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STABILITY OF CIRCULAR ORBITS IN STATIONARY, AXISYMMETRIC SPACE-TIMES*

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ABSTRACT

Stability criteria are derived for circular particle orbits in the equatorial plane of a stationary, axisymmetric space-time in general relativity. They are applied to show that the orbits of the particles making up an infinitesimally thin disk are stable as long as the angular momentum per unit mass of the particles increases monotonically outward from the axis of symmetry.

I. INTRODUCTION

The stability of circular orbits in the equatorial plane of a stationary, axisymmetric space-time in general relativity is of physical interest in at least two contexts. First, an idealized model for a highly flattened galaxy is an infinitesimally thin disk made up of particles (stars) traveling in circular orbits in the gravitational field due to the smoothed-out matter distribution of the particles themselves. Such disks have been studied in Newtonian theory by Toomre (1964) and Hunter (1963, 1965), among others; and Bardeen and Wagoner (1969) have begun an investigation of their properties in general relativity. A necessary condition for these disks to be an approximation to a physically realizable system is that the individual particle orbits be stable (see Chandrasekhar 1942 for a Newtonian analysis). Second, circular particle orbits in the exterior metric of a collapsed object or "black hole" in general relativity have played an important role in theories (Salpeter 1964; Lynden-Bell 1969) of how matter being accreted by the black hole can release energy. So far these theories have been based on the Schwarzschild metric, the exterior metric of a collapsed object with zero angular momentum. However, one would expect the collapsed object to possess an appreciable amount of angular momentum in general, in which case the appropriate metric is probably the stationary, axisymmetric vacuum metric of Kerr (1963).[1]

In general relativity, as in Newtonian theory, perturbations to a circular orbit involving motions in the equatorial plane decouple from those involving motions perpendicular to the plane. Both types of stability are considered in this paper.

Units are chosen such that the gravitational constant G and the speed of light c are equal to one. The signature of the metric tensor is $(-+++)$. The coordinate $x^0 = t$ is the time coordinate for which the metric is stationary, normalized so $g_{00} = -1$ at spatial infinity in an asymptotically flat space-time. The coordinate $x^1 = \phi$ is the axial angle around the axis of symmetry. The metric coefficients depend only on x^2 and x^3, which I take to be cylindrical coordinates ρ and z such that the axis of symmetry is $\rho = 0$ and the equatorial plane of symmetry is $z = 0$. Greek indices range from 0 to 3; Latin indices, from 2 to 3. Repeated indices imply a summation over the appropriate range.

II. THE METRIC

The geometry of space-time and the energy-momentum tensor are assumed to be invariant under a simultaneous change of sign of ϕ and t. This will be true for the

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[1] See Thorne (1970) for a discussion of the physical significance of the Kerr metric.

gravitational field of any purely rotating perfect fluid (Carter 1969). A general form for the line element of a stationary, axisymmetric space-time satisfying this additional symmetry is

$$ds^2 = -(e^{2\nu} - \omega^2\rho^2 e^{2\beta-2\nu})dt^2 - 2\omega\rho^2 e^{2\beta-2\nu} + \rho^2 e^{2\beta-2\nu}d\phi^2 + e^{2\mu}(d\rho^2 + dz^2)\,. \tag{1}$$

Of the four metric functions ν, β, ω, and μ, only ν, β, and ω are involved in a discussion of circular orbits. They are assumed to be continuous functions with continuous first derivatives in the vicinity of $z = 0$, except that the partial derivative with respect to z may be discontinuous across $z = 0$ in the limit that an infinitesimally thin surface density is present there. The metric functions are even functions of z.

The function $\omega(\rho, z)$ is the angular velocity of the "local inertial frame"; the congruence of timelike lines with $d\phi/dt = \omega$ has an angular-velocity four-vector (vorticity) equal to zero and thus is locally nonrotating. The physical velocity v of a rotating observer with $d\phi/dt = \Omega$, relative to the local inertial frame, is

$$v = (\Omega - \omega)\rho e^{\beta-2\nu}\,. \tag{2}$$

The velocity four-vector is

$$U^0 = e^{-\nu}(1 - v^2)^{-1/2}\,, \quad U^1 = \Omega U^0\,, \quad U^2 = U^3 = 0\,. \tag{3}$$

The covariant component $U_1 = \rho e^{\beta-\nu}v/(1 - v^2)^{1/2}$ is the angular momentum per unit mass of the observer.

