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HOT WHITE DWARFS

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ABSTRACT

A sequence of models of white dwarfs including corrections for nonideality in the equation of state has been computed for a typical white-dwarf luminosity. The mass-radius relations found by Hamada and Salpeter are appreciably modified by thermal effects, particularly for white dwarfs of low mass. Our results for 40 Eri B, together with a recent determination of its radius, indicate a chemical composition heavier than helium. We find that some white dwarfs exhibit a zone where electrons are strongly coupled, such that determination of the central temperature becomes difficult.

I. INTRODUCTION

Hamada and Salpeter (1961) discovered that the mass-radius diagram of zero-temperature white dwarfs exhibits fine structure dependent on chemical composition. The fine structure, which consists of a small difference in radius for a given mass, arises because of interaction terms in the zero-temperature equation of state which depend on composition parameters other than μ_e, the mean number of electrons per nucleon. Given a sufficiently accurate determination of the mass and radius of a white dwarf (which is in fact very difficult to accomplish), one can thus, in principle, deduce its internal chemical composition. Attempts to perform such a determination have been made, e.g., by Greenstein and Trimble (1967) and by Matsushima and Terashita (1969).

In this paper, we have calculated the effect of a finite internal temperature on a number of Hamada-Salpeter models in terms of the (in principle) observable parameter, the total luminosity. We find that the interaction terms in the pressure may remain reasonably temperature-dependent even at typical central temperatures of white dwarfs, in contrast to the dominant pressure of an ideal Fermi gas, and the chemical fine structure in the mass-radius curve can thus vary considerably with the luminosity of the white dwarf.

In particular, we consider the case of 40 Eri B, whose mass and radius are now considered to be rather accurately determined. We also have reinvestigated the problem of the determination of the central temperature of a white dwarf from its total luminosity, which is of some importance in the interpretation of crystallizing white dwarfs (Van Horn 1968). Finally, we briefly consider the effect of finite temperature on the pulsation period of a Hamada-Salpeter white dwarf.

II. INPUT PHYSICS AND METHODS

a) *Equation of State*

Salpeter (1961) calculated the Coulomb, Thomas-Fermi, exchange, and correlation corrections to the Chandrasekhar (1935) equation of state for electrons at zero temperature. For typical white dwarfs of about 1 $M_\odot$, the Coulomb and exchange corrections are the most important, and the other two corrections can generally be neglected. At finite temperature, only the exchange correction can still be calculated analytically. If one assumes an incompressible electron fluid (equivalent to infinite electron density), Monte Carlo data (Brush, Sahlin, and Teller 1966) may be used to obtain the Coulomb correction at finite temperature. The Thomas-Fermi correction could be calculated at finite temperature by using the method of Feynman, Metropolis, and Teller (1949), but this method is not applicable to finite-temperature white dwarfs since it assumes that electrons are excited to states above the Fermi sea while the ion lattice remains in its zero-temperature configuration. In fact, the ion lattice melts long before an appreciable fraction of electrons are excited (Van Horn 1968). The behavior of the correlation term at finite temperature is apparently unknown, although it is known that it tends to the electron Debye-Hückel correction in the limit of nondegenerate electrons (Kadanoff and Baym 1962).

The finite-temperature equation of state which we have adopted is as follows:

$$P(\rho,T) = P_0(\rho,T) + P_{\mathrm{BST}}(\rho,T) + P_x(\rho,T) , \tag{1}$$

where $P_0(\rho,T)$ is the well-known pressure of an ideal gas of electrons, valid for any (including relativistic) density and temperature, $P_{\mathrm{BST}}(\rho,T)$ is the Brush, Sahlin, and Teller pressure according to the analytic approximation of Van Horn (1969), and $P_x(\rho,T)$ is the exchange pressure of electrons. For the latter, we have used the zero-temperature form of Zapolsky (Salpeter 1961) for high electron density, and we have switched over to the finite-temperature, nonrelativistic form (Hubbard 1969) before the electrons become appreciably nondegenerate. This procedure produces a discontinuity in the exchange pressure because relativistic corrections are apparently important even at low density ($\rho \sim 10^3$ g cm^{-3}); however, the exchange pressure is such a small correction that the error is unimportant, and it did not prevent convergence of the models. Apparently, a fully relativistic, finite-temperature result for the exchange pressure of electrons is not available, but for our application it would probably produce very little improvement in accuracy.

