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THE EFFECT OF WAVE-PARTICLE INTERACTIONS ON THE PROPAGATION OF COSMIC RAYS

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ABSTRACT

The interaction between galactic cosmic rays and Alfvén waves in the interstellar medium is investigated. They may interact adiabatically through magnetic mirror scattering or non-adiabatically through gyration frequency resonance. The equations describing the latter are derived. The growth rates for the waves are given, and the Fokker-Planck equation for the diffusion of cosmic rays in velocity space is derived. These two equations are applied to a model of the cosmic rays consisting of a uniform tube of magnetic field with open ends. An equation of spatial diffusion is derived in the limit of strong wave-particle scattering, and this equation is compared with the observed properties of the galactic cosmic rays to derive a mean free path for scattering of about 10 pc. It is shown that when the interstellar damping of the Alfvén waves is included, the waves are probably marginally stable. Finally, a self-consistent model is specified in which the sources of turbulence and cosmic rays are given and the cosmic-ray densities are to be determined. This model is solved in the crude approximation where all particles have effectively the same energy and all waves the same wavelength. It is shown that the cosmic rays can have an appreciable effect on their confinement to the Galaxy. It is shown that if the inhomogeneous distribution of mass is taken into account, the confinement of cosmic rays is determined primarily by the low-density regions between the clouds. An attempt is made to evaluate the efficiency of heating of cosmic rays, and it appears that their energy changes very little during their galactic confinement. However, because there is a non-linear relation between the source and the cosmic-ray density, the observed energy spectrum does not necessarily represent the emitted spectrum.

I. INTRODUCTION

The general problem of the origin of cosmic rays divides into two parts. The first part concerns the actual origin or injection of the cosmic rays into the Galaxy, while the second part concerns the subsequent behavior of the cosmic rays, their motion and acceleration in the Galaxy. This paper is concerned with the second part of this problem only.

It is generally recognized that the cosmic rays propagate along the galactic magnetic field. Because of the observed isotropy and age of the cosmic rays, it seems clear that the cosmic rays cannot propagate freely along the lines of force but must be continually scattered and slowly diffuse out of the galaxy. Further, the scattering cannot be by particles since the energies of the cosmic rays are much higher than nuclear binding energies and such collisions would destroy all nuclear species heavier than protons. Thus, the most likely scattering mechanism is off waves, and this paper is concerned with the behavior of particles as they scatter off magnetic inhomogeneities or waves.

Cosmic rays can scatter off large-amplitude magnetic-field inhomogeneities. This was first proposed by Fermi (1949, 1954), who showed that they would be accelerated as well. Morrison, Olbert, and Rossi (1954) investigated this process more quantitatively and showed that it was difficult to get quantitative agreement with the observed diffusive properties of cosmic rays unless the magnetic field was very tangled. They assumed that the field was confined to clouds and that the cosmic rays were scattered off clouds. However, it now seems likely, not that the field has such a tangled structure, but rather that it is more or less uniform over distances of several kiloparsecs with smaller fluctuations with scales of order of a parsec (Spitzer 1968; Serkowski 1962).

Another possibility is that of cyclotron resonance scattering of cosmic rays off small-wavelength Alfvén waves. (The Alfvén waves propagate in the cold interstellar medium, and their propagation rate is only slightly affected by cosmic rays.) If the wavelength of the Alfvén waves is of order of the gyration radius of the cosmic ray, the Doppler-shifted wave frequency seen by the cosmic ray is of the order of its cyclotron frequency. The result of such a resonance is that the cosmic ray gets scattered in pitch angle with little change in energy. In this paper we concentrate on this interaction between Alfvén waves and cosmic rays and show that the scattering rate can be appreciable if there is any energy in this wavelength range. Further, we show that the energy in Alfvén waves is indeed large, owing to an instability found by Lerche (1966, 1967) and whose importance has been emphasized by Wentzel (1968) and Parker (1968). These results show that a small cosmic-ray anisotropy makes these waves unstable. Such an anisotropy is produced naturally in the Galaxy by the cosmic rays escaping out of the Galaxy along the lines of force. A velocity of the order of the Alfvén velocity leads to instability. Thus, the cosmic rays themselves can produce waves which in turn react on the cosmic rays and scatter them.

The importance of scattering particles in pitch angles by waves was first demonstrated by Kennel and Petschek (1966). They considered the effect of whistler-mode scattering on the trapped particles in the Van Allen belt. The interaction of high-energy particles with Alfvén modes by cyclotron resonance has been considered by Cornwall (1966). Tidman (1966) showed that Alfvén waves are efficiently damped by cosmic rays because of cyclotron resonance. Jokipii (1966, 1967) calculated the scattering of cosmic rays in interplanetary space by Alfvén waves. The general theory for scattering by plasma turbulence has been worked out by Kennel and Engelmann (1966).

This paper is divided into two main parts: § II and § III. In § II we develop the physical basis for the interaction between cosmic rays and Alfvén waves. By solving the relativistic Vlasov equation we compute the linearized growth rate of Alfvén waves in the presence of a general cosmic-ray distribution. Then by going to second order, we derive a Fokker-Planck equation for the cosmic rays, whose scattering coefficients are expressed in terms of the energy density of the Alfvén waves, whose amplitudes we assume to be random. The general properties of these equations are discussed.

In § III the results of § II are applied to the Galaxy and other systems in which cosmic rays are found. A model is postulated in which the cosmic rays are assumed to be restricted to a single magnetic tube of force of length $2L$. In the limit of large scattering rates the Fokker-Planck equation is reduced to a one-dimensional diffusion equation along this tube of force. A self-consistent problem is then set up in which sources of cosmic rays with their initial energy spectra are given. Also, all sources of Alfvén wave turbulence other than the cosmic rays are assumed given. The cosmic-ray densities and spectra and the turbulent energy spectra of the Alfvén waves are then to be solved for. The cosmic-ray densities depend on the energy of the Alfvén waves, while the energy of the Alfvén waves depends on the cosmic rays through their effect on the damping. (In the absence of cosmic rays, the damping is by friction between the charged and neutral parts of the interstellar gas.)

This model is solved approximately for the case of homogeneous sources in a uniform interstellar medium. The solution is expressed in terms of a standard source of cosmic rays, a standard source of turbulence, and a standard cosmic-ray density. It is shown that (*a*) the behavior of the solution (of the density in terms of the cosmic-ray source) is different according to whether the source of turbulence is weaker or stronger than the standard source and (*b*) for a fixed source of turbulence the dependence of the cosmic rays on their source is non-linear, being different as the source is weak, intermediate, or strong (compared with the standard source). This result makes it clear that it is not possible to infer the nature of the sources directly from the observed cosmic-ray spectra and density without taking into account their confinement. It is further possible to calculate the mean age and isotropy of the cosmic rays.

Using this result, one can compare the self-consistent solution with the region of the

Galaxy near us and solve for the sources. Taking $L = 3$ Kpc, we find that the sources are weak and that the density is below the standard one.

However, if one takes into account the granular nature of interstellar matter, it is shown that the main diffusion of cosmic rays occurs in between the clouds, where the interstellar density is low. In this case the source of cosmic rays is strong and the turbulence sources are somewhat indeterminate.

Finally, the case where the field is not uniform is considered so that we can have magnetic pumping (Fälthammar 1963). The amount of acceleration of the cosmic rays by this method is then computed. It is found that the amount of acceleration of the cosmic rays in the interstellar medium is negligible in the galactic case, although it may not be in the other cases.

II. THE BASIC EQUATIONS FOR WAVES AND COSMIC RAYS

a) Hydromagnetic Waves in H I *Regions*

In this section we give the equations which govern the interaction between the Alfvén waves and cosmic rays. The Alfvén waves are carried by the cold interstellar medium, and they change their amplitude by ordinary damping and by interaction with cosmic rays. We restrict ourselves to the H I regions of interstellar space. These regions have two components: the neutral component, made up of hydrogen and helium which is not ionized by interstellar radiation; and a charged component of much lower density, made up mostly of ionized carbon and electrons (and perhaps a small fraction of the hydrogen ionized by cosmic rays). At the wavelengths of interest, the collisions between the ionized and neutral components are relatively infrequent and therefore the hydromagnetic waves of interest are essentially carried by the ionized component alone. However, occasional collisions between the ionized and neutral components do provide a damping mechanism for the Alfvén waves.

Let us first discuss the nature of the hydromagnetic waves. We assume the interstellar magnetic field $\boldsymbol{B}_0$ is uniform and in the z-direction. The ratio of the pressure of the ionized interstellar matter to the magnetic pressure is very small ($\sim 10^{-5}$), so we may neglect it and take the interstellar matter as cold. Therefore, the hydromagnetic waves are quite simple. Of the two modes, the magnetosonic has a velocity $V_A = B_0/(4\pi\rho^*)^{1/2}$ in all directions, where ρ^* is the density of the ionized interstellar matter, while the Alfvén mode has a phase velocity $V_A \cos\zeta$, where ζ is the angle between the direction of propagation and $\boldsymbol{B}_0$. The cosmic rays do not influence the speed of propagation of hydromagnetic waves except for propagation nearly perpendicular to $\boldsymbol{B}_0$, $|\pi/2 - \zeta| \sim V_A/c \approx 3 \times 10^{-4}$, and we disregard waves propagating in this direction. The Alfvén waves are linearly polarized except for propagation nearly along $\boldsymbol{B}_0$, $\zeta \sim (\omega/\Omega_i)^{1/2} \approx (V_A/c)^{1/2}$, and we disregard propagation in this direction also.

In addition to the cosmic-ray interaction (which we discuss below) and the collisions between charged particles and neutral particles, there are several other possible damping mechanisms for the waves. These are resistivity, electron viscosity, electron thermal conductivity, and ion viscosity. They are discussed in Appendix A, where it is shown that none of them can be important compared with ion-neutral collisions. The effect of ion-neutral collisions is discussed in Appendix C. Further, the electric field in the waves is almost exactly perpendicular to the magnetic field because of the large electrical conductivity of the interstellar medium. It is shown in Appendix B that the minute E_z field does not lead to any significant modification of the interaction between cosmic rays and waves.

b) The Effect of the Cosmic Rays on the Waves

We now consider the effect of cosmic rays on the amplitude of Alfvén waves. In order to do this, we use a collisionless theory and solve the relativistic Vlasov equation for the distribution function $f(r, p, t)$, where $\boldsymbol{p}$ is the relativistic momentum and $\boldsymbol{r}$ is the position, as an expansion in the electric field amplitude $\boldsymbol{E}$. To lowest order,

$$f_0 = n^* \delta(p) + F(p) , \tag{1}$$

where n^* is the density of the cold charged particles and F is the distribution function of the cosmic rays. To first order we can solve for F (assuming $E \sim \exp[i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t]$), compute the perturbed current, and substitute in Maxwell's equation to obtain a dispersion relation. The details are the same as given by Lerche (1967). Taking into account the low density of cosmic rays, we can neglect them in the real part of the dispersion relation, and we have for the two modes:

$$\omega(k) = \omega_{\boldsymbol{k}} + i\Gamma_{\boldsymbol{k}} , \tag{2}$$

$$\omega_{\boldsymbol{k}} = \begin{Bmatrix} k V_A \\ k V_A \cos \zeta \end{Bmatrix} , \tag{3}$$

$$\Gamma_{\boldsymbol{k}} = \Sigma 2\pi^2 q^2 \left(\frac{V_A}{c}\right)^2 \sum_{n=-\infty}^{\infty} \int d^3 p v_\perp^2 \delta(\omega_{\boldsymbol{k}} - k_z v_z - n\Omega) \begin{Bmatrix} J_n'^2(x) \\ \dfrac{n^2 J_n^2(x)}{x^2} \end{Bmatrix} \left[\frac{\partial F}{\partial \epsilon} + \frac{k_z}{\omega_{\boldsymbol{k}} p} \frac{\partial F}{\partial \mu}\right] , \tag{4}$$

$$x = k_\perp v_\perp / \omega .$$

The first sum is over all the various species of cosmic rays, and q is the charge, c the speed of light, v the velocity, $\Omega = qBc/\epsilon$ the relativistic cyclotron frequency, ϵ the relativistic energy, J_n the Bessel function, $\mu = \cos\theta = p_z/p$, and the subscript $\perp$ refers to the component perpendicular to $\boldsymbol{B}_0$. The upper line refers to the magnetosonic mode, denoted by m.s., and the lower line to the Alfvén mode, denoted by a. We have given only the real and imaginary parts for $\omega > 0$, corresponding to waves propagating in the direction of $\boldsymbol{k}$. (The other root is obtained by merely reversing the sign of $\omega_{\boldsymbol{k}}$.)

