ostriker *et al.* (1966) in the Newtonian context.

If the constraints are applied through the Lagrange multipliers Λ and Φ in equation (13), care must be taken that these are consistent with hydrostatic equilibrium. For instance, if the specific entropy is independent of position in the star, the equations of hydrostatic equilibrium (20), (21) can be satisfied only if the surfaces of constant "injection energy" Φ, constant angular velocity $\Omega = \Lambda$, and constant angular momentum per baryon j coincide. This is the relativistic analogue of rotation in cylindrical shells in the Newtonian case. Boyer (1966) has obtained related results regarding surfaces of constant pressure.

The alternative to the action principle is to solve the initial-value equations for the functions λ and ω, equation (68) for ν, and then find the matter configuration and the function β which extremize the total gravitational mass. As far as just calculating the equilibrium configuration is concerned, it would be simpler to go ahead and directly integrate all four Einstein equations and the equation of hydrostatic equilibrium. However, in extremizing the total energy one can learn something about the stability of the equilibrium configuration. Also, for a given matter configuration (given functions ξ and η and given rest-mass, angular-momentum, and entropy distributions), extremizing the total energy with respect to the function β should produce a local minimum, which would be the momentarily stationary matter configuration without any gravitational waves present. It may be of physical interest to explore how gravitational waves affect the geometry in the presence of rotating matter.

Since the total gravitational mass has a definite value for a given matter configuration and given functions β and ν, the form of the integral used to calculate the gravitational mass is not important to the extremization process. Either I'_1 as given by equations (45) and (50) or the integral (66) may be used to evaluate the gravitational mass of a trial configuration, as long as the functions λ, ω, and (ν) are obtained from the initial-value equations (and eq. [68]). The simplest integral is (66); so this is likely to be the best to use.

Even if not used to calculate the structure of the star directly, the action form of the variational principle gives relationships between neighboring equilibrium models which are important checks on the accuracy of numerical calculations. Equations (18) and (19) define $\Lambda(x^k)$ and $\Phi(x^k)$ for an equilibrium model. The equilibrium configuration is perturbed by adding a small number of baryons δdA to the ring of matter at x^k containing dA baryons and by changing the angular momentum of the ring dJ by an amount δdJ. Then from the fact that the functional I'_2 defined above is stationary to first order in variations of an equilibrium configuration and $I'_1 = M$ for both the old and the new equilibrium configurations, the change in M between the two equilibrium configurations is

$$\Delta M = \int \Phi(x^k)\delta dA + \int \Lambda(x^k)\delta dJ \,, \tag{74}$$

summing over the changes in all the rings. The identification of rings between the old and new configurations must be such that the specific entropy is constant in the perturbation, but is otherwise arbitrary within the general constraint that the change in x^k be small.

The condition that the specific entropy be constant in the perturbation for each ring can be relaxed by modifying the variational integral to include a Lagrange multiplier for the entropy contained in the ring. If this Lagrange multiplier is denoted by $\tau(\xi, \eta)$, the new variational integral is

$$\begin{aligned} I_4 &= I'_1 - \int \Lambda(\xi, \eta)dJ - \int \Psi(\zeta, \eta)dA - \int \tau(\xi, \eta)dS \\ &= I'_1 - 2\pi \int \Lambda(\epsilon + p)U_1 U^0(-g)^{1/2}dx^2dx^3 - 2\pi \int \Psi n U^0(-g)^{1/2}dx^2dx^3 \\ &\quad - 2\pi \int \tau s n U^0(-g)^{1/2}dx^2dx^3 \,, \end{aligned} \tag{75}$$

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where dS denotes the total entropy in a ring and s denotes the specific entropy. The Lagrange multiplier for the baryon number, Ψ, is different from before because now the baryon density n and the temperature T are the independent thermodynamic variables and are varied independently in the variational principle. The variations of n are carried out at constant T instead of constant s.