In summary, the equation of state (1) is probably valid to within a fraction of a percent through the bulk of a typical white dwarf. It fails in the small zone where the electrons are becoming nondegenerate, but the cumulative effect on the radius of the star is negligible.

b) Opacity

For the radiative opacity, we have used the tables of Cox and Stewart (1965), interpolating where necessary between chemical composition. For the conductive opacity, we have used the tables of Hubbard and Lampe (1969), together with tables calculated as necessary for additional chemical compositions.

c) Atmospheres

In all cases, we have assumed that the atmosphere of the white dwarf is of solar chemical composition (corresponding to a DA star), and have used the grid of DA model atmospheres of Terashita and Matsushima (1969) to integrate inward to a point where the diffusion approximation holds. For a specific luminosity and mass, we guess the radius, which then specifies the model atmosphere. Starting from the base of the model atmosphere, we then integrate a solar-composition envelope inward until the energy release due to the proton-proton reaction equals the total luminosity, which thus sets an upper limit to the thickness of a hydrogen-rich envelope. For the *p-p* reaction rate, we have used the result of Bahcall and May (1968), neglecting factors of order unity due to electron screening and precise chemical composition. We then continue integrating inward, using the assumed chemical composition of the core.

d) Core Models

The structure of any finite-temperature star is, of course, an initial-value problem. Therefore, the correct approach is to calculate a series of evolutionary models, culminating in the desired model. However, since we are interested in the radius of a white dwarf of given mass and luminosity without making any restrictive assumptions as to its previous history, such as the possibility of mass loss (which is difficult to include in an evolutionary calculation in a realistic manner), we have simply assumed that all energy sources are contained within the essentially isothermal interior. For a white dwarf which is highly degenerate and whose reservoir of thermal energy is large compared with its reservoir of gravitational energy, this approximation is valid. This assumption, of course, restricts us to consideration of models whose increase in radius due to thermal expansion is small compared with their zero-temperature radius. We have considered the proton-proton reaction only to establish an upper limit to the extent of a hydrogenic envelope, and no models have been calculated which actually derive their luminosity from nuclear reactions.

The procedure we have used is to specify the thickness of the hydrogenic envelope (up to the maximum), and to integrate inward from that point (assuming constant luminosity) to the point where the temperature gradient becomes small, such that the temperature varies by only a few percent per thousand kilometers. At this point we could either set the temperature equal to a constant the rest of the way to the center or assume homologous energy release, such that the energy release is proportional to the local temperature (Hayashi, Hōshi, and Sugimoto 1962). Either procedure yields the same central temperature to within a few percent.

With the temperature-pressure relation thus established, white-dwarf models are then integrated for the prescribed mass. If the radius differs greatly from the initially guessed radius, another iteration is necessary, but this result indicates that the assumptions concerning the location of energy sources are vitiated in any case.

III. RESULTS

a) *Internal Temperatures*

For all of the models, we have chosen a typical white-dwarf luminosity of $10^{-2} L_{\odot}$. For luminosities much greater than this, our assumptions concerning location of energy may be invalid (depending to some extent on the mass), while for lower luminosities we have encountered difficulties with the opacities (see below). The central temperature appears to be insensitive to the thickness of the hydrogenic envelope, but it is somewhat more sensitive to the chemical composition of the underlying material. In some cases, our results are in fair agreement with a crude estimate of the central temperatures of white dwarfs given by Schwarzschild (1958). Schwarzschild's technique consists of integrating an ideal-gas envelope inward by the use of Kramers's opacities and radiative zero boundary conditions to the point of 50 percent electron degeneracy, and assuming isothermality below this point. It is known that this technique tends to predict internal temperatures which are somewhat too high. However, the Hubbard-Lampe conductive opacities tend to be higher than the older conductive opacities in the envelope and lead to an upward revision of the internal temperature, bringing it back into fair agreement with Schwarzschild's result in some instances.

In the paper by Hubbard and Lampe, it was pointed out that, for a given chemical composition, there is a zone in the temperature-density plane where conventional methods of calculating the conductive opacity can be expected to fail. This zone, which we denote in this paper as the electron-coupling zone, corresponds roughly to the region where the mean kinetic energy of the electrons is comparable to or smaller than the mean interaction energy of the electrons with other charged particles, and is roughly coincident with the region of incomplete ionization. Evidently this zone is also the region where the equation of state (1) may be expected to fail. As discussed by Hubbard and Lampe, the boundaries of the electron-coupling zone are only qualitative, and it is not known how rapidly the assumption of free electrons fails as the boundaries are approached.