Equation (4) gives the rate of growth or damping of the wave amplitudes produced by the cosmic rays. It is clear from the δ-function factor that only those cosmic rays which see a Doppler-shifted frequency $\omega_{\boldsymbol{k}} - k_z v_z$ which is an integral multiple of their cyclotron frequency Ω can resonate with the wave. Since $\omega_{\boldsymbol{k}} = kV_A \ll k_z v_z$ for $n \neq 0$, we can replace this resonance condition by

$$k_z = n\Omega / v_z \approx n / r_L , \tag{5}$$

where $r_L = v/\Omega$ is essentially the cyclotron radius. For $n \neq 0$, the resonance occurs for waves with wavelength of order the cyclotron radius (times 2π).

For $n = 0$, the resonance condition is that the wave has the same phase velocity as the particle. This only occurs for waves propagating nearly perpendicular to $\boldsymbol{B}_0$ or for particles with $v_z \approx V_A$. It is zero-gyration-radius effect and is in fact the small-wave-amplitude limit of the Fermi interaction. We postpone discussion of this resonance until the end of § III.

From the form of equation (4) we see that an isotropic cosmic-ray distribution leads to damping if $\partial F / \partial \epsilon$ is negative, but a slight anisotropy can lead to instability, since it is enhanced by a factor of order $kv/\omega_{\boldsymbol{k}} \approx c/V_A$. To see this more quantitatively, let us assume a cosmic-ray distribution,

$$F = F_0(\epsilon) + \mu F_1(\epsilon) + \tfrac{1}{2}\mu^2 F_2(\epsilon) , \tag{6}$$

$$F_0 = K_1/p^m , \quad F_1 = K_2/p^q , \quad F_2 = K_3/p^r , \quad \epsilon > \epsilon_0 , \tag{7}$$

and zero for $\epsilon < \epsilon_0$. Let us further consider waves with $k_\perp \ll k_z$ so we may replace all

Bessel functions by their zero-argument values and drop terms with $|n| > 1$. Then we easily find (for $F_1, F_2 \ll F_0$)

$$\begin{aligned}\Gamma(k) &= -\Gamma_1 + \Gamma_2 \\ &= \pi^2 \frac{(m-3)}{(m-2)} \Omega_0 m_{\mathrm{H}} \frac{\Phi(>\epsilon_1)}{m^* n^* c} \left[-1 + \frac{c}{V_A} \frac{k_z}{|k|} \frac{(m-2)}{q(q-2)} \frac{F_1(\epsilon_1)}{F_0(\epsilon_1)} \right],\end{aligned} \tag{8}$$

where $\Omega_0 = qB/m_{\mathrm{H}}c$ is the non-relativistic cyclotron frequency of the cosmic rays, m_{H} is the proton mass (we assume all the cosmic rays are protons), $\Phi(> \epsilon_1)$ is the total cosmic-ray flux in particles cm^{-2} sterad^{-1} with $\epsilon > \epsilon_1$, $\epsilon_1 = qB/k \gg m_{\mathrm{H}}c^2$ is that energy for which $kr_L \approx 1$, and m^* is the cold ion mass. We denote the isotropic damping by $-\Gamma_1$ and the anisotropic term by Γ_2. Note that the symmetric anisotropy F_2 does not contribute, its effect being canceled by particles with opposite v_z. (It can lead to growth for waves with $\zeta \lesssim [\omega/\Omega_0]^{1/2}$.) We see that the cosmic rays lead to growth of the waves for $F_1/F_0 \gtrsim V_A/c$. Thus if F_1 corresponds to a mass drift of the cosmic rays U, $U \gtrsim V_A$ leads to instability of the waves propagating in the same direction. This conforms to a simple physical picture. If a cosmic ray interacts with a wave traveling in the same direction, it is mainly scattered in pitch angle θ by $\Delta\theta$ but gives a small amount of energy to the wave, $(V_A/c)\epsilon\Delta\theta$. For an isotropic distribution the opposite process occurs almost equally often and this effect nearly cancels. However, since the absorbing particles are slightly more numerous (having less energy) damping of the wave occurs. This gives Γ_1. If the distribution is anisotropic, the cancellation fails by the degree of the anisotropy, and the average gain per cosmic ray of the wave is $(V_A/c)(F_1/F_0)\epsilon\Delta\theta$.

c) The Effect of the Waves on Cosmic Rays

We now turn to the effect of the hydromagnetic waves on the cosmic rays. We assume that the waves of different wavelengths and frequencies are random and uncorrelated. We write:

$$E(r,t) = \int E(k,\omega) \exp\,(i k\cdot r - i\omega t) dk d\omega\,, \tag{9}$$

and the statistics of the waves are given by

$$\begin{aligned}\langle E^*(k',\omega') E(k,\omega) \rangle &= [I_a(k,\omega) k_\perp k_\perp / k_\perp^2 + I_{\mathrm{m\,s}}(k,\omega) k \times B_0 k \times B_0/(k \times B_0)^2] \\ &\quad \times \delta(\omega' - \omega)\delta(k' - k)\,.\end{aligned} \tag{10}$$

Further, for $\omega > 0$,

$$I_{\substack{\mathrm{m\,s} \\ a}} = I_{\substack{\mathrm{m.s} \\ a}}(k) \begin{Bmatrix} \delta(\omega - \omega_{k\mathrm{m.s}}) \\ \delta(\omega - \omega_{ka}) \end{Bmatrix}, \tag{11}$$

so the frequencies are peaked about the natural frequency ω_k. The angular brackets denote an average over an ensemble of such systems. Since the energy in a hydromagnetic wave is $2c^2/V_A^2$ times the electrostatic energy, the energy in the waves is

$$(c^2/V_A^2) \int_{\omega>0} d\omega \int dk \frac{(I_{\mathrm{m.s.}} + I_a)}{2\pi} = \frac{c^2}{V_A^2} \int dk \frac{I_{\mathrm{m.s.}}(k) + I_a(k)}{2\pi}\,. \tag{12}$$

We can solve the Vlasov equation to second order in E, and after ensemble averaging we can find the effect of the waves on the time development of the cosmic-ray distribution. This has already been done by Kennel and Engelmann (1966) for the non-relativistic case, and we generalize their results to the relativistic case to obtain

$$\frac{\partial F}{\partial t} + v_z \frac{\partial F}{\partial z} = \frac{\partial}{\partial p}\cdot\left(D \cdot \frac{\partial F}{\partial p} \right) + S(p,r,t)\,, \tag{13}$$

where we have kept the convective term along the lines. S is an arbitrary source, and

$$D = 2\pi q^2 \sum_{n=-\infty}^{\infty} \int d\boldsymbol{k} \int_{\omega>0} d\omega \left[I_{\mathrm{m\,s.}} J_n'^2 + I_a \frac{J_n^2 n^2}{x^2} \right] \delta(\omega - k_z v_z - n\Omega)$$
$$\times \left[\left(1 - \frac{k_z v_z}{\omega}\right) \boldsymbol{e}_\perp + \frac{k_z v_\perp}{\omega} \boldsymbol{e}_z \right] \left[\left(1 - \frac{k_z v_z}{\omega}\right) \boldsymbol{e}_\perp + \frac{k_z v_\perp}{\omega} \boldsymbol{e}_z \right], \qquad (14)$$

where $\boldsymbol{e}_\perp$ and $\boldsymbol{e}_z$ are unit vectors in the $p_\perp$- and p_z-directions. Equation (14) represents the Fokker-Planck coefficients for accumulative weak collisions off Alfvén waves. It is easy to show that equations (13) and (14) conserve the energy of cosmic rays plus waves when combined with equation (12). However, there is no H-theorem, and equation (13) is not satisfied by a distribution of cosmic rays and waves in thermal equilibrium. This results from the fact that only terms corresponding to absorption and induced emission have been kept, while the spontaneous-emission term has been dropped. In practice, however, the effective temperature of the waves is much higher than that of the cosmic rays, so the spontaneous-emission term is completely negligible. (It is of order of the Coulomb-scattering term.)

An important simplification can be made in equation (14) by dropping the 1 in comparison with $k_z v_z/\omega \approx c/V_A$. Each bracket becomes $(-k_z v/\omega)\boldsymbol{e}_\theta$, and $\boldsymbol{D}$ represents pure scattering in θ. Equation (13) then reduces to

$$\frac{\partial F}{\partial t} + v_z \frac{\partial F}{\partial z} = \pi^2 \Omega \frac{\partial}{\partial \mu} \left[\frac{(1-\mu^2)}{|\mu| B_0^2 r_L} \mathfrak{E}\left(\frac{1}{r_L \mu}\right) \frac{\partial F}{\partial \mu} \right] + S\,,$$
$$\mathfrak{E}(k_z) = \sum_{n \neq 0} \mathfrak{E}_n(nk_z) \qquad (15)$$

where

$$\mathfrak{E}_n(k_z) = \frac{1}{2\pi} \frac{c^2}{V_A^2} \int d^2 k_\perp \left[I_{\mathrm{m\,s.}} 4 J_n'^2(x) \cos^2 \zeta + I_a 4 \frac{J_n^2(x) n^2}{x^2} \right]. \qquad (16)$$

The quantities $\mathfrak{E}_n(k_z)dk_z$ thus represent the energy in waves with k_z in dk_z except for the Bessel-function form factors in $k_\perp$, as can be seen on comparison with equation (12).