The only nonzero components of the energy-momentum tensor allowed by the symmetries assumed are T_{00}, T_{01}, T_{11}, and the T_{ab}. Since the only possible energy flux is in the ϕ-direction, one can define a "comoving" reference frame such that the physical energy flux relative to this frame is zero. If v_m is the physical velocity of the comoving frame with respect to the local inertial frame, and if E and P_1 are the energy density and stress (pressure) in the ϕ-direction in the comoving frame, then

$$e^{2\nu}(T^{00} - \tfrac{1}{2}g^{00}T_a{}^a) = \tfrac{1}{2}(E + P_1)\frac{1 + v_m{}^2}{1 - v_m{}^2} + \tfrac{1}{2}T_a{}^a \tag{4}$$

and

$$T_1{}^0 = (E + P_1)\frac{v_m}{1 - v_m{}^2}\rho e^{\beta-2\nu}\,. \tag{5}$$

For a perfect fluid $\frac{1}{2}T_a{}^a = P_1 = P$, the isotropic pressure in the comoving frame.

The Einstein equations for a line element of the type (1) have been derived by van Stockum (1937), Hartle and Sharp (1967), Cohen and Brill (1968), and Levy (1968), among others. They are equivalent to the following set of equations for ν, β, and ω:

$$\Delta\nu = -\nabla\beta\cdot\nabla\nu + \tfrac{1}{2}\rho^2 e^{2\beta-2\nu}\nabla\omega\cdot\nabla\omega + 4\pi e^{2\mu}\left[(E + P_1)\frac{1 + v_m{}^2}{1 - v_m{}^2} + T_a{}^a\right]; \tag{6}$$

$$\frac{1}{\rho}\nabla\cdot(\rho\nabla\beta) = -\nabla\beta\cdot\nabla\beta + 8\pi e^{2\mu}T_a{}^a\,; \tag{7}$$

$$\frac{1}{\rho^2}\nabla\cdot(\rho^2\nabla\omega) = 4\nabla\nu\cdot\nabla\omega - 3\nabla\beta\cdot\nabla\omega - 16\pi e^{2\mu}(E + P_1)\frac{\Omega_m - \omega}{1 - v_m{}^2}\,. \tag{8}$$

The differential operators Δ, $\nabla\cdot$, and ∇ are the flat-space Laplacian, divergence, and gradient in cylindrical coordinates. The metric function μ is not involved in a discussion of the stability of circular orbits.

III. GEODESIC EQUATIONS

Let the tangent vector to a timelike geodesic representing the orbit of a particle, normalized to be a unit vector, be denoted by $p^a = dx^a/ds$. Physically, p^a is the momentum per unit mass four-vector of the particle. Because of the time independence and axial symmetry of the metric, the energy per unit mass $E = -p_0$ and the angular momentum per unit mass about the axis of symmetry $\Phi = p_1$ are constants of the motion. From $p_a p^a = -1$, the equations of motion for a particle confined to the equatorial plane are

$$dt/ds = p^0 = e^{-2\nu}(E - \omega\Phi) \; ; \tag{9}$$

$$d\phi/ds = p^1 = \Omega p^0 = \frac{1}{\rho^2 e^{2\beta-2\nu}}\Phi + \omega e^{-2\nu}(E - \omega\Phi) \; ; \tag{10}$$

$$e^{2\mu}(d\rho/ds)^2 = Q(\rho, E, \Phi) = e^{2\nu}(E - \omega\Phi)^2 - \frac{1}{\rho^2 e^{2\beta-2\nu}}\Phi^2 - 1 \, . \tag{11}$$

The geodesic equations proper are most conveniently derived from

$$dp_a/ds = -\tfrac{1}{2}(g^{\alpha\beta})_{,a}p_\alpha p_\beta \, . \tag{12}$$

The parameter s is the proper time along the geodesic. For $a = 2$, $p_2 = e^{2\mu}(d\rho/ds)$, and equation (12) is just the partial derivative with respect to ρ of equation (11), except for the term containing $(p_3)^2$, which is second order in the deviation from a circular orbit. We will consider only first order perturbations in this paper.