In the white-dwarf models which we have calculated, many of the envelopes pass through a portion of the electron-coupling zone. This result would not be serious if the zone were to occur in a region where radiative opacities are much less than the conductive opacities, or if the region were essentially isothermal. In Table 1, we give the results of calculations of helium and carbon envelopes, for several masses, with an assumed luminosity of $10^{-2}\,L_\odot$ in all cases. The symbol W denotes the width of the electron-coupling zone in kilometers, κ_R/κ is the ratio of the radiative opacity to the total opacity, T is the temperature in degrees Kelvin, and u, l refer to the upper and lower boundaries of the zone. The integration of the envelopes was effected by smoothly interpolating conductive opacities in the zone. It may be seen that the effect of the electron-coupling zone is small in the case of helium, but for carbon it may become quite significant, particularly for the white dwarfs of higher mass. For elements with $Z \gtrsim 6$, the zone becomes so large that it is impossible to obtain a meaningful central temperature except for very high luminosities ($L \sim L_\odot$), in which case the star is quite far from its final white-dwarf configuration. We have compared our models with Van Horn's (1968) models of crystallizing

TABLE 1

PROPERTIES OF THE ELECTRON-COUPLING ZONE FOR HELIUM AND CARBON WHITE DWARFS

Parameter	Mass ($M_\odot$) 0.20	0.43	0.60	0.80	1.05
He:					
W (km)	442	98	52	26	16
$(\kappa_R/\kappa)_u$	0.985	0.952	0.980	0.970	0.967
$(\kappa_R/\kappa)_l$	0.983	0.862	0.960	0.950	0.939
T_u (° K)	3.38×10^6	3.00×10^6	2.38×10^6	2.28×10^6	2.22×10^6
T_l (° K)	4.92×10^6	4.66×10^6	4.17×10^6	4.00×10^6	3.90×10^6
C:					
W (km)	2151	417	197	116	71
$(\kappa_R/\kappa)_u$	0.962	0.953	0.965	0.957	0.948
$(\kappa_R/\kappa)_l$	0.800	0.732	0.607	0.516	0.488
T_u (° K)	5.63×10^6	4.48×10^6	3.77×10^6	3.62×10^6	3.47×10^6
T_l (° K)	1.40×10^7	1.25×10^7	1.13×10^7	1.05×10^7	1.00×10^7

NOTE.—A luminosity of $10^{-2}\,L_\odot$ is assumed in all cases.

white dwarfs, and we find that, for helium or carbon composition, the electron-coupling zone begins to be important at about the point at which ion crystallization is beginning in the center. The possibility thus remains that the time scale for cooling of white dwarfs may be affected by changes in the opacity law near the surface at the same time that the heat of fusion is being released near the center. It has also been suggested that superconductivity may occur in the electron-coupling zone in the case of white dwarfs of extremely low temperature (Ginzburg and Kirzhniz 1968).

b) Radii

Table 2 gives the results of model calculations for a range of masses and chemical compositions. In the table, $R_{\max}$ and $R_{\min}$ denote, respectively, the radius obtained when one assumes that the hydrogen envelope reaches the maximum extent permitted by nuclear reactions, and the radius obtained when one assumes that the hydrogen envelope terminates at mean optical depth of unity; the percentages are the increase in radius over the zero-temperature Hamada-Salpeter radius. The quantities T_c and Γ_c are the central temperature and the central ion-coupling parameter (defined, e.g., in Van Horn 1968). All quantities are for a total luminosity of $10^{-2}\,L_\odot$. It was not possible to calculate central temperatures for magnesium models, for reasons discussed above, and therefore a carbon envelope has been assumed to exist down to a depth where isothermality obtains. For this reason, the central temperatures of the magnesium models are the same as those of the carbon models.

Greenstein and Trimble (1967) have carried out a statistical analysis of the relation between gravitational redshift and radius for a number of white dwarfs, and have concluded that the median mass, if helium composition is assumed, is 0.86 $M_{\odot}$. However, the corresponding photometric radius is 10–15 percent too large for a Hamada-Salpeter white dwarf of such a mass. In the case of an assumed composition of iron, the discrepancy is much greater. This has been cited by Ostriker and Hartwick (1968) as evidence for magnetic fields and/or rotation in white dwarfs. However, if the Greenstein and Trimble white dwarfs have an actual composition of helium, fairly extensive hydrogen envelopes, and a mean luminosity of about 10^{-2} $L_{\odot}$, then much of the discrepancy could be explained by purely thermal effects. However, helium is possibly an unlikely chemical composition for such massive white dwarfs.