Examination of the Euler-Lagrange equations obtained by varying Ω, n, and T shows that as before Λ can be identified as the angular velocity Ω of the ring in the equilibrium configuration; that

$$\Psi = \left(\frac{\epsilon + p}{n} - Ts\right)(U^0)^{-1}\,, \tag{76}$$

the thermodynamic potential for adding baryons at constant *temperature* and constant pressure, redshifted to infinity; and that

$$\tau = T(U^0)^{-1} \tag{77}$$

in the equilibrium configuration. The equations of structure, etc., obtained by requiring that I_4 be stationary under independent variations of the $g_{\alpha\beta}$, ξ, and η are, of course, the same as before. The equation of hydrostatic equilibrium takes on the form

$$\frac{\partial\Omega}{\partial x^k} j + \frac{\partial\Psi}{\partial x^k} + \frac{\partial\tau}{\partial x^k} s = 0\,, \tag{78}$$

with Ψ and τ evaluated by equations (76) and (77).

A formula for the difference in energy between two neighboring equilibrium configurations analogous to equation (74) can be obtained from equation (75). Since $I'_1 = M$ and I_4 is stationary with respect to changes δdA in the baryon number, δdJ in the angular momentum, and δdS in the entropy of a ring containing dA baryons, etc., the change in M is

$$\Delta M = \int \Omega\delta dJ + \int \Psi\delta dA + \int \tau\delta dS\,. \tag{79}$$

The total entropy dS in a ring is the specific entropy s times the baryon number dA, so an alternative form of equation (79) is

$$\Delta M = \int \Omega\delta dJ + \int \Phi\delta dA + \int \tau\delta s dA\,, \tag{80}$$

where we have used the fact that $\Psi + \tau s = \Phi$. Now there are no constraints on the identification of corresponding rings in two neighboring equilibrium models, so the rings defined by the same coordinate range dx^2dx^3 can be compared regardless of the (infinitesimally different) entropy distributions in the stars.

Exact formulae like equation (80) are very helpful in checking the accuracy of difficult numerical computations. The version of this result obtained by Hartle and Sharp (1967) for uniformly rotating, isentropic stars has been used to check numerical calculations of uniformly rotating disks (Bardeen and Wagoner 1969). Hartle (1970) calculates the rotational energy of a differentially rotating star from the change in gravitational mass with angular momentum implied by equation (74) or (80).

The criterion for the stability against convection in a rotating star is, like the overall dynamical stability of the star, intrinsically second order in the deviation from hydrostatic equilibrium. Some partial information about the stability to convection can be obtained from the first-order variational principle by considering *nonlocal* changes in the structure of equilibrium configurations which might come about through the action of convection. If the nonlocal change results in a lower gravitational mass for the new configuration, then energy is available to drive convection and it is reasonable to conjecture that the original configuration is convectively unstable. The nonlocal changes must be "smooth" ones in which the angular momentum and entropy of the fluid elements are

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not changed; otherwise, the detailed processes for the adjustment of the angular momentum and entropy will affect the change in energy of the star.

Consider a star in which there are two rings of matter, 1 and 2, which have the same angular momentum per baryon $j_1 = j_2 = j_0$ and specific entropy $s_1 = s_2 = s_0$. Transfer of matter between these two rings can occur via a convective element which conserves angular momentum and entropy. The change in energy of the star when an amount δA of baryons is transferred from ring 1 to ring 2 is, from equation (74),

$$\Delta M = (\Omega_2 - \Omega_1)j_0\delta A + (\Phi_2 - \Phi_1)\delta A \,. \tag{81}$$

This expression can be simplified by using the equation of hydrostatic equilibrium in the form of equation (20) and (21). When integrated over a path from ring 1 to ring 2 in the star, equations (20) and (21) give

$$\int_1^2 j d\Omega + \Phi_2 - \Phi_1 - \int_1^2 (T/U^0) ds = 0 \,. \tag{82}$$

This is used to eliminate $\Phi_2 - \Phi_1$ from equation (81), so

$$\begin{aligned} \Delta M &= \left[(\Omega_2 - \Omega_1)j_0 - \int_1^2 j d\Omega + \int_1^2 (T/U^0) ds\right]\delta A \\ &= \left[\int_1^2 \Omega dj + \int_1^2 (T/U^0) ds\right]\delta A \,. \end{aligned} \tag{83}$$

The sign of equation (83) can be estimated as follows. If T/U^0 decreases monotonically going from 1 to 2 and if s increases to a maximum before decreasing back to s_0 along the path, then $\int_1^2 (T/U^0) ds \geq 0$. If the path is along a surface of constant j, then the other integral doesn't contribute. The change in gravitational mass $\Delta M > 0$ if the matter is transferred from 1 to 2, but $\Delta M < 0$ if matter is transferred from 2 to 1.