The equation governing perturbations perpendicular to the plane is (12) with $a = 3$ and can be written

$$\begin{aligned} \frac{d}{ds}\left(e^{2\mu}\frac{dz}{ds}\right) = -\nu_{,3}\left[e^{-2\nu}(E - \omega\Phi)^2 + \frac{\Phi^2}{\rho^2 e^{2\beta-2\nu}}\right] - \omega_{,3}e^{-2\nu}\Phi(E - \omega\Phi) \\ + \beta_{,3}\frac{\Phi^2}{\rho^2 e^{2\beta-2\nu}} + \mu_{,3}e^{-2\mu}(p_2^2 + p_3^2) \, . \end{aligned} \tag{13}$$

Both p_2 and p_3 are first order in the perturbation from a circular orbit. Furthermore, except in the extreme limit that matter in the equatorial plane has an infinite volume density, the derivative with respect to z of ν can be approximated by $\nu_{,3} \simeq z\nu_{,33} = z\nu_{,zz}$ near the equatorial plane. To first order, then,

$$\begin{aligned} e^{2\mu}\frac{d^2z}{ds^2} + z\left\{\nu_{,zz}\left[e^{-2\nu}(E - \omega\Phi)^2 + \frac{e^{2\nu-2\beta}}{\rho^2}\Phi^2\right] + \omega_{,zz}e^{-2\nu}\Phi(E - \omega\Phi) \right. \\ \left. - \beta_{,zz}\frac{e^{2\nu-2\beta}}{\rho^2}\Phi^2\right\} = 0 \, . \end{aligned} \tag{14}$$

The motion in the z-direction is not coupled to motion in the ρ-direction.

IV. UNPERTURBED CIRCULAR ORBITS

Equation (11) governs all orbits in the equatorial plane. The condition that $d\rho/ds$ is momentarily zero is that

$$Q(\rho, E, \Phi) = 0 \, . \tag{15}$$

In order that $d\rho/ds$ be permanently zero, the condition for a circular orbit, the equation

$$\frac{\partial Q}{\partial \rho} = 0 \tag{16}$$

must be satisfied simultaneously with equation (15). For a given value of ρ, these are two simultaneous equations for the energy E and angular momentum Φ of the circular orbit. From equations (9) and (10) the angular velocity Ω relative to a distant observer is given by

$$\Omega = \omega + \frac{e^{4\nu-2\beta}}{\rho^2}\frac{\Phi}{E-\omega\Phi}\,. \tag{17}$$

The physical velocity v relative to the local inertial frame is

$$v = \frac{e^{2\nu-\beta}}{\rho}\frac{\Phi}{E-\omega\Phi}\,. \tag{18}$$

On the other hand, one can express Φ and E in terms of v:

$$\Phi = \rho e^{\beta-\nu}v/(1-v^2)^{1/2}\,; \tag{19}$$

$$E - \omega\Phi = e^{\nu}/(1-v^2)^{1/2}\,. \tag{20}$$

Equations (15) and (16) combined with (19) and (20) determine the velocity of rotation v of a circular orbit,

$$(1+\rho\beta_{,\rho}-\rho\nu_{,\rho})v^2 - \rho e^{\beta-2\nu}(\rho\omega_{,\rho})v - \rho\nu_{,\rho} = 0\,. \tag{21}$$

The two solutions to this equation will usually give the velocity of rotation of both a "direct" ($\omega v > 0$) and a "retrograde" ($\omega v < 0$) circular orbit. In extremely relativistic regions one or both of these solutions may be unphysical, either because the solutions for v are complex or more likely because $v^2 > 1$ for one or both of the orbits, implying a velocity of rotation greater than the speed of light. In the Newtonian limit the two solutions are "degenerate," differing only in sign, but the dragging of inertial frames in relativity breaks the degeneracy. In a region of strong gravitational fields ($\nu \ll -1$) generated by a rapidly rotating body, typically only the direct circular orbit will be physically allowed.