TABLE 2

PROPERTIES OF THE WHITE-DWARF MODELS

Parameter	Mass ($M_{\odot}$) 0.20	0.43	0.60	0.80	1.05
He:					
R_{max}* (km)	23300 (67%)	11920 (16%)	...	...	5530 (9%)
R_{min} (km)	19130 (37%)	11060 (8%)	...	...	5350 (5%)
T_c (° K)	2.39×10^7	1.29×10^7	...	...	1.33×10^7
Γ_c	1.3	4.8	...	...	13.7
C:					
R_{max}* (km)	21670 (63%)	11120 (12%)	9180 (10%)	7180 (5%)	...
R_{min} (km)	17460 (30%)	10570 (7%)	8660 (3%)	6880 (<1%)	...
T_c (° K)	3.09×10^7	2.24×10^7	2.03×10^7	1.86×10^7	...
Γ_c	6.9	17.9	27.5	40.6	...
Mg:					
R_{max}* (km)	20700 (62%)	10820 (11%)	8930 (9%)	7030 (5%)	5000 (3%)
R_{min} (km)	16500 (29%)	10260 (6%)	8330 (2%)	6720 (<1%)	4870 (≪1%)
T_c (° K)	3.04×10^7	2.24×10^7	2.03×10^7	1.86×10^7	1.66×10^7
Γ_c	23.0	53.4	89.8	141.4	329.0

NOTE.—Percentages in parentheses are the amount of expansion over the zero-temperature, zero-hydrogen models.
* Including maximum hydrogen envelope.

Terashita and Matsushima (1969) have carried out another analysis of the mass-radius distribution for DA white dwarfs, and, although the scatter is great, obtain a mean mass of ~0.6 $M_{\odot}$ and a mean radius of ~0.0140 $R_{\odot}$. Again, for helium, this is about 10 percent too large a radius, but it can be comfortably explained by thermal expansion. For carbon, thermal expansion could probably not explain such a discrepancy completely.

The most classic example of a white dwarf which does not obey the standard mass-radius relation is Sirius B, which has a well-determined mass of 1.05 $M_{\odot}$ and a radius of about 0.010 $R_{\odot}$ (Eggen and Greenstein 1965). We have calculated a model of Sirius B using the estimated luminosity of 10^{-2} $L_{\odot}$ and assuming an improbable chemical composition of helium. The maximum radius of this model (see Table 2) is 0.008 $R_{\odot}$, which falls far short of explaining the discrepancy.

The white dwarf whose mass and radius are best determined is 40 Eri B. Matsushima and Terashita (1969) have given its radius as 0.0150 ± 0.0003 $R_{\odot}$, based upon a detailed model-atmosphere analysis. Direct determination of its mass gives the result 0.43 ±

$0.04\,M_\odot$ (Popper 1954). We have constructed a number of models of 40 Eri B in an attempt to determine its chemical composition from the above precise determination of its radius. In Figure 1 are plotted the locus in the temperature-pressure plane of the hydrogen and helium envelopes of 40 Eri B. The starting point for the integration is the nongray atmosphere of Matsushima and Terashita, and it can be seen that our envelope, which uses interpolated Cox and Stewart opacities, matches the model atmosphere smoothly. The locus of the helium envelope is practically coincident with the locus of the hydrogen envelope, a result which implies that the central temperature is highly insensitive to the thickness of the hydrogen zone. If the result of Matsushima and Terashita is used, then the radius of 40 Eri B is between 10240 and 10650 km. If their error estimate is realistic, this is sufficient to eliminate pure helium as a possible chemical composition, since the smallest helium model has a radius of 11060 km. As may be seen from Table 2, even a pure-carbon model of 40 Eri B only falls just within the error bars, such that an