The actual flow of matter will be in the direction for which $\Delta M < 0$, and it will be able to continue until *all* the matter from the part of the path of constant j along which s decreases is transferred to the part along which s increases, at least if T/U^0 continues to decrease monotonically from 1 to 2 in the modified equilibrium configurations. What this argument suggests, then, is that if the specific entropy decreases along a surface of constant j in the direction in which T/U^0 is decreasing (the direction of heat flow, presumably outward from the equatorial plane of the star), there is an instability to convective motions until the specific entropy monotonically increases in this direction. If s increases as T/U^0 decreases, the star should be stable to this type of axisymmetric, adiabatic convection. This criterion is identical with the Schwarzschild criterion for convective instability.

On the other hand, along a surface of constant specific entropy a similar argument shows that if in going from 1 to 2 j has a maximum and the angular velocity Ω decreases monotonically, as one would expect in the direction outward from the axis of symmetry, then $\Delta M < 0$ for mass transfer from 2 to 1, suggesting instability to convection if the angular momentum per baryon decreases outward from the axis of symmetry along a surface of constant s, which is similar to Rayleigh's criterion for instability of rotational motions in Newtonian theory.

The complex effects of angular-momentum conservation on convective motions make it doubtful that these simple injection-energy arguments of the type developed by Thorne (1967) for spherically symmetric stars can be extended to more general types of convective motions in rotating stars. A more complete treatment of convective instability could be based on a relativistic adaptation of the *local* analysis of Goldreich and Schubert (1967) in Newtonian theory.

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APPENDIX A

An orthonormal tetrad is a set of four mutually orthogonal unit vectors at a point in space-time, one timelike and one spacelike, which give the directions of the four axes of a locally Minkowskian coordinate system (see Synge 1960, p. 8). The tetrad vectors can be used to express tensors in an arbitrary coordinate system in terms of physical quantities measured in this local reference frame. Indices in parentheses will be used to denote components in the local reference frame, and are raised and lowered by the Minkowski metric. The tetrad vectors for the locally nonrotating reference frame discussed in § VI are, giving both contravariant and covariant components in terms of the metric functions defined by equation (35),

$$\lambda_{(0)}: \quad \lambda_{(0)}{}^0 = e^{-\nu}\,, \quad \lambda_{(0)}{}^1 = \omega e^{-\nu}\,, \quad \lambda_{(0)0} = -e^{\nu}\,, \quad \lambda_{(0)1} = 0\,; \tag{A1}$$

$$\lambda_{(1)}: \quad \lambda_{(1)}{}^0 = 0\,, \quad \lambda_{(1)}{}^1 = e^{-\psi}\,, \quad \lambda_{(1)0} = -\omega e^{\psi}\,, \quad \lambda_{(1)1} = e^{\psi}\,; \tag{A2}$$

$$\lambda_{(2)}: \quad \lambda_{(2)}{}^2 = e^{-\mu_2}\,, \quad \lambda_{(2)2} = e^{\mu_2}\,; \tag{A3}$$

$$\lambda_{(3)}: \quad \lambda_{(3)}{}^3 = e^{-\mu_3}\,, \quad \lambda_{(3)3} = e^{\mu_3}\,. \tag{A4}$$

The rule for calculating the physical components of a tensor $T_{\alpha\beta}$ is

$$T_{(\sigma)(\tau)} = T_{\alpha\beta}\lambda_{(\sigma)}{}^{\alpha}\lambda_{(\tau)}{}^{\beta}\,, \tag{A5}$$

and conversely

$$T_{\alpha\beta} = \lambda^{(\sigma)}{}_{\alpha}\lambda^{(\tau)}{}_{\beta}T_{(\sigma)(\tau)}\,. \tag{A6}$$

The physical components in a frame moving with rotational velocity v with respect to the locally nonrotating frame can be found by applying a Lorentz transformation to the physical components of the tensor in the locally nonrotating frame.