Since Q is zero for all circular orbits in the equatorial plane, its total derivative with respect to ρ must be zero. This gives the equation

$$dE/d\rho = \Omega d\Phi/d\rho \tag{22}$$

which governs the way the energy and angular momentum of circular orbits vary with ρ.

V. STABILITY IN THE EQUATORIAL PLANE

From equation (11) it is clear that a circular orbit is stable against perturbations in the equatorial plane if and only if the discriminant $Q(\rho, E, \Phi)$ is a *maximum* as a function of ρ at fixed E and Φ when E and Φ satisfy equations (15) and (16). If Q is a maximum, an infinitesimally small perturbation in E and/or Φ will inevitably result in a turning point in the perturbed orbit when the deviation of ρ from that of the circular orbit is still small. On the other hand, if Q is a minimum, a small deviation in ρ will grow. Therefore, a circular orbit is stable against this type of perturbation if and typically only if

$$\frac{\partial^2 Q(\rho, E, \Phi)}{\partial\rho^2} < 0 \tag{23}$$

when equations (15) and (16) are satisfied.

A straightforward calculation gives

$$\begin{aligned}\frac{(1-v^2)}{2}\frac{\partial^2 Q}{\partial\rho^2} = &-(1+v^2)\nu_{,\rho\rho} + v^2\beta_{,\rho\rho} - v^2\frac{\omega_{,\rho\rho}}{\Omega-\omega} + 4v^2\nu_{,\rho}\frac{\omega_{,\rho}}{\Omega-\omega} + v^4\frac{\omega_{,\rho}}{\Omega-\omega}\\ &- 2v^2(\beta_{,\rho}{}^2 - 2\beta_{,\rho}\nu_{,\rho}) - \frac{4}{\rho}\beta_{,\rho}v^2 - \frac{3}{\rho^2}v^2\,.\end{aligned} \tag{24}$$

I have used (19) and (20) to express E and Φ in terms of v after taking the derivatives.

The second derivatives in equation (24) can be eliminated by using the result of differentiating equation (21) with respect to ρ. The physical velocity v is considered a function of Ω and ρ via equation (2). After a considerable amount of algebra the result can be put in the form

$$\frac{\rho^2(1-v^2)}{2}\frac{\partial^2 Q}{\partial\rho^2} = -X\left[\frac{\rho d\Omega/dp}{\Omega-\omega} + \frac{1-v^2}{v^2}X\right], \tag{25}$$

where X is a symbol for the expression

$$X = v^2(1+\rho\beta_{,\rho}) + (1-v^2)\rho\nu_{,\rho}\,. \tag{26}$$

From this form it is immediately clear that the stability condition (23) is satisfied in a region of the equatorial plane where the angular velocity Ω of the circular orbits is uniform, if it is assumed that timelike orbits exist ($v^2 < 1$). This establishes the stability of the individual orbits of the particles making up the uniformly rotating disk of Bardeen and Wagoner (1969) against perturbations in the equatorial plane, even in the limit that the redshift from the center of the disk is infinite. The same applies to the orbits of the particles of the uniformly counterrotating disk of Morgan and Morgan (1969).

A third useful form of the stability criterion is obtained by differentiating expression (19) for the angular momentum of a circular orbit. The result is

$$\rho d\Phi/d\rho = \frac{\Phi}{1-v^2}\left[\frac{\rho d\Omega/d\rho}{\Omega-\omega} + \frac{1-v^2}{v^2}X\right]. \tag{27}$$

Comparison with equation (25) gives

$$\tfrac{1}{2}\rho^2\frac{\partial^2 Q}{\partial\rho^2} = -\frac{\rho d\Phi/d\rho}{\Phi}X\,. \tag{28}$$