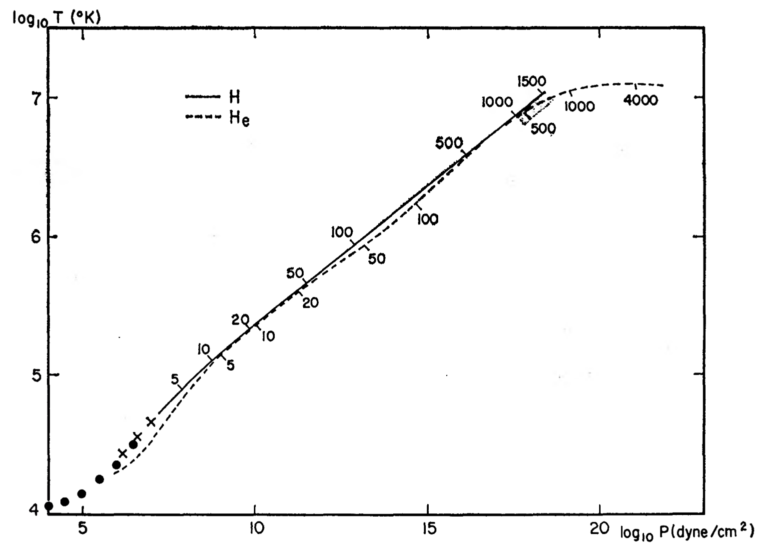

FIG. 1.—Hydrogen and helium envelopes for 40 Eri B. Numbers are the depths from the surface in km. *Closed circles*, points from the model atmosphere of Matsushima and Terashita; *crosses*, top three mesh points of our hydrogen envelope. *Hatched area*, electron-coupling zone for helium.

exceedingly thin hydrogen envelope would have to be postulated for this case. We have also calculated a model of 40 Eri B for a chemical composition of 50 percent carbon, 50 percent oxygen by weight. In this case, the electron-coupling zone is still small enough to have a negligible effect on the internal temperatures. We find, for this case, that the increase in opacity is enough to raise the internal temperature by such an amount that the carbon-oxygen model has a radius only about 50 km smaller than the pure-carbon model. From these results, we conclude that there is weak evidence for a composition of 40 Eri B of elements heavier than carbon or carbon-oxygen. A composition of mainly helium seems to be definitely ruled out. These conclusions, of course, depend upon the absence of systematic errors in the radius determination. We have not investigated composite models with shells of differing composition, since the number of free parameters then becomes very large.

c) Pulsation Periods

We have investigated the effect of a finite temperature on the pulsation period of a Hamada-Salpeter white dwarf for the case of a mass of $0.43\,M_\odot$. The calculation assumes adiabatic, linear pulsations, and the period of the fundamental mode is determined by integrating the eigenvalue equation outward from the central singularity and inward from the surface singularity, and requiring continuity of the eigensolution and its first derivative at an interior point, in the manner described by Bardeen, Thorne, and Meltzer (1966) (without including general-relativistic effects). For a zero-temperature white dwarf of pure helium, the period of the fundamental mode is 17.39 seconds. For a helium white dwarf with a luminosity of $10^{-2}\,L_\odot$, the period becomes 18.04 seconds. For a helium white dwarf of $10^{-2}\,L_\odot$ and the maximum permissible hydrogen envelope, the period becomes 18.06 seconds. The period thus increases by only 4 percent in either case, and we conclude that the period increases more slowly than the radius, as a function of luminosity. We have not investigated whether any of our models are actually pulsationally unstable.

IV. SUMMARY

We have found that, even for fairly luminous white dwarfs, there is a shell near the surface where physical conditions cannot be described by the conventional free-electron theory. At very low luminosities and/or high central densities, the present theory of opacity of white-dwarf matter may not permit a determination of the central temperature from the total luminosity. Discussions of the cooling time scales of white dwarfs (Van Horn 1968) implicitly assume the validity of an extrapolation of the present theory into the electron-coupling zone of the temperature-density plane.

At typical white-dwarf luminosities, thermal effects can produce deviations from the zero-temperature mass-radius curve which are of the same order as the Hamada-Salpeter corrections to the standard Chandrasekhar mass-radius curve. Any attempt to deduce the chemical composition of a white dwarf from its position on the mass-radius curve should therefore include such effects.

We have shown that the radius determination of 40 Eri B by Matsushima and Terashita implies a chemical composition heavier than helium. According to the calculations of Cox and Salpeter (1964) and L'Ecuyer (1966), a star with mass greater than about 0.3 $M_\odot$ will eventually burn helium; however, Paczynski (1969) has calculated that a star with an ultimate mass of 0.43 $M_\odot$ could conceivably reach a white-dwarf state with a significant abundance of helium. We achieve a slightly better agreement with the radius determination for a magnesium composition than for carbon or carbon-oxygen; we thus cite this as weak evidence for mass loss, which seems quite likely for 40 Eri B in any case, considering that the main-sequence lifetime of a star of 0.43 $M_\odot$ would be approximately 10^{11} years (Clayton 1968).

We find that the pulsation period of a white dwarf increases more slowly than its radius, as a function of total luminosity.

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