If we drop terms in equation (15) with $|n| \neq 1$ (corresponding to higher cyclotron resonances), assume $\mathfrak{E}_1(k_z)$ even in k_z, and compare with the standard form of the Fokker-Planck equation, we see that the rate of pitch-angle scattering is

$$\frac{\langle \Delta\theta^2 \rangle}{t} = \frac{\pi}{2} \Omega \left. \frac{8\pi k_z \mathfrak{E}_1(k_z)}{B_0^2} \right|_{k_z = (r_L|\mu|)^{-1}}. \qquad (17)$$

This result has already been given by Jokipii (1966) assuming a time-independent magnetic field *ab initio* (our $\omega = 0$ limiting case) and in the small-gyration-radius limit. The time-dependent result can be obtained by keeping the extra terms in equation (14). (This leads to the additional term $(\pi/8)(v_A/c)^2 p^{-2} \partial[p^2 \mathfrak{E} \partial F/\partial p]/\partial p$ in eq. [15]). The finite-gyration-radius result has been included in the Bessel-function factors and the $|n| \neq 1$ terms in equations (15) and (16). However, in the rest of this paper we essentially restrict ourselves to his case.

If we understand $k_z \mathfrak{E}$ as representing the energy "at k_z," then we can say that the rate of pitch-angle scattering of a cosmic ray is smaller than its cyclotron frequency by roughly the ratio of the energy in waves at its cyclotron radius to the magnetic energy. One can arrive at this result by a simple physical argument. If we regard all the turbulence at the cyclotron radius as being lumped into one wavelength, the magnetic field makes an angle $\Delta\theta$ with $\boldsymbol{e}_z$, with $(\Delta\theta)^2$ of the order of the above ratio. If we regard different wavelengths as uncorrelated and note that the particle encounters Ω waves per second and that it is scattered in pitch angle by $\Delta\theta$ per wavelength, we arrive at the above result.

We see from equation (16) that the wave energies enter into equation (15) with a weight depending on $k_\perp$. However, from equation (4) we see that waves with different $k_\perp$ have different growth rates, the rate being faster for smaller $k_\perp$. We thus make the simplification throughout the rest of this paper that the weighted-average equation (16) has the growth rate given by the $k_\perp = 0$ mode. This is accurate if the wave amplitude decreases rapidly with increasing $k_\perp$, as one might expect.

III. APPLICATION TO GALACTIC COSMIC RAYS

a) Introduction

We now wish to apply the results of § II to galactic cosmic rays. We should like to arrive at a diffusion theory similar to that of Morrison *et al.* (1954), but for diffusion along the magnetic lines of force. For a model of the cosmic rays we assume that they are confined to a tube of force of the galactic magnetic field and along this tube of force equation (15) applies. We assume this tube of force has a finite length $2L$, after which it leaves the galactic disk and goes into the halo, and the cosmic rays are lost. From observations (Serkowski 1962; Spitzer 1968) one knows that the field is roughly parallel to the arm and closely parallel to the plane of the disk.

The mean angle the field makes with the plane of the Galaxy, α, is less than 5° and could be 0°. If α is 5° and we take 100 pc for the half-thickness of the Galaxy, we would have $L = 1.1$ kpc. On the other hand, if α is very small, we would hardly expect L to be larger than 10 kpc, so the value of L is quite uncertain. For the purpose of numerical investigations in our model we set

$$L = 3 \text{ kpc} . \tag{18}$$

The mean value of B is also uncertain, but for our numerical examples we take B_μ, the field in microgauss, to be

$$B_\mu = 3 \text{ microgauss} . \tag{19}$$

The Alfvén speed for the shorter wavelengths is that of the charged component alone (see Appendix C). It is generally accepted (Spitzer 1968) that the main ionized constituent is carbon, with a relative abundance to hydrogen by number of 5×10^{-4}. If ρ^* is the density of the charged component and ρ of the neutral component,

$$\rho^*/\rho = 6 \times 10^{-3} . \tag{20}$$

This value applies in the clouds. In between, the ionized component may be more massive because of the ionization of hydrogen by low-energy cosmic rays (Spitzer 1968).

We take the mean hydrogen density to be $n_{\text{H}} = 1$ hydrogen atom cm^{-3}. These values lead to

$$V_A = 28 B_\mu \text{ km sec}^{-1} = 85 \text{ km sec}^{-1} . \tag{21}$$

The gyration radius of relativistic cosmic rays is

$$r_L = \frac{c}{\Omega} = 3.2 \frac{\gamma}{B_\mu} \times 10^{12} \text{ cm} = 10^{12}\gamma \text{ cm} , \tag{22}$$

where $\gamma = \epsilon/m_{\text{H}}c^2$. We restrict ourselves to only the proton component of cosmic rays. The cosmic rays behave differently at different energies. The mean energy of the relativistic cosmic rays is 5 BeV kinetic energy (Ginzburg and Syrovatskii 1964), and for our numerical examples we take cosmic rays with $\gamma = 10$. Thus we are primarily interested in turbulent wavelengths of 10^{13} cm $\approx$ 1 a.u.

In this section we first reduce the Fokker-Planck equation of the last section to an equation of spatial diffusion whose diffusion coefficient depends on the wave energy density. Then we apply this diffusion equation to our model for galactic cosmic rays, deriving a mean free path by comparison with the observed age. Next we investigate the stability of hydromagnetic waves, computing the growth rate produced by cosmic rays and comparing with the damping rate produced by friction with the neutral gas.

Next we attempt to develop our model self-consistently. We assume a given source of cosmic rays and of hydromagnetic turbulence. Then we solve for the actual turbulence and cosmic-ray energy density. We investigate the non-linear dependence of the density of cosmic rays on source strengths and show that there are natural units for these strengths such that weaker sources lead to one behavior and stronger sources to another. We investigate such things as age, rates of hydromagnetic wave dissipation, and heating and cooling of the cosmic rays, on the basis of this model. Finally, we extend the model to take into account the observed non-uniform distribution of interstellar matter in clouds and the heating effect of large-scale magnetic-field inhomogeneities on the cosmic rays.

b) The Diffusion Equation

From the estimated age of the cosmic rays one knows that they must not stream freely along the disk or else they would be lost in a time of the order of 10000 years. Thus, they must be strongly diffused, and we take the diffusion term on the right-hand side as the dominant one.

We write equation (15) as

$$\frac{\partial F}{\partial t} + \mu v \frac{\partial F}{\partial z} = \frac{\partial}{\partial \mu}\left[\frac{(1-\mu^2)}{2}\nu(\mu)\frac{\partial F}{\partial \mu}\right] + S(\epsilon,z) , \tag{23}$$

where

$$\nu(\mu) = \frac{\pi}{4}\Omega \frac{8\pi}{|\mu| r_L B_0^2}\mathfrak{E}[(r_L|\mu|)^{-1}] . \tag{24}$$

Since ϵ is constant, we may regard it as a parameter in equation (23). We regard ν as a large quantity and expand in $1/\nu$:

$$F = F_0 + F_1 + F_2 + \ldots . \tag{25}$$

To lowest order F_0 is independent of μ:

$$F_0 = F_0(z) . \tag{26}$$

Integrating the first-order equation with respect to μ and dividing by ν, we have

$$\frac{\partial F_1}{\partial \mu} = -\frac{v}{\nu}\frac{\partial F_0}{\partial z} , \tag{27}$$

and to second order,

$$\frac{\partial F_0}{\partial t} + \mu v \frac{\partial F_1}{\partial z} = \frac{\partial}{\partial \mu}\left[\frac{(1-\mu^2)}{2}\nu \frac{\partial F_2}{\partial \mu}\right] + S . \tag{28}$$

We get a condition that one can solve for F_2 by averaging this equation in μ:

$$\frac{\partial F_0}{\partial t} + \left\langle \mu v \frac{\partial F_1}{\partial z}\right\rangle = S . \tag{29}$$

But

$$\left\langle \mu v \frac{\partial F_1}{\partial z} \right\rangle = -\tfrac{1}{2} \frac{\partial}{\partial z} v \left\langle (\mu^2 - 1) \frac{\partial F_1}{\partial \mu} \right\rangle = \tfrac{1}{2} \frac{\partial}{\partial z} v^2 \left\langle \frac{\mu^2 - 1}{\nu} \frac{\partial F_0}{\partial z} \right\rangle , \tag{30}$$

so finally

$$\frac{\partial F_0}{\partial t} = \frac{\partial}{\partial z}\left(D \frac{\partial F_0}{\partial z} \right) + S , \tag{31}$$

$$D = \frac{v\lambda}{3} = v^2 \int_0^1 \frac{1 - \mu^2}{2\nu} d\mu = \frac{v^3 B_0^2}{4\pi^2 \Omega^2} \int \frac{(\mu - \mu^3) d\mu}{\mathfrak{E}[(r_L \mu)^{-1}]} = \frac{c^3 B_0^2}{16\pi^2 \beta \mathfrak{E}(r_L^{-1})\Omega^2} . \tag{32}$$

Jokipii (1966, 1968) has also derived an expression for a diffusion coefficient which differs from equation (32). The difference arises because Jokipii assumed that F_1 was proportional to μ. This clearly does not satisfy equation (27) unless ν is independent of μ. Our expression is generally valid for any $\nu(\mu)$. An important difference between his diffusion coefficient and ours is that ours has a singularity at $\mu = 0$ which his expression does not exhibit. This singularity might be expected on physical grounds. It is discussed below. The quantity λ is the effective mean free path, and

$$\beta = \left[4 \int_0^1 \frac{(\mu - \mu^3) d\mu}{\mathfrak{E}[(r_L \mu)^{-1}]} \right]^{-1} \Big/ \mathfrak{E}(r_L^{-1}) . \tag{33}$$

One does not know $\mathfrak{E}(k)$ without solving a self-consistent problem. However to get an idea of β, we assume $\mathfrak{E} \sim k^{-r} \sim \mu^r$. Then β is given in Table 1.