A reasonable (centrally concentrated) mass distribution with positive energy density and pressure will generate a gravitational field through equations (6) and (7) with $\nu_{,\rho} \geq 0$ and $\beta_{,\rho} \geq 0$ everywhere, so it is reasonable to expect that $X > 0$. Certainly this is true for a thin-disk model of the type considered by Bardeen and Wagoner. Therefore, one can conclude with a fair degree of generality that the circular particle orbits will be stable against perturbations in the equatorial plane in a region where the magnitude of the angular momentum of the particle orbits increases outward from the axis of symmetry and will be unstable in a region of the equatorial plane where it decreases outward. A firm conclusion is that any extremum in the angular momentum as a function of ρ corresponds to a change in the stability of circular orbits. Equation (22) shows that an extremum in Φ implies an extremum in the energy E. The inverse statement, that an extremum in E implies an extremum in Φ, will typically, but not necessarily, be true. It is conceivable that $\Omega = 0$ for some circular orbit in a highly relativistic metric.

VI. STABILITY PERPENDICULAR TO THE EQUATORIAL PLANE

Equation (14) governing perturbations perpendicular to the equatorial plane has the form of a simple harmonic oscillator. The stability criterion can be written

$$W = \nu_{,zz}(1+v^2) + \frac{\omega_{,zz}}{\Omega-\omega}v^2 - \beta_{,zz}v^2 > 0\,, \tag{29}$$

The second derivatives with respect to z can be expressed in terms of derivatives with respect to ρ and the energy-momentum tensor by using equations (6)–(8). Note that all first derivatives with respect to z are zero on the equatorial plane, since the energy-momentum tensor is included explicitly without any surface-density terms. The revised form of the stability discriminant becomes

$$\rho^2 W = 4\pi e^{2\mu}\rho^2\left[(E+P_1)\frac{1+v_r^2}{1-v_r^2}+T_a{}^a\right]+\tfrac{1}{2}\frac{(1-v^2)}{v^2}X^2 - X\left[\frac{d\Omega/d\rho}{\Omega-\omega}+\frac{1-v^2}{v^2}X\right]. \tag{30}$$

The quantity X is still defined by expression (26), and the velocity v_r is the velocity of the matter element relative to a freely falling observer following the circular orbit whose stability is in question,

$$v_r = \frac{v_m - v}{1 - vv_m}. \tag{31}$$

Note that $v_r^2 < 1$ if $v^2 < 1$ and $v_m^2 < 1$.

Outside the matter distribution the circular orbits are stable against perturbations both in and perpendicular to the equatorial plane if and only if

$$\tfrac{1}{2}\frac{(1-v^2)}{v^2}X < -\frac{\rho d\Omega/d\rho}{\Omega-\omega}X < \frac{1-v^2}{v^2}X. \tag{32}$$

Inside the matter distribution the stability criterion for perturbations perpendicular to the plane can be stated simply if the matter distribution is concentrated in a thin disk. If the terms from the energy-momentum tensor in equation (30) dominate those containing radial derivatives of the potentials, the orbits are stable if $E + P_1 > 0$ and if $E + P_1 + T_a{}^a > 0$. The condition is essentially that the gravitational force of the matter on itself be attractive, the same as the condition imposed in the singularity theorems of Hawking and Ellis (1968) and others. For the zero-pressure disks of Bardeen and Wagoner (1969) the condition for stability is just that the energy density be greater than zero.

VII. CONCLUSION

Which of the different forms for the stability criteria are most convenient to apply to the circular orbits in a given metric will depend on the particular case, on whether the metric is known analytically or just numerically, for instance. The circular orbits of the particles making up a "zero pressure" disk in general relativity will be stable if and only if the angular momentum per unit mass increases monotonically outward in the disk, regardless of how strong the gravitational fields generated by the disk are. Equilibrium models of a uniformly rotating disk exist in which the magnitude of the relativistic analogue of the Newtonian gravitational potential, ν, is arbitrarily large in the vicinity of the disk ($\nu \to -\infty$). The stability of the individual particle orbits says nothing about stability of a disk against collective motions in which the gravitational field is affected by the perturbations. Even in the Newtonian limit such a disk is unstable against fragmentation (Toomre 1963; Hunter 1963).

The properties of circular orbits in the equatorial plane of the Kerr metric will be treated in detail in a separate paper. Some aspects have been considered by Felice (1968), and Carter (1968) has obtained equations for general particle orbits in the Kerr metric.

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