APPENDIX B

The physical components of the Riemann tensor in the locally nonrotating frame take on a relatively simple form when expressed in terms of the metric functions defined by equation (35). Direct calculation gives

$$R_{(0)(1)(0)(1)} = (\nu_{,2}\psi_{,2}e^{-2\mu_2} + \nu_{,3}\psi_{,3}e^{-2\mu_3}) + \tfrac{1}{4}(\omega_{,2}\omega_{,2}e^{-2\mu_2} + \omega_{,3}\omega_{,3}e^{-2\mu_3})e^{2\psi-2\nu}\,,$$

$$R_{(0)(2)(0)(2)} = e^{-\mu_2-\nu}[e^{-\mu_2}(e^{\nu})_{,2}]_{,2} + e^{-2\mu_3}\nu_{,3}\mu_{2,3} - \tfrac{3}{4}e^{-2\mu_2}\omega_{,2}\omega_{,2}e^{2\psi-2\nu}\,,$$

$$R_{(0)(3)(0)(3)} = e^{-\mu_3-\nu}[e^{-\mu_3}(e^{\nu})_{,3}]_{,3} + e^{-2\mu_2}\nu_{,2}\mu_{3,2} - \tfrac{3}{4}e^{-2\mu_3}\omega_{,3}\omega_{,3}e^{2\psi-2\nu}\,,$$

$$R_{(1)(2)(1)(2)} = -e^{-\mu_2-\psi}[e^{-\mu_2}(e^{\psi})_{,2}]_{,2} - e^{-2\mu_3}\psi_{,3}\mu_{2,3} - \tfrac{1}{4}e^{-2\mu_2}\omega_{,2}\omega_{,2}e^{2\psi-2\nu}\,,$$

$$R_{(1)(3)(1)(3)} = -e^{-\mu_3-\psi}[e^{-\mu_3}(e^{\psi})_{,3}]_{,3} - e^{-2\mu_2}\psi_{,2}\mu_{3,2} - \tfrac{1}{4}e^{-2\mu_3}\omega_{,3}\omega_{,3}e^{2\psi-2\nu}\,,$$

$$R_{(2)(3)(2)(3)} = -e^{-\mu_2-\mu_3}\{[e^{-\mu_2}(e^{\mu_3})_{,2}]_{,2} + [e^{-\mu_3}(e^{\mu_2})_{,3}]_{,3}\}\,,$$

$$R_{(0)(2)(0)(3)} = e^{-\mu_2-\mu_3}[e^{-\nu}(e^{\nu})_{,23} - \nu_{,2}\mu_{2,3} - \nu_{,3}\mu_{3,2} - \tfrac{3}{4}\omega_{,2}\omega_{,3}e^{2\psi-2\nu}]\,,$$

$$R_{(1)(2)(1)(3)} = -e^{-\mu_2-\mu_3}[e^{-\psi}(e^{\psi})_{,23} - \psi_{,2}\mu_{2,3} - \psi_{,3}\mu_{3,2} + \tfrac{1}{4}\omega_{,2}\omega_{,3}e^{2\mu-2\nu}]\,,$$

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$$R_{(0)(2)(1)(2)} = \tfrac{1}{2}e^{-2\psi-\mu_2}[e^{3\psi-\nu-\mu_2}\omega_{,2}]_{,2} + \tfrac{1}{2}e^{-2\mu_3}\omega_{,3}\mu_{2,3}e^{\psi-\nu} ,$$

$$R_{(0)(3)(1)(3)} = \tfrac{1}{2}e^{-2\psi-\mu_3}[e^{3\psi-\nu-\mu_3}\omega_{,3}]_{,3} + \tfrac{1}{2}e^{-2\mu_2}\omega_{,2}\mu_{3,2}e^{\psi-\nu} ,$$

$$R_{(0)(2)(1)(3)} = \tfrac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}[\omega_{,23} + 2\omega_{,2}\psi_{,3} + \omega_{,3}\psi_{,2} - \omega_{,2}\nu_{,3} - \omega_{,2}\mu_{2,3} - \omega_{,3}\mu_{3,2}] ,$$

$$R_{(0)(1)(2)(3)} = \tfrac{1}{2}e^{\psi-\nu-\mu_2-\mu_3}[\omega_{,2}(\psi - \nu)_{,3} - \omega_{,3}(\psi - \nu)_{,2}] .$$