TABLE 1

DEPENDENCE OF β ON r

r	β
0	1
1	3/8
1 5	1/6 4
2	0

We see from this table that for $r > 2$, β is zero. Physically this means that the scattering past the angle $\theta = 90°$ is done by very short wavelengths. If there is no energy in the short wavelengths, then the particles cannot reverse v_z, and the diffusion is infinite. But particles with θ near 90° are very easily mirror-scattered, so that a cutoff should occur for μ near zero in equation (33), and β actually is not zero. On the other hand, if $r \leq 1$, the energy in short wavelengths diverges. The value of β is a major uncertainty in our diffusion theory. For this paper we take the value corresponding to $r = 1.5$,

$$\beta = 1/6.4 .$$

Since the age of cosmic rays is also of interest, one can define F' depending on the age τ, such that F' is the number of cosmic rays in the age range $d\tau$. Equation (23) then becomes

$$\frac{\partial F'}{\partial t} + \frac{\partial F'}{\partial \tau} + \mu v \frac{\partial F'}{\partial z} = \frac{\partial}{\partial \mu}\left[(1 - \mu^2) \frac{\nu}{2} \frac{\partial F'}{\partial \mu} \right] + S\delta(\tau) . \tag{34}$$

Then setting

$$G = \int_0^\infty F' \tau d\tau , \quad F = \int_0^\infty F' d\tau , \tag{35}$$

we see $\langle \tau \rangle = G/F$ is the mean age. One obtains an equation for G by multiplying equation (34) by τ and integrating over τ from 0 to ∞,

$$\frac{\partial G}{\partial t} + \mu v \frac{\partial G}{\partial z} = \frac{\partial}{\partial \mu}\left[\frac{(1-\mu^2)}{2} \nu \frac{\partial G}{\partial \mu}\right] + F , \tag{36}$$

which is identical with equation (23) with S replaced by F, and thus by the same procedure we obtain

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial z}\left(D \frac{\partial G}{\partial z}\right) + F . \tag{37}$$

c) *Cosmic Rays in the Galaxy*

To apply our diffusion equation to cosmic rays in the Galaxy, we need to know D (or $\mathfrak{E}$) and S as a function of z. For a first comparison, let us take D and S constant. The steady-state solution to equation (31) for constant D and S and for loss out the ends at $z = \pm L$ is

$$F_0 = \frac{S}{2D}(L^2 - z^2) , \tag{38}$$

$$G = \frac{F_0}{12D}(5L^2 - z^2) , \tag{39}$$

so the average age is

$$\langle \tau \rangle = G/F = \frac{L^2}{12D}\left(5 - \frac{z^2}{L^2}\right) . \tag{40}$$

For the anisotropy δ we have, from equations (25) and (27),

$$\delta = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = -\frac{\lambda}{F_0}\frac{\partial F_0}{\partial z} = \frac{2\lambda z}{L^2 - z^2} . \tag{41}$$

The known parameters are δ and $\langle \tau \rangle$ at the Sun, and $\langle \tau \rangle$ is given in terms of Σ, the amount of mass per unit area through which the cosmic rays have passed. Σ is in the range 3–7 g cm^{-2} (Ginzburg and Syrovatskii 1964). We take

$$\Sigma = \rho v \langle \tau \rangle = 7 \text{ g cm}^{-2} . \tag{42}$$

The unknowns are z, the distance of the Sun from the center of its line of force, and λ, the mean free path. Since the Sun is close to the galactic plane, we assume $z \ll L$. We can solve for λ and L:

$$\begin{aligned} \lambda &= \tfrac{5}{4}\frac{L^2}{\Sigma}\rho = 0.9 L_{\text{kpc}}^2 \text{ pc} = 8 \text{ pc} , \\ z &= \tfrac{2}{5}\frac{\delta \Sigma}{\rho} = 550\left(\frac{\delta}{10^{-3}}\right) \text{pc} . \end{aligned} \tag{43}$$

We can get an idea of the wave energy by combining equations (24) and (32). If we take $k\mathfrak{E}$ as an index of the energy in the range k,

$$k\mathfrak{E} = \frac{3cB_0^2}{16\pi^2\Omega\beta\lambda}$$
$$= \frac{1.9 \times 10^{-20}\gamma B_\mu}{\lambda_{pc}\beta} \text{ erg cm}^{-3} = 4.8 \times 10^{-19} \text{ erg cm}^{-3} . \quad (44)$$

The assumption has been made that the magnetic field is essentially stationary over the age of the cosmic rays. If the lines of force move out of the Galaxy as a whole, carrying the cosmic rays with them, then the lifetimes could be much longer, λ could be decreased and $\mathfrak{E}$ substantially increased.

Let us compare our simple model with the stability criteria derived in § II. Γ is given by equation (4). Let us replace the Bessel-function factors by their values for $k_\perp = 0$. Then the growth rate depends on the anisotropy $\partial F/\partial\mu$, which is given in terms of $\mathfrak{E}$ by equations (25) and (27). It is easily seen that the argument of $\mathfrak{E}$ is unchanged under the integration in equation (4), and, further, if F_0 is given by the power law in equation (7), then the calculation reduces to the same case as that in equation (8), with $q = m - 1$ and $F_1 = -\lambda\partial F_0/\partial z$. If we use equation (38) to express $\partial F_0/\partial z$ in terms of F_0, we find at $z = L/2$

$$\Gamma_{\text{c.r.}} = -\Gamma_1 + \Gamma_2 = \pi^2 \frac{m-3}{m-2}\frac{\rho}{\rho^*}\Omega_0\frac{\Phi(>\epsilon_1)}{n_{\text{H}}c}$$
$$\times\left(-1 + \frac{m-2}{(m-1)(m-3)}\frac{8}{3}\beta\lambda\frac{c}{V_A}\frac{1}{F_0}\frac{\partial F_0}{\partial z}\right)$$
$$= 0.9 \times 10^{-12}\rho/\rho^* B_\mu\gamma_1^{-1.4}\left(-1 + 1.7\beta\frac{\lambda}{L}\frac{c}{V_A}\right) \text{ sec}^{-1} \quad (45)$$
$$= 1.8 \times 10^{-11}\{-1 + 2.5\} = 2.7 \times 10^{-11} \text{ sec}^{-1} ,$$

where we have taken Φ from Table 4 (p. 36) of Ginzburg and Syrovatskii (1964) (we choose their case $\gamma - 1 = 1.4$, $m = 4.4$), and $\lambda_{pc} = 8$. If this exceeds the damping rate of the waves, then these waves will grow with an e-folding rate faster than once every 1000 years. This will lead to an increased wave amplitude and thus ν will increase, leading to a longer life than is observed for the cosmic rays. On the other hand, if the damping rate is much higher than this, then one must find a source of energy for the turbulence greater than 2.6×10^{-29} erg cm^{-3} sec^{-1}. This is to be compared with the energy lost by the cosmic rays (greater than 10 BeV) of 3×10^{-27} erg cm^{-3} sec^{-1}, assuming a lifetime of 4.5 million years. Thus the power involved in turbulence is much smaller than the power involved in supplying energy to the cosmic rays.

We now consider the damping rate of the Alfvén waves. It is shown in Appendix C that

$$\Gamma^* = \tfrac{1}{2}\nu_0 = \tfrac{1}{2}n_{\text{H}}\langle\sigma v_{\text{H}}\rangle\frac{m_{\text{H}}}{m_c} , \quad (46)$$

where σ is the cross-section for scattering and v_{H} is the thermal velocity of the hydrogen atoms ($m_{\text{H}}/m_c = 1/12$). Taking $\sigma = 15 \times 10^{-15}$ cm^2, $v_{\text{H}} = 1.4 \times 10^5$ cm sec^{-1} (for $T = 100°$) from Spitzer (1968) and $n_{\text{H}} = 1$ cm^{-3}, we get

$$\Gamma^* = 0.9 \times 10^{-10} \text{ sec}^{-1} , \quad (47)$$

a value in order-of-magnitude agreement with $\Gamma_{\text{c.r.}}$.

If we consider the uncertainties associated with the values of the physical quantities in equations (45) and (46), as well as the crudeness of our model, we cannot decide from the numbers whether the waves in the Galaxy are unstable or not.

What would happen if the waves were unstable? The result would be that $\mathfrak{E}$ would grow, and from equations (24) and (32) ν would increase and D and λ would decrease. Because of the large value of $\nu \approx 30\ \mathrm{yr}^{-1}$, equation (27) would still be valid, and from equation (45) Γ would decrease, becoming stable. (We assume that F in equation [45] does not change in the time for $\mathfrak{E}$ to change since the time constant for F changing from equation [31] is $S/F \sim \langle\tau\rangle \sim 10^6$ yr and is much longer than $\Gamma_{\mathrm{c.r.}}^{-1}$, so $\mathfrak{E}$ changes much faster than F.) Similarly, if $\Gamma_{\mathrm{c.r.}} < \Gamma^*$, $\mathfrak{E}$ decreases, leading to an increase in λ and $\Gamma_{\mathrm{c.r.}}$ Thus, we expect the amplitude of the waves to stabilize at a level which makes $\Gamma_{\mathrm{c.r.}} \sim \Gamma^*$, so the agreement of $\Gamma_{\mathrm{c.r.}}$ and Γ^* for the Galaxy is not merely fortuitous. The time to stabilize is of order Γ_1^{-1}, as a simple linearized calculation shows.

d) *A Self-Consistent Model*

In the first part of this section we have used the observed properties of galactic cosmic rays to determine the diffusive properties of the interstellar medium. Let us now take a more general point of view and develop a model which will apply to other objects containing high-energy particles as well.

We assume that specific sources of cosmic rays S are specified, as well as sources of hydromagnetic turbulence T. We assume also that our model of a uniform tube of force of length $2L$ containing a very partially ionized gas still applies. Then we expect the energy density of the turbulence, as well as the cosmic-ray density, to be uniquely determined. The level of the turbulence is given by the damping of the waves, which is controlled by the cosmic rays themselves. On the other hand, the density of the cosmic rays depends on the waves through the diffusion coefficient. Thus, we should have a self-consistent model.

The full self-consistent model should lead to an equation of non-linear diffusion in energy and z. We simplify this problem by ignoring diffusion in energy and applying our model to one point in space, $z = L/2$, and one energy, $\gamma = 10$. We take into account the particles at other energies by assuming a power-law spectrum, the waves at other wavelengths by introducing the proper mean value through the parameter β, and the behavior at other z's by our homogeneous-solution equation (38). This simplification of the self-consistent model, while quite crude, enables us to get an idea of the full self-consistent problem.

The relevant equations are as follows: assume that we know $\mathfrak{E}$ and F at some time. Equation (32) gives $D = v\lambda/3$. Equation (31) then shows how F changes in time, while the equation for $\mathfrak{E}$ is

$$\frac{\partial \mathfrak{E}}{\partial t} = 2(\Gamma_{\mathrm{c.r.}} - \Gamma^*)\mathfrak{E} + T = 2(\Gamma_2 - \Gamma_1 - \Gamma^*)\mathfrak{E} + T\,, \tag{48}$$

where $\Gamma_{\mathrm{c.r.}}$ is given by equation (45) and Γ^* by equation (46). Equations (27) and (32) are valid for a time-dependent system, since the time constant in equation (27), ν^{-1}, is short compared to the time constant in equation (31), $L^2/\lambda v$. Since T and $\mathfrak{E}$ refer to energy density per unit wavenumber (somewhat unfamiliar units since k is so small), we give our results in terms of $k\mathfrak{E}$ and kT which are roughly the total energy density in a band with $\Delta k \sim k$ about k. We speak of these quantities as total energy densities at k. Then from equation (32) we have

$$\lambda = r_L/u\,, \tag{49}$$

where

$$u = \frac{16\pi^2\beta}{3B_0^2}\,k\mathfrak{E}\,. \tag{50}$$

If we assume a steady state for F in equation (48), we have, from equations (32), (45), and (48)–(50),

$$u = \frac{(8\pi^2\beta kT/3B_0^2) + (\alpha'\beta\gamma\epsilon N/12n^*)(c^2/LV_A)}{\Gamma^* + \alpha(\epsilon N/12n^*)\Omega_0}\,, \tag{51}$$

where N is the number density of cosmic rays per unit energy,

$$\alpha = \frac{\pi}{4(m-2)}\,, \quad \alpha' = \frac{32}{9}\frac{m-2}{(m-1)(m-3)}\alpha\,, \tag{52}$$

and m is the exponent in equation (7). We have expressed $\Phi(>\epsilon)$ in terms of $N(\epsilon)$ by equation (7).

If we again use equation (38) (assuming a steady F), we have at $z = L/2$

$$\epsilon N = \tfrac{9}{8}\frac{\epsilon S' L^2}{\lambda v}\,, \tag{53}$$

where S' is the source of N, i.e., $S' d\epsilon = 4\pi p^2 S dp$. We may solve equations (49) and (51)–(53) for N. We obtain

$$\tfrac{8}{9}\frac{v r_L N}{L^2 S'} = \frac{8\pi^2\beta kT/3B_0^2 + \alpha'\beta c^2\gamma\epsilon N/(12n^* L V_A)}{\Gamma^* + \alpha\Omega_0(\epsilon N/12n^*)}\,, \tag{54}$$

a quadratic equation for N. We imagine a fixed model with variable sources of turbulence T and cosmic rays S, and we wish to find out how N varies with T and S. This is most easily done by expressing S, T, and N in terms of standard values characteristic of the model. Set

$$\epsilon N_0 = \frac{12n^*\Gamma^*}{\alpha\Omega_0} = \tfrac{1}{2}\frac{nn^*}{\alpha\Omega_0}\langle\sigma v_{\rm H}\rangle = 1.7 \times 10^{-10}\frac{n_{\rm H}^2\delta}{B_\mu}\ {\rm cm^{-3}}\,.$$

$$\epsilon S_0 = \frac{8}{9}\frac{\alpha}{\alpha'}\frac{V_A}{L}\frac{\epsilon N_0}{\beta} = 8.0 \times 10^{-26}\frac{n_{\rm H}^{3/2}\delta^{1/2}}{\beta L_{\rm kpc}}\ {\rm cm^{-3}\ sec^{-1}} \tag{55}$$

$$kT_0 = \frac{3}{8\pi^2}\frac{\alpha'}{\alpha}\Gamma^* B_0^2\frac{r_L}{L}\frac{c}{V_A} = 6.6 \times 10^{-29}\frac{\gamma n_{\rm H}^{3/2}\delta^{1/2}}{L_{\rm kpc}}\ {\rm ergs\ cm^{-3}\ sec^{-1}}\,,$$

where in the numerical expressions we assume all the charged particles are carbon ($m^*/m = 12$) and $n^* = \delta 5 \times 10^{-4}\, n_{\rm H}$, so $\delta = 1$ corresponds to the galactic value. If we set

$$N = N_0 f\,, \quad S' = S_0 s\,, \quad T = T_0 t\,, \tag{55a}$$

equation (54) becomes

$$\frac{f}{s} = \frac{t+f}{1+f}\,, \tag{56}$$

or

$$f^2 + (1-s)f - st = 0\,, \tag{56a}$$

a quadratic equation for f. Thus, N_0, S_0, and T_0 are natural units for the problem.

Let us assume t fixed and ask how f varies with s. Equation (56) represents a hyperbola in the (s,f)-plane, passing through the origin with slope t and having asymptotes $f = -1$ and $f = s + (t - 1)$. It is sketched in Figure 1 for the two cases $t \ll 1$ and $t \gg 1$. Let us take the sources of turbulence to be weak ($t \ll 1$). Then for a very weak source of cosmic rays

$$f = st(s \ll 1) . \tag{57}$$

By comparing with equation (56), we see that both f-terms on the right-hand side are negligible, which corresponds to the cosmic rays not contributing to the turbulence. The turbulence is given by balancing the source against the damping Γ^*. This gives λ and thus the diffusion. (This behavior is given by the straight-line portion near the origin

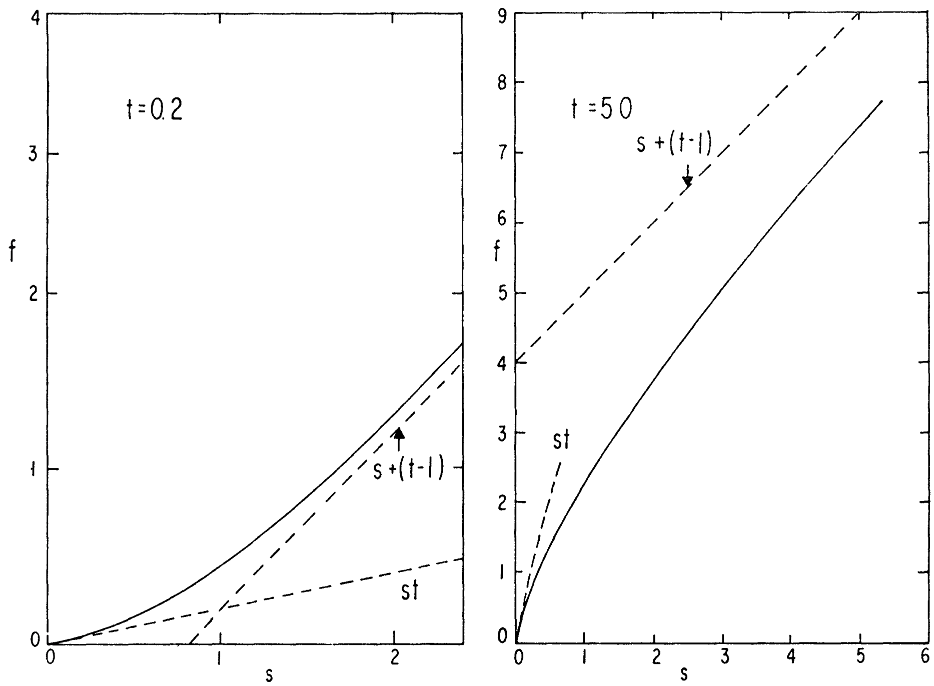

FIG. 1.—Normalized cosmic-ray density f versus normalized source strength s for the case of weak and strong turbulent sources: $t = 0.2$ and $t = 5$.

of the hyperbola in Fig. 1.) It can be shown that

$$\frac{\Gamma_2}{\Gamma^*} = s , \quad \frac{\Gamma_1}{\Gamma^*} = f , \tag{58}$$

so $s < f$ corresponds to stable waves. As s increases toward 1, Γ_2 approaches Γ^*. The waves are enhanced, since the cosmic rays reduce the effective damping. This decreases λ, increasing the confinement and thus the number of cosmic rays. This is represented by the increasing slope of the hyperbola in Figure 1. For $s \sim 1$,

$$f \approx \frac{st}{|1 - s|} (s \sim 1, 1 - s > \surd t) , \tag{59}$$

and the slope increases rapidly until $f \sim 1$. In this region f becomes independent of t. For this regime we have $\Gamma_2 \approx \Gamma^*$, $\Gamma_1 < \Gamma^*$, and the waves are marginally stable at a level which keeps them marginally stable. As s increases further Γ_1 becomes larger than Γ^*, and Γ_1 and Γ_2 must balance, giving

$$f = s - 1 , \quad s \gg 1 . \tag{60}$$

Thus, in summary, we have three regimes in the weak-turbulence case:

$$\begin{aligned} &a) \text{ the weak-source regime,} \\ &b) \text{ the intermediate-source regime,} \\ &c) \text{ the strong-source regime.} \end{aligned} \tag{61}$$

For (a), the waves are damped; for (b) and (c), they are marginally stable, and the density and turbulent energy must adjust to keep them marginally stable. For the case of a large turbulent source $t \gg 1$, one again gets three regimes:

$$\begin{aligned} &a) \text{ weak: } f = st\,, \quad s \ll 1/t\,, \\ &b) \text{ intermediate: } f = \sqrt{(st)}\,, \quad 1/t \ll s \ll t\,, \\ &c) \text{ strong: } f = s + t - 1\,, \quad t \ll s\,. \end{aligned} \tag{62}$$

From equation (58) one sees that $\Gamma_1 \lesseqgtr \Gamma_2$ according to whether $f \gtrless s$, so if $f < s$, the cosmic rays reduce the damping and increase the turbulence. Being better confined, their density is increased. Thus the curve below $f = s$ is convex upward. The curve above is convex downward since cosmic rays enhance the damping.

Two other quantities of interest for our model are the age of the cosmic rays and the rate of dissipation of wave energy (which goes to heat the interstellar hydrogen). From equations (40) and (49)–(55),

$$\langle \tau \rangle = \tfrac{5}{4}\tfrac{8}{9}\frac{N_0 f}{S_0\, s} = \tau_0 \frac{f}{s}\,, \tag{63}$$

where τ_0 is the standard unit of time,

$$\tau_0 = \tfrac{10}{9}\frac{N_0}{S_0} = 7.8 \times 10^7 n_{\rm H}^{1/2} L_{\rm kpc} \delta^{1/2} \beta B_\mu^{-1} \text{ yr}\,. \tag{64}$$

For the dissipation of turbulent energy, we find from the same equations

$$2\Gamma^* k\mathfrak{E} = \frac{f}{s} kT_0\,, \tag{65}$$

so T_0 is the standard unit for dissipation of energy into the interstellar medium as well as for the source of turbulent energy. It is seen from equation (57) that for a weak source the energy dissipated equals the source of turbulent energy, tT_0, while for a strong source it approaches T_0. (In no case can it exceed the maximum of T_0 and tT_0.) Finally, this should be compared with the total rate of energy loss of the cosmic rays leaving the Galaxy:

$$\frac{2\Gamma^* k\mathfrak{E}}{\gamma m_{\rm H} c^2 S_0} = \frac{27a}{16\pi}\left(\frac{a'}{a}\right)^2 \frac{f}{s^2} = 0.56\beta \frac{f}{s^2}\,. \tag{66}$$

The efficiency of conversion of cosmic-ray energy into turbulence energy is thus

$$\frac{\Delta\epsilon}{\epsilon} = 0.56\beta \frac{f - st}{s^2} = 0.56\beta \frac{sf - f^2}{s^2} < \tfrac{1}{4}(0.56)\beta \tag{67}$$

(using equation [56a]), where $\Delta\epsilon$ is the total decrease in energy ϵ during the life of the particle. If $f < s$, the cosmic rays are cooled, as they should be since from the above argument they are enhancing the waves. Since $\beta < 1$, they are never cooled by more than 15 per cent of their total energy. For $f > s$, they are heated, part of the turbulent energy provided by T going into them and the rest into heating the interstellar gas. The fraction of T going into cosmic-ray heating is, from equation (65),

$$\frac{\epsilon S' \Delta\epsilon}{kT} = \left(1 - \frac{f}{st}\right), \quad (f > s), \tag{68}$$

and can be very close to one for s small.

Although our treatment of the self-consistent model is quite rough, it makes one thing clear. The relation between the source S' and the cosmic-ray density is not necessarily a linear one, and is different in different regimes, i.e., a weak or a strong source of turbulence. Further, as can be seen from equation (55), T_0 depends on γ, so one can be in different regimes at different energies. Thus, if s falls off faster than T/γ, one may move from a strong-source regime to a weak-source regime, while if T falls off slower than $\gamma \sim k^{-1}$, one may move from a weak-turbulence regime $t < 1$ to a strong-turbulence regime in which the behavior of the cosmic rays (with respect to their confinement) is different. Thus a power-law source of both cosmic rays and turbulence need not lead to a power-law energy spectrum of cosmic rays, but it will if the regime is the same and is linear. Thus, on the basis of a more careful treatment one must exercise some care in interpreting the source energy spectrum from the observed spectrum.