The remaining nonzero components are equivalent to these through the symmetries of the Riemann tensor. Note that $R_{(0)(1)(2)(3)}$ is invariant under Lorentz transformations in the ϕ-direction and therefore is an invariant measure of the "twisting" of space-time due to the dragging of inertial frames. If ∇ is the gradient operator in the locally nonrotating frame and v_0 is the velocity of the stationary frame with respect to the locally nonrotating frame, $v_0 = -\omega e^{\psi-\nu}$, then

$$R_{(0)(1)(2)(3)} = \tfrac{1}{2}(\nabla\omega \times \nabla v_0)_{(1)} .$$

APPENDIX C

The Einstein equations for a metric of the type (35) have been written down by a number of authors (see Hartle 1967), and the Ricci tensor in the locally nonrotating frame has been given by Cohen and Brill (1968). The physical components of the Ricci tensor are easily calculated from

$$R_{(\alpha)(\beta)} = R^{(\gamma)}{}_{(\alpha)(\gamma)(\beta)} ; \tag{C1}$$

so, we will give only those forms used in the body of the paper.

The scalar curvature

$$R = R^{(\alpha)}{}_{(\alpha)} \tag{C2}$$

is

$$\begin{aligned} R = {} & -2e^{-\nu-\psi-\mu_2-\mu_3}\{[e^{-\mu_2}(e^{\nu+\psi+\mu_3})_{,2}]_{,2} + [e^{-\mu_3}(e^{\nu+\psi+\mu_2})_{,3}]_{,3}\} \\ & + 2\{e^{-2\mu_2}[\psi_{,2}\mu_{3,2} + \nu_{,2}(\psi_{,2} + \mu_{3,2})] + e^{-2\mu_3}[\psi_{,3}\mu_{2,3} + \nu_{,3}(\psi_{,3} + \mu_{2,3})]\} \\ & + \tfrac{1}{2}[\omega_{,2}\omega_{,2}e^{-2\mu_2} + \omega_{,3}\omega_{,3}e^{-2\mu_3}]e^{2\psi-2\nu} . \end{aligned} \tag{C3}$$

The coordinate component

$$R_0{}^0 = -R_{(0)(0)} + \omega e^{\psi-\nu}R_{(0)(1)} \tag{C4}$$

is

$$\begin{aligned} R_0{}^0 = {} & e^{-\nu-\psi-\mu_2-\mu_3}\{-[e^{\nu+\psi+\mu_3-\mu_2}\nu_{,2}]_{,2} - [e^{\nu+\psi+\mu_2-\mu_3}\nu_{,3}]_{,3} \\ & + \tfrac{1}{2}[e^{3\psi-\nu+\mu_3-\mu_2}\omega\omega_{,2}]_{,2} + \tfrac{1}{2}[e^{3\psi-\nu+\mu_2-\mu_3}\omega\omega_{,3}]_{,3}\} . \end{aligned} \tag{C5}$$

The initial-value equation

$$R_{(0)(0)} + \tfrac{1}{2}R = R_{(1)(2)(1)(2)} + R_{(1)(3)(1)(3)} + R_{(2)(3)(2)(3)} = 8\pi T_{(0)(0)} \tag{C6}$$