It should be noted that the numerical values given in equation (55) are not accurate at very high energies (long wavelengths) since equation (46) was used for Γ^* and this equation breaks down for large wavelengths (see Appendix C). Γ^* decreases, and the values in equation (55) should be decreased by a ratio of Γ^* to the value given in equation (46). This tends to make the effective source s stronger. It is possible that this effect, together with a naturally decreasing s, could explain the change in slope of the cosmic-ray curve between 10^{15} and 10^{18} eV. For example, assume the source to be a power-law spectrum which was strong at low energies and weak at higher energies. Then if the turbulence source was weak and also had a power law, we see from equations (60) and (57) that at low energies $f = s$, at intermediate energies $f = st$, and at high energies $f = s$ again. This would correspond to the observation that at low and high energies the cosmic-ray spectrum has the same power law, while in the intermediate range it is steeper.

In what regime are the galactic cosmic rays in the neighborhood of the Sun? This is difficult to judge because we have no direct evidence of the size of T and S and because the values of other physical parameters are uncertain. However, if we assume $B_\mu = 3$, $n_{\rm H} = \delta = 1$, $\beta = 1/6.4$, and $L_{\rm kpc} = 3$, we get the standard values

$$\begin{aligned} (\epsilon N_0)_{\rm Gal} &= 5.7 \times 10^{-11}\ {\rm cm^{-3}}, \quad (\epsilon S_0)_{\rm Gal} = 1.7 \times 10^{-25}\ {\rm cm^{-3}\ sec^{-1}}, \\ (kT_0)_{\rm Gal} &= 2.2 \times 10^{-28}\ {\rm ergs\ cm^{-3}\ sec^{-1}}, \quad (\tau_0)_{\rm Gal} = 1.2 \times 10^{7}\ {\rm yr}. \end{aligned} \tag{69}$$

Then assuming $\Sigma = 7$ g cm^{-2}, $\epsilon = 10$ BeV, we get from equations (55a), (63), and (56):

$$f = 0.20, \quad f/s = \langle\tau\rangle/\tau_0 = 0.37, \quad s = 0.52, \quad t = 0.25. \tag{70}$$

e) An Inhomogeneous Model

So far we have assumed S' and $n_{\rm H}$, as well as other parameters, constant in z. We now take into account the granular nature of the interstellar matter, in our self-consistent model. We ignore the H II regions of ionized hydrogen. Interstellar matter in H I regions is concentrated in clouds with densities roughly 10 times the mean density and occupying 10 per cent of the volume. In between the clouds we take the density one-tenth the mean density. In our model we distinguish two regions: A, the clouds, which we assume distributed densely in z; and B, the rest. We set $m_{\rm A} = \mu_{\rm A} n_{\rm H}$ in A and $n_{\rm B} = \mu_{\rm B} n_{\rm H}$ in B. We assume that all other physical parameters of the model, B, $n^*/n_{\rm H}$, S, etc., are constant.

We must now take into account the local variation in z of F, at least distinguishing between regions A and B. (We expect heavy damping of waves in the clouds so that $\mathfrak{E}$ is small and λ large there.) If we set

$$\frac{\partial F_0}{\partial z} = -\frac{1}{\ell(z)} F_0 , \tag{71}$$

where we expect l to be quite different in the two regions, we can again solve for F_1 from equation (27) and substitute the result in equation (8) to find Γ_1 and Γ_2. Equation (45) is now modified by replacing L by l. Equations (49) and (51) are still valid, with λ and μ depending on z and L replaced by l. However, equation (53) is not valid, since it was derived on the assumption of constant λ. Instead, we integrate equation (31) in z to obtain

$$zS = -D\frac{\partial F_0}{\partial z} = \frac{c\lambda}{3}\frac{F_0}{\ell} . \tag{72}$$

Now by combining equations (49)–(51) and taking the standard quantities in equation (55) relative to region B and setting $\mu = n/n_{\rm B}$, we find that equation (56) is replaced by

$$\frac{3}{8}\frac{L^2}{\ell z}\frac{f}{s} = \frac{t + \mu^{-3/2}(\frac{3}{4}L/\ell)f}{\mu + \mu^{-1}f} . \tag{73}$$

We can solve equation (73) for l/L and find

$$l/L = \tfrac{3}{4}\frac{f}{st}\left(\frac{L\mu}{2z} + \frac{Lf\mu^{-1}}{2z} - s\mu^{-1/2}\right) . \tag{74}$$

This is to be combined with equation (72). If we average equation (72) over several regions A and B (still for a small range in z), we find

$$\frac{df}{dz} = \frac{f}{\ell_{\rm B}}\left(\tfrac{9}{10} + \tfrac{1}{10}\frac{\ell_{\rm B}}{\ell_{\rm A}}\right) = \frac{f}{\langle \ell \rangle} . \tag{75}$$

Thus, if $\ell_{\rm B} < 10\,\ell_{\rm A}$, we have $\langle \ell \rangle = \ell_{\rm B}$, the regions A can be completely ignored without serious error, and the diffusion problem can be solved completely in region B. In this case $\ell_{\rm B}/L = \frac{3}{4}$, and, since f, t, and s are the same in A and B, the condition $\ell_{\rm B} < 10\,\ell_{\rm A}$ becomes at $z = L/2$:

$$\frac{\ell_{\rm A}}{L} = \frac{f}{st}\left(\mu_{\rm A} + \frac{f}{\mu_{\rm A}} - \frac{s}{\mu_{\rm A}{}^{1/2}}\right) \gg \tfrac{1}{10} . \tag{76}$$

Since $\mu_{\rm A} \approx 100$, this condition is satisfied unless $s \approx 1000$ or $f/st \ll 10^{-3}$. Neither case occurs in the Galaxy, so we have the result: *We can ignore the interstellar clouds completely and study the cosmic rays in a uniform medium with the reduced density equal to that in between the clouds.* This result follows physically from the fact that the damping of waves is sufficiently strong in the clouds that the mean free path is long. Combining this with the fact that the clouds occupy a small volume, we see that the cosmic-ray density should be essentially the same on each side of the cloud.

In order to compare with galactic values, we have to inquire how the age is changed in the inhomogeneous model. It is easy to derive the analogous equation to equation (36) for inhomogeneous densities, diffusion coefficients, and sources, and to solve it by an integral. One finds that, as far as the aging is concerned, equation (42) holds with ρ the mean density, so our estimate for τ is still valid. However, equations (63) and (64) are valid when the standard quantities are referred to region B alone. Thus, if we take $n_B/\langle n\rangle = \frac{1}{10}$, N_0 is multiplied by 10^{-2}, S_0 by $10^{-1.5}$, and τ by $10^{-0.5}$. Therefore, new values of f, s, and t are 20, 17, and 4.5, respectively, although the last is quite uncertain. Using these values, one finds from equation (76) $\ell_A/\ell_B = 28$, verifying our conclusion that region B is the important region for diffusion.

f) Acceleration of the Cosmic Rays

So far in our model we have assumed that the magnetic field is uniform with only fluctuations due to small-scale turbulence. (We have neglected the $n = 0$ modes in equation [14.]) We now consider the effect of larger-scale fluctuations. We can treat these most simply by a non-diffusion theory; that is, we do not assume complete randomness of the larger-scale fluctuations. Since for the larger-scale fluctuations the gyration radius of the cosmic rays is small, we can use the conventional guiding-center equations for these effects. One rewrites equation (23) with these guiding-center terms on the left-hand side in the ultrarelativistic limit:

$$\frac{\partial F}{\partial t} + \mu v \frac{\partial F}{\partial z} - \frac{(1-\mu^2)}{2}\left(v\frac{\partial \ln B}{\partial z} + \mu\frac{\partial \ln B}{\partial t}\right)\frac{\partial F}{\partial \mu} + p\frac{(1-\mu^2)}{2}\frac{\partial \ln B}{\partial t}\frac{\partial F}{\partial p} = \frac{\partial}{\partial \mu}\left[\frac{(1-\mu^1)}{2}\nu\frac{\partial F}{\partial \mu}\right] + S\,. \quad (77)$$

The two additional terms on the left-hand side represent magnetic mirror scattering off moving large-scale inhomogeneities (Fermi acceleration) and the betatron effect in increasing and decreasing magnetic fields. Moreover, the small-scale scattering produces just the "collisions" necessary to make Fermi acceleration and magnetic pumping work. To get an idea of the magnitude of the acceleration, let us again pass to the limit in which collisions dominate: $\nu \gg c\,\partial \ln B/\partial z$, $\partial \ln B/\partial t$. Then we develop F in a series in $1/\nu$ as in equation (25). F_0 is again independent of μ. Averaging equation (77) over μ, we get the condition we can solve for F_1:

$$\frac{\partial F_0}{\partial t} + \frac{1}{3}\frac{\partial \ln B}{\partial t}p\frac{\partial F_0}{\partial p} = 0, \quad (78)$$

and

$$\frac{\partial F_1}{\partial \mu} = \frac{\mu a_t}{3}\frac{p}{\nu}\frac{\partial F_0}{\partial p} - \frac{v}{\nu}\frac{\partial F_0}{\partial z}, \quad (79)$$

where $a = \ln B$. In second order we again average over μ and use the equations analogous to equation (30). Averaging over the time variations of a, we get for the secular rate of change of F_0,

$$\frac{\partial F_0}{\partial t} = B\frac{\partial}{\partial z}\left(\frac{D}{B}\frac{\partial F}{\partial z}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D'\frac{\partial F}{\partial p}\right]$$

$$D' = \frac{p^2}{8}\langle a_t^2\rangle\int_0^1 \frac{(\mu^2-\mu^4)}{\nu}\,d\mu\,. \quad (80)$$

For the case $\mathfrak{E} \sim k^{-1.5}$,

$$D' = \frac{1}{36}\frac{p^2}{c^2}\langle a_t^2\rangle D \approx \frac{\langle \Delta p^2\rangle}{t}, \quad (81)$$

and the rate of increase of energy is

$$\frac{\langle\dot{\epsilon}\rangle}{\epsilon} = \frac{1}{108}\frac{\lambda\omega^2}{c}\left\langle\frac{\delta B}{B}\right\rangle^2 . \tag{82}$$

If we set $\omega = kV_A = 2\pi V_A/\lambda_0$,

$$\frac{\langle\dot{\epsilon}\rangle}{\epsilon} = 3.1 \times 10^{-17} B_\mu^2 \delta^{-1} n_H^{-1} \frac{\lambda_{pc}}{\lambda_{0pc}^2}\left\langle\frac{\delta B}{B}\right\rangle^2 \sec^{-1} . \tag{83}$$

If we set $B_\mu = 3$, $n_H = 0.1$ cm^{-3}, $\lambda = 8$ pc (characteristic of region B), $\lambda_0 = 1$ pc, and $\delta B/B \approx 0.1$, we obtain 1.1×10^{-16} sec^{-1}. This is to be compared with the age $\tau = 1.4 \times 10^{14}$ sec, so the energy change is very small. We can conclude from equations (67) and (83) that the observed cosmic rays have the same energy as when they left the source. (However, the application of eq. [82] to more compact systems shows that acceleration can be very efficient.)

IV. SUMMARY AND DISCUSSION

We have attempted to develop a theory of the diffusion of cosmic rays through a medium such as the Galaxy which takes into account the effect of hydromagnetic waves. One of the key assumptions of the theory is that the model we are treating has a uniform magnetic field, at least over many wavelengths.

We found that two classes of wavelengths are important for the cosmic rays: the short wavelengths of the order of the gyration radius for which the waves can interact by cyclotron resonance, and longer wavelengths with which the cosmic rays interact adiabatically by magnetic mirror scattering and the betatron effect. We treated the first class of waves by a stochastic small-amplitude theory. This is developed in § II, where we gave the dispersion relation characterizing the effect of the cosmic rays on the waves. It was shown that a small drift of the cosmic rays relative to the background interstellar plasma of the order of the Alfvén velocity can produce an instability which increases the amplitude of the waves. The growth rate is given in equation (4). This instability was discovered by Lerche (1966, 1967) and Wentzel (1968).

By proceeding to next order in the amplitude of the wave, we were able to exhibit a Fokker-Planck equation, equation (13), which describes the effect of the hydromagnetic waves on the particles. The Fokker-Planck coefficients are given in terms of the energy density in $(\boldsymbol{k},\boldsymbol{x})$-space of the hydromagnetic wave. They are generalizations of similar coefficients given by Jokipii (1966, 1967), Cornwall (1966), and others. It is shown that the main interaction with the cosmic rays is to scatter them in pitch angle with only a small change in energy. This is because the waves are much slower than the particles, and the energy change is on the average $(V_A/c)^2$ smaller than the change of pitch angle. The Fokker-Planck equation is then simplified by expanding it in small V_A/c and is given in equation (15). In order to apply this equation, one must know the wave amplitudes. These are given by balancing the growth rate of the waves due to their instabilities against the damping rate of the waves due to friction between the ionized and neutral portions of the interstellar gas. The instability arises naturally as the cosmic rays attempt to rush out the ends of the line of force. The combined growth due to cosmic rays and damping due to the neutral gas must be balanced against all external sources of waves as in equation (48). As the growth rate of the waves depends on the cosmic-ray anisotropy, this relation is only an implicit one and must be combined with a diffusion equation. In order to carry this out, the simplification of lumping all cosmic rays into the same energy and all waves into the same wavelength is carried out. This lumping is carried out by taking the observed spectra for cosmic rays and introducing the parameter β for the waves.

So far the theory is fairly accurate and is valid for magnetic fields uniform over several gyration radii and for waves of not too large amplitude. We now attempt to apply these results to the galactic cosmic rays and to other systems in which high-energy particles are trapped. This is done in § III. The basic equations for this application are equation (4), giving the effect of cosmic rays on the waves, and equation (15), giving the effect of the waves on the cosmic rays. To do this, we make a simple model for the cosmic rays and the Galaxy, consisting of a single tube of force of length $2L$ with open ends out of which the cosmic rays escape. We write down a diffusion equation, including the wave scattering as a collision term. For strong collisions, this can be reduced to an equation of spatial diffusion of the standard type, where the diffusion coefficient is given in terms of the wave energy density by equation (32).

With this simplification, a relation between the source of cosmic rays, the sources of turbulence, and the density of cosmic rays is derived in equation (54). This is most easily investigated by introducing a standard source of cosmic rays S_0, turbulent energy T_0, and the cosmic-ray density N_0 in equation (55). In terms of these quantities the relation reduces to equation (56). On the basis of this model one finds two regimes for the turbulent source (according to whether it is strong or weak compared with the standard source).

Assume a weak turbulent source, and gradually increase the cosmic-ray source S'. Then at first the waves are stable to cosmic rays, $\Gamma_{\text{c.r.}} < \Gamma^*$, and the turbulence is independent of the cosmic-ray density N, so the latter increases linearly with S'. As S' and N increase further, $\Gamma_{\text{c.r.}}$ starts to approach Γ^*, the damping is reduced, and the turbulence rises. This leads to increased confinement, and N increases faster than linearly with S'. As $\Gamma_{\text{c.r.}}$ gets close to Γ^*, the turbulence and density increase rapidly, so the cosmic rays are less anisotropic, and the damping due to the body of the cosmic rays becomes comparable to the instability due to anisotropy, Γ_2. When Γ_1 and Γ_2 are large compared with Γ^*, a balance between the two of them is reached, and N increases linearly with S' but with a larger coefficient. This is illustrated in Figure 1. For strong turbulence, Γ_1 is always larger than Γ_2, so the cosmic rays at first rise linearly with the source, then slower than linearly, and then linearly again. Thus, the cosmic rays control their own diffusion to a certain degree, and it is possible that the observed energy spectrum does not reflect the source energy spectrum exactly.

Our model so far has been homogeneous, with all the interstellar matter smeared out uniformly. An attempt is made to treat the actual distribution of interstellar matter in clouds by an inhomogeneous model. It is shown that in general the diffusion is faster in clouds (due to increased damping associated with this higher density). This leads to the consequence that in most cases, as far as diffusion is concerned, one can ignore the clouds completely and concentrate solely on the regions between the clouds, treating them by a homogeneous model. Thus, the cosmic rays are diffusing in a more rarefied medium than is generally supposed. However, the age inferred from the amount of matter through which the cosmic rays pass is still the same as for a homogeneous model.

Finally, an attempt is made in the last subsection to take into account larger-scale waves, such as those treated by Fermi (1949) and Fälthammar (1963). The small-scale waves now provide a collision mechanism which might make the methods work. It is found that for $\delta B/B \sim 0.1$ and wavelengths of the order of 1 pc the heating of cosmic rays in interstellar space is very weak, so they seem to have acquired their energy at the source. The model may be applied to the source itself, where magnetic pumping may not be small.

We have presented a very simple model to explore the effects of waves on cosmic-ray diffusion. It leaves many things out which could easily be included, and it is treated in a very elementary way. For example, only one species of cosmic rays is included. This could easily be generalized simply by introducing many f's in equation (56), with sums over f's in the numerator and denominator but not on the left. The different species should be taken at equal values of rigidity r_L. It should be easy to extend the model to non-uniform sources T and S and interstellar densities varying in z as the observed variation in height above the galactic plane. A more careful treatment of the model, not lumping the energies together, should be carried out. Also, an estimate of the turbulent sources should be made. The emission of waves by stellar winds and flares is an obvious possible source. The energy variation has been treated in only the briefest fashion. However, in spite of the defects, we feel the model has demonstrated that the relationship between sources of cosmic rays and their density is more intricate than at first supposed, but that it is still quite amenable to a quantitative treatment by the methods introduced in this paper.

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APPENDIX A

DAMPING OF HYDROMAGNETIC WAVES IN H I REGIONS

The simplest way to compute the rate of dissipation of waves is to calculate the entropy production of the waves, θ, multiply it by the temperature T, and equate it to the energy loss of the waves:

$$2\Gamma\left(\frac{2c^2}{V_A{}^2}\right)\frac{E^2}{8\pi} = \theta T \, . \tag{A1}$$

The details of the calculation are carried out by Braginskii (1965, eqs [8.39]–[8.43]). For Alfvén waves he finds

$$\Gamma = \Gamma_{\text{Joule}} + \Gamma_{\text{vis}} = \tfrac{1}{2}\left(\frac{c^2k_\perp{}^2}{4\pi\sigma_\|} + \frac{c^2k_\|{}^2}{4\pi\sigma_\perp}\right) + \frac{1}{2\rho^*}\left(k_\perp{}^2\eta_1 + k_\|{}^2\eta_2\right) , \tag{A2}$$

where the first term is resistive damping, the second is viscous damping, and σ is the conductivity, given by

$$\frac{\sigma_\|}{1.96} = \sigma_\perp \approx 10^{13}T^{3/2}\ \text{sec}^{-1} \tag{A3}$$

for T in electron volts. The viscosity coefficients are

$$\tfrac{1}{4}\eta_2 = \eta_1 = \frac{0.3n^*T_i\tau_i}{(\Omega\tau)_i{}^2} , \tag{A4}$$

where

$$\tau_i = 3 \times 10^6\left(\frac{m^*}{2m_{\text{H}}}\right)^{1/2}\frac{T_i{}^{3/2}}{n^*} \tag{A5}$$

is the ion-ion collision term. The ratio of Γ_{vis} to Γ^*, the damping produced by ion-neutral collisions, is

$$\frac{\Gamma_{\text{vis}}}{\Gamma^*} \approx \frac{k_\perp{}^2a_i{}^2}{\nu_0\tau_i} \approx k^2a_i{}^2\,\frac{n^*\langle\sigma v\rangle_{\text{Coul}}}{n_{\text{H}}\langle\sigma v\rangle}\left(\frac{m_{\text{H}}}{m^*}\right)^{1/2} , \tag{A6}$$

where a_i is the ion gyration radius, and $\langle\sigma v\rangle_{\text{Coul}}$ is the mean velocity times the Coulomb cross-section. But $\sigma_{\rm Coul} \approx 10^{-8}$ for $T = 0.01$ eV, and $n^*/n_{\rm H} \approx 10^{-3}$, so from equation (46) and $k \sim 1/r_L$ we have

$$\frac{\Gamma_{\rm vis}}{\Gamma^*} \approx \frac{1}{5} 10^3 \frac{a_i^2}{r_L^2} \approx 10^{-7}/\epsilon^2(\text{BeV}) , \tag{A7}$$

which shows $\Gamma_{\rm vis}$ is completely negligible. Similarly, for $k = 10^{-13}$ cm^{-1} we have

$$\Gamma_{\rm Joule} \approx 10^{-16} \text{ sec}^{-1} , \tag{A8}$$

which on comparison with Γ^* in equation (47) is completely negligible.