can be written

$$\begin{aligned} & - e^{-\psi-\mu_2-\mu_3}\{[e^{\mu_3-\mu_2}(e^{\psi})_{,2}]_{,2} + [e^{\mu_2-\mu_3}(e^{\psi})_{,3}]_{,3}\} \\ & - e^{-\mu_2-\mu_3}\{[e^{-\mu_2}(e^{\mu_3})_{,2}]_{,2} + [e^{-\mu_3}(e^{\mu_2})_{,3}]_{,3}\} \\ & - \tfrac{1}{4}[\omega_{,2}\omega_{,2}e^{-2\mu_2} + \omega_{,3}\omega_{,3}e^{-2\mu_3}]e^{2\psi-2\nu} = 8\pi\frac{\epsilon + pv^2}{1 - v^2} , \end{aligned} \tag{C7}$$

while the other initial-value equation

$$R_{(0)(1)} = 8\pi T_{(0)(1)} \tag{C8}$$

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is

$$\tfrac{1}{2}e^{-2\psi-\mu_2-\mu_3}\{[e^{3\psi-\nu+\mu_3-\mu_2}\omega_{,2}]_{,2} + [e^{3\psi-\nu+\mu_2-\mu_3}\omega_{,3}]_{,3}\} = -8\pi\frac{(\epsilon + p)v}{1 - v^2}\,. \quad \text{(C9)}$$

An expression for the physical component $R_{(0)(0)}$ used in § VIII is

$$\begin{aligned} R_{(0)(0)} &= e^{-\nu-\psi-\mu_2-\mu_3}\{[e^{\nu+\psi+\mu_3-\mu_2}\nu_{,2}]_{,2} + [e^{\nu+\psi+\mu_2-\mu_3}\nu_{,3}]_{,3}\} \\ &\quad - \tfrac{1}{2}[\omega_{,2}\omega_{,2}e^{-2\mu_2} + \omega_{,3}\omega_{,3}e^{-2\mu_3}]e^{2\psi-2\nu}\,. \end{aligned} \quad \text{(C10)}$$

APPENDIX D

In the body of the paper we have not gone to great lengths to write expressions in an explicitly coordinate invariant form. For those more interested in the formal structure of the variational principle than ease of application, we show how to remedy this. In particular, we analyze in detail the coordinate freedom associated with a momentarily stationary hypersurface.

The action form of the variational principle involves an integral over a trial stationary, axisymmetric configuration. The space-time geometry is, like the equilibrium configuration, characterized by two Killing vector fields, $\zeta_{(\phi)}{}^a$ and $\zeta_{(t)}{}^a$. The vectors are defined unambiguously in an asymptotically flat space-time by requiring that $\zeta_{(t)}{}^a$ be a timelike unit vector at spatial infinity and that $\zeta_{(\phi)}{}^a$ be spacelike everywhere, with a magnitude equal to the circumferential radius of a circle around the axis of symmetry at spatial infinity. Furthermore, there are invariant spacelike hypersurfaces which have as a unit normal n_a at each point the timelike unit vector which is a linear combination of $\zeta_{(\phi)a}$ and $\zeta_{(t)a}$ orthogonal to $\zeta_{(\phi)}{}^a$ (see Carter 1969; Thorne 1970). The action I_1, as defined by equation (9), can be written in an invariant way as an integral over an invariant hypersurface,

$$I_1 = \int\left[-\frac{1}{16\pi}R\zeta_{(t)}{}^\beta - \zeta_{(t)}{}^a T_a{}^\beta\right]dS_\beta\,. \quad \text{(D1)}$$

The hypersurface element dS_β is a timelike vector equal to $n_\beta dS$, where dS is the proper three-volume element in the hypersurface. If the coordinates are chosen as in equation (1), equation (D1) reduces to equation (9).

In the energy-extremization interpretation of the variational principle the integral (45) defining I'_1 is interpreted as an integral over an only momentarily stationary hypersurface in an axisymmetric space-time obeying the Einstein equations instead of an integral over an invariant hypersurface in a stationary space-time which does not necessarily obey the Einstein equations. The definition of a momentarily stationary hypersurface can be made mathematically precise in a way independent of coordinates. Following Wheeler (1964), a spacelike hypersurface $t = 0$ is characterized by its intrinsic geometry as represented by the spatial metric tensor $g_{\alpha\beta}$ (now Greek indices will range from 1 to 3), and by its extrinsic geometry as represented by the extrinsic curvature three-tensor $K_{\alpha\beta}$. In terms of the lapse function

$$N = (-g^{00})^{-1/2} \quad \text{(D2)}$$

and the shift function

$$N_\alpha = g_{0\alpha}\,, \quad \text{(D3)}$$

a spacelike three-vector in the hypersurface, the extrinsic curvature tensor is

$$K_{\alpha\beta} = \frac{1}{2N}[N_{\alpha/\beta} + N_{\beta/\alpha} - g_{\alpha\beta,0}]. \quad \text{(D4)}$$

A slash now denotes a covariant derivative in the three-space. The three-space is axisymmetric, so there is a Killing vector $\zeta_{(\phi)}{}^a$.

In the absence of rotation a momentarily static, or time-symmetric, hypersurface is defined

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by $K_{\alpha\beta,} = 0$. In the presence of rotation the tensor $K_{\alpha\beta}$ is not zero, but a momentarily stationary hypersurface can be defined in an invariant way from the conditions

$$\zeta_{(\phi)}{}^{\alpha}\zeta_{(\phi)}{}^{\beta}K_{\alpha\beta} = 0 \tag{D5}$$

and

$$P^{\alpha\sigma}K_{\sigma\tau}P^{\tau\beta} = 0 \,, \tag{D6}$$

where $P^{\alpha\sigma}$ is a tensor which projects perpendicular to the Killing vector $\zeta_{(\phi)}{}^{\alpha}$,

$$P^{\alpha\sigma} = g^{\alpha\sigma} - \zeta_{(\phi)}{}^{\alpha}\zeta_{(\phi)}{}^{\sigma}/\zeta_{(\phi)}{}^{\gamma}\zeta_{(\phi)\gamma} \,. \tag{D7}$$

The conditions (D5) and (D6) imply through the initial-value equations that the energy flux in the momentarily stationary hypersurface is entirely in the direction of $\zeta_{(\phi)}{}^{\alpha}$.

To see how conditions (D5) and (D6) reduce to the conditions on the four-dimensional metric tensor used to define the momentarily stationary hypersurface in § VII, let x^1 be the axial angle coordinate ϕ for which $\zeta_{(\phi)}{}^{\alpha} = \delta_1{}^{\alpha}$. The metric-tensor components on the hypersurface are then functions only of the x^k, $k = 2, 3$. By coordinate transformation of the type

$$\phi' = \phi + F(x^2, x^3) \,, \qquad x^{k'} = x^{k'}(x^2, x^3) \,, \tag{D8}$$

we make the $g_{1k} = 0$ on the $t = 0$ hypersurface. If the coordinates in the vicinity of the $t = 0$ hypersurface are chosen so

$$N^{\alpha} = -\omega(x^2, x^3)\delta_1{}^{\alpha} \tag{D9}$$

in equation (D4), the condition (D5) becomes

$$-\frac{1}{2N}\, g_{11,0} = 0 \,, \tag{D10}$$

and equation (D6) is

$$-\frac{1}{2N}\, g_{km,0} = 0 \,. \tag{D11}$$

The only nonzero components of $K_{\alpha\beta}$ are the

$$K_{1m} = -\frac{1}{2N}\,(g_{11}\omega_{,m} + g_{1m,0}) \,. \tag{D12}$$

The alternative definition of momentarily stationary is that the space-time in the vicinity of the $t = 0$ hypersurface is symmetric under simultaneous change of sign of ϕ and t. We now show that this is equivalent to the above. The general transformation of coordinates in the space-time which leaves the spatial coordinates at $t = 0$ unchanged, preserves the explicit axial symmetry, and preserves equations (D9)–(D11) can be written as a Taylor expansion in t. To order t^2,

$$t' = f_0 t + \tfrac{1}{2}h_0 t^2 \,, \qquad \phi' = \phi + f_1 t + \tfrac{1}{2}h_1 t^2 \,, \qquad x^{k'} = x^k + \tfrac{1}{2}f_k t^2 \,. \tag{D13}$$

The f_0, f_1, f_2, f_3, h_0, and h_1 are functions of the x^k.

The new and old metric-tensor components are related to first order in t by

$$g_{00} = g_{00}'(f_0 + h_0 t)^2 + 2g_{01}'(f_0 + h_0 t)(f_1 + h_1 t) + g_{11}'(f_1 + h_1 t)^2 \,, \tag{D14}$$

$$g_{01} = g_{01}'(f_0 + h_0 t) + g_{11}'(f_1 + h_1 t) \,, \tag{D15}$$

$$g_{11} = g_{11}' \,, \tag{D16}$$

$$g_{0k} = f_0 g_{0k}' + f_1 f_{1k}' + g_{km}' f_m t + (g_{00}' f_0 + g_{01}' f_1) f_{0,k} t + (g_{01}' f_0 + g_{11}' f_1) f_{1,k} t \,, \tag{D17}$$

$$g_{1k} = g_{1k}' + g_{01}' f_{0,k} t + g_{11}' f_{1,k} t \,, \tag{D18}$$

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$$g_{km} = g_{km}' \, . \tag{D19}$$

The metric functions ν and ω defined by $e^{-\nu} = (-g^{00})^{1/2}$ and $\omega = -g_{01}/g_{11}$ transform according to

$$e^{\nu'} = e^{\nu}/(f_0 + h_0 t) \, , \tag{D20}$$

$$\omega' = (\omega + f_1 + h_1 t)/(f_0 + h_0 t) \, . \tag{D21}$$

It is always possible to choose h_0 and h_1 to make g_{00} and g_{01} (or ν and ω) have zero first time derivatives as required by the symmetry under $(t, \phi) \to (-t, -\phi)$.

Both g_{0k} and g_{1k} are zero at $t = 0$. While their first time derivatives are not required to be zero by the symmetry, equations (D17) and (D18) give

$$g_{0k,0} = f_0^2 g_{0k,0}' + f_1 g_{1k,0} + g_{km} f_m + g_{00}' f_0 f_{0,k} + g_{01}' f_0 f_{1,k} \tag{D22}$$

and

$$g_{1k,0} = f_0 g_{1k,0}' + g_{11}[f_{1,k} - \omega' f_{0,k}] \, . \tag{D23}$$

(Note that $K_{\alpha\beta}$ is invariant under the transformation (D13), as it must be as a tensor on the $t = 0$ hypersurface.) The four functions f_0, f_1, f_2, f_3 can always be chosen to make the $g_{0k,0}$ and the $g_{1k,0}$ equal to zero. Now all first time derivatives of the metric-tensor components are zero, and the initial-value equations, which can be written in terms of the extrinsic curvature tensor (D12) and the intrinsic scalar curvature ${}^3R$, have the same form on the momentarily stationary $t = 0$ hypersurface as they do for the globally stationary equilibrium configuration.

There is still some coordinate freedom left. The $g_{1k,0}$ remain zero as long as f_0 and f_1 satisfy

$$f_{1,k} = \omega' f_{0\ k} \, , \tag{D24}$$

and the f_k can be chosen to keep the $g_{0k,0}$ equal to zero. Therefore, a change of scale of the time coordinate which gives

$$\nu' = \nu + F(\omega) \, , \tag{D25}$$

$$\omega' = \int_0^{\omega} e^{F(\omega)} d\omega \tag{D26}$$

leaves the form of the initial-value equations unchanged on the $t = 0$ hypersurface. The function $F(\omega)$ must be zero when $\omega = 0$ if the space-time is to remain asymptotically flat, but is otherwise arbitrary. Since the surfaces of constant ν and constant ω will not coincide in general, even for the globally stationary equilibrium configuration, ν is not completely arbitrary as a result of this coordinate freedom. This is a qualitative difference from the momentarily static case, where $\omega = \omega' = 0$ and equation (D24) can be satisfied by an arbitrary f_0 (Harrison *et al.* 1965). The surfaces of constant ω and constant ν will coincide to first order in the angular velocity of a slowly rotating star.

The transformation (D25), (D26) leaves invariant the combination $e^{-\nu}\omega_{,k}$ appearing in the initial-value equations (59) and (60), which is consistent with the invariance of the spatial metric under the transformation.

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