For magnetosonic waves Braginskii finds

$$\Gamma = \Gamma'_{\rm Joule} + \Gamma'_{\rm vis} + \Gamma'_{\rm ther} , \tag{A9}$$

where $\Gamma'_{\rm Joule}$ is of the same order as $\Gamma_{\rm Joule}$ and $\Gamma'_{\rm vis}$ is the sum of a term of the same order as $\Gamma_{\rm vis}$ and an additional term,

$$\Gamma'_{\rm vis} = \ldots + \tfrac{1}{6} \frac{k_\perp^2 \eta_0}{\rho^*} , \tag{A10}$$

where η_0 is the longitudinal ion viscosity,

$$\eta_0 = 0.96 n^* T_i \tau_i . \tag{A11}$$

Thus,

$$\frac{\Gamma'_{\rm vis}}{\Gamma^*} \approx \frac{n^*}{n_0} \frac{k_\perp^2 \lambda^2_{\rm Coul}}{6} \frac{\langle \sigma v \rangle_{\rm Coul}}{\langle \sigma v \rangle} \left(\frac{m_{\rm H}}{m^*}\right)^{1/2} \approx 10^3 k_\perp^2 \lambda^2_{\rm Coul} . \tag{A12}$$

Now $\lambda_{\rm Coul} \approx 10^{11}$ cm, so viscous damping of magnetosonic waves is comparable with damping by collisions with neutrals for $k_\perp \lesssim 10^{-12}$. We are mainly concerned with longer wavelengths in this paper. Finally, $\Gamma_{\rm ther}$ is the damping due to thermal conductivity. It is given by

$$\Gamma_{\rm ther} = \frac{k_\perp^2 T}{k^2 \rho^* V_A^2} (k_\parallel^2 \kappa_\parallel + k_\perp^2 \kappa_\perp) , \tag{A13}$$

where $\kappa_\perp$ and $\kappa_\parallel$ are the thermal conductivities perpendicular and parallel to the magnetic field. The quantity $\kappa_\perp$ is smaller than $\kappa_\parallel$ by a factor of $1/(\Omega\tau)^2$ for each species, so we can neglect it; $\kappa_\parallel$ is the electron thermal conductivity and is

$$\kappa_\parallel^e = \frac{3.16 n_e T_e \tau_e}{m_e} , \tag{A14}$$

so

$$\Gamma'_{\rm ther} \approx 3 \left(\frac{m^*}{m_e}\right)^{1/2} \left(\frac{n^* T}{\rho^* V_A^2}\right) k_\perp^2 \lambda^2_{\rm Coul} n_e \langle \sigma v \rangle_{\rm Coul} . \tag{A15}$$

Since $n^*T/\rho^* V_A^2$ is the ratio of the plasma pressure to the magnetic pressure, β^*, which is of the order of 10^{-5}, we see on comparison with $\Gamma'_{\rm vis}$ that $\Gamma'_{\rm ther} (\approx 10^3 \beta^* \Gamma'_{\rm vis})$ is negligible.

It is to be noted that we have assumed a strong-collision limit in treating the damping of hydromagnetic waves. This is valid since $\lambda_{\rm Coul} \approx 10^{11}$ cm is smaller than the typical wavelengths. Thus, the results on collisionless damping such as those given by Barnes (1966) do not apply. It is interesting that even if they did, the damping would be smaller than Γ^*.

APPENDIX B

THE COMPONENT OF $\boldsymbol{E}$ PARALLEL TO $\boldsymbol{B}_0$

In the paper we have neglected any component of the electric field parallel to $\boldsymbol{B}_0$. However, because of finite resistivity, etc., there will be a small one, and it is of interest to investigate whether this small component can have any effect on the scattering of cosmic rays. From the paper of Kennel and Engelmann (1966, eqs. [2.25]–[2.27]) we see that E_z is more efficient at scattering than $E_\perp$ by a factor of the order of c/V_A, so a small component could be important.

From the generalized Ohm's law (Spitzer 1962) we have

$$E_z = \frac{j_z}{\sigma} - \frac{1}{n^* e} \nabla_z p_e + \frac{m_e}{n^* e^2} \frac{\partial j}{\partial t} , \tag{B1}$$

where n^* is the plasma density. Now

$$j_z \approx \frac{k_\perp j_\perp}{k_z} \approx \frac{k_\perp c^2}{k_z V_A{}^2} \frac{\partial E_\perp}{\partial t} , \tag{B2}$$

so the E_z from finite resistivity, $E_z{}^\sigma$, gives

$$\frac{c}{V_A} \frac{E_z{}^\sigma}{E_\perp} = \left(\frac{c}{V_A}\right)^3 \frac{\omega}{\sigma} \approx \frac{\omega}{10^{10}} \left(\frac{c}{V_A}\right)^3 . \tag{B3}$$

Now $c/V_A \approx 3 \times 10^3$ and $\omega \approx kV_A = 10^{-6}$ for $k \approx 10^{-13}$, so the E_z-component can be neglected. Similarly, the contribution from $\partial j/\partial t$, $E_z{}^t$, gives

$$\frac{c}{V_A} \frac{E_z{}^t}{E_\perp} = \frac{c m_e \omega}{V_A n^* e^2 E_\perp} j_z \approx \left(\frac{c}{V_A}\right)^3 \frac{m_e}{n^* e^2} \omega^2 \approx 2 \times 10^{-7} , \tag{B4}$$

so $E_z{}^t$ may also be neglected.

To find the term due to ∇p_e, we write

$$\nabla_z \delta p_e \approx k_z T \delta n \approx k_z T \frac{1}{\omega} k_\perp \left(\frac{cE_\perp}{B}\right) . \tag{B5}$$

Thus the contribution $E_z{}^p$ is

$$\frac{c}{V_A} \frac{E_z{}^p}{E_\perp} = \frac{c}{V_A} \frac{k_\perp k_z v^{*2}}{\Omega \omega} = \left(\frac{c}{V_A} ka\right)^2 \lesssim 10^{-4} \tag{B6}$$

so all contributions to E_z are negligible relative to $E_\perp$.

APPENDIX C

THE PROPAGATION OF ALFVÉN WAVES IN A PARTIALLY IONIZED MEDIUM

Let us consider a gas made of n_0 neutrals of atomic mass m_0 and n^* singly-charged ions of atomic mass m^*. Then if $\boldsymbol{v}_0$ and $\boldsymbol{v}^*$ are the neutral and ionized mass velocities in an Alfvén wave with frequency ω and wave number $\boldsymbol{k}$, for the neutrals

$$\rho \omega^2 \boldsymbol{v}_0 = -i\nu^* \omega \rho (\boldsymbol{v}_0 - \boldsymbol{v}^*) , \tag{C1}$$

$$\nu^* = n^* \langle \sigma v_{\rm H} \rangle . \tag{C2}$$

We replace the friction between the ions by a simple collision term, and σ is the cross-section for scattering of neutrals by ions. We assume $m_0 < m^*$.

For the ions we must also include the magnetic force of an Alfvén wave:

$$\rho^* \omega^2 \boldsymbol{v}^* = \rho^* \omega_k{}^2 \boldsymbol{v}^* - i\nu_0 \omega \rho^* (\boldsymbol{v}^* - \boldsymbol{v}_0) , \tag{C3}$$

$$\nu_0 = n_0 \langle \sigma v_{\rm H} \rangle \frac{m_0}{m^*} , \tag{C4}$$

so $\nu^*/\nu_0 = \rho^*/\rho_0$, where ρ^* and ρ_0 are the mass densities of the ions and neutrals. The quantity $\omega_{\boldsymbol{k}}^2$ is the natural frequency of a hydromagnetic wave in the charged medium without the friction term ($\omega_{\boldsymbol{k}}^2 = k^2 V_A^{*2}$ or $k_z^2 V_A^{*2}$ for a magnetosonic or Alfvén mode, $\boldsymbol{B}$ is in the z-direction and $V_A^* = B(4\pi\rho^*)^{-1/2}$. Combining equations (C1) and (C3), we obtain the dispersion relation,

$$(\omega^2 - \omega_{\boldsymbol{k}}^2)\omega + i\nu_0[(1+\epsilon)\omega^2 - \omega_{\boldsymbol{k}}^2\epsilon] = 0\,, \tag{C5}$$

where $\epsilon = \rho^*/\rho \ll 1$. Thus, when $\omega \gg \nu_0$, collisions are not frequent enough to make the neutrals move with the ions, and $\omega^2 \approx \omega_{\boldsymbol{k}}^2$. The charged particles move as though the neutrals were absent. On the other hand, if $\omega \ll \nu_0\epsilon$, the neutrals collide often enough that $v_0 \approx v^*$ and the entire medium moves. The increased mass reduces the Alfvén speed, and $\omega \approx \omega_{\boldsymbol{k}}\epsilon^{1/2} = kB(4\pi\rho_0)^{-1/2}$. In between, the particles move independently; the friction is strong, and the waves do not propagate.

In detail equation (C5) is a cubic equation for ω. If ϵ is small, we can solve the equation asymptotically. If $\nu_0 \gg \omega_{\boldsymbol{k}}$, we may neglect the ω^3 term and obtain

$$\omega = \pm\left(\omega_{\boldsymbol{k}}^2\epsilon - \frac{\omega_{\boldsymbol{k}}^4}{4\nu_0^2}\right)^{1/2} - i\,\frac{\omega_{\boldsymbol{k}}^2}{2\nu_0}\,, \qquad \omega_{\boldsymbol{k}} \ll \nu_0\,. \tag{C6}$$

If $\omega_{\boldsymbol{k}} \gg \nu_0\sqrt{\epsilon}$, the constant term may be neglected, so that

$$\omega = \pm\left(\omega_{\boldsymbol{k}}^2 - \frac{\nu_0^2}{4}\right)^{1/2} - \frac{i\nu_0}{2}\,, \qquad \omega_{\boldsymbol{k}} \gg \nu_0\sqrt{\epsilon}\,. \tag{C7}$$

These two expressions overlap and agree in the overlapping region $\nu_0\sqrt{\epsilon} \ll \omega_{\boldsymbol{k}} \ll \nu_0$ if one expands them out.

Equation (C6) corresponds to long wavelengths (small k and $\omega_{\boldsymbol{k}}$). We see that, as $\omega_{\boldsymbol{k}}$ tends to zero, the wave tends to an Alfvén wave with the whole medium moving, while the damping goes to zero as ν_0 goes to infinity. (Strong collisions tie the two species together and reduce the damping.) Further, from equations (C6) and (C7), we see that no waves propagate when $2\nu_0\sqrt{\epsilon} < \omega_{\boldsymbol{k}} < \frac{1}{2}\nu_0$. Finally, for short waves we see from equation (C7) that $\omega = \omega_{\boldsymbol{k}}$; only the charged particles participate in the wave, and the damping is proportional to ν_0, more collisions increasing the friction. The damping rate is

$$\Gamma^* = \frac{\nu_0}{2}\,.$$

Using the values given in the text for the interstellar medium, we set

$$\lambda_1 = \frac{2\pi}{k_1} = \frac{\pi V_A^*}{\nu_0\sqrt{\epsilon}} = 0.22 B_\mu \delta^{-1}\ \text{pc} = 0.63\ \text{pc}$$

$$\lambda_2 = \frac{2\pi}{k_2} = \frac{4\pi V_A^*}{\nu_0} = 0.065 B_\mu \delta^{-1}\ \text{pc} = 0.20\ \text{pc}\,.$$

For $\lambda < \lambda_2$, $\omega_{\boldsymbol{k}} = kV_A^*$; for $\lambda_2 < \lambda < \lambda_1$ there is no propagation; for $\lambda > \lambda_1$, $\omega_{\boldsymbol{k}} = kV_A^0 = kB(4\pi\rho_0)^{-1/2}$.

Note added in proof.—After this paper went to press it was pointed out to the author by Melrose and Wentzel that the neglect of the 1 in equation (14) is not valid for small anisotropies if all the waves are traveling in the same direction. This is obviously the case for those waves created by the cosmic rays. As a consequence, equations (15) and (23) are valid only in the frame of reference moving with the waves. For this case one finds that equation (31) is modified by an addition of the term $V_A \partial F/\partial z$ on the right-hand side if the waves are moving to the right (in the rest frame), while the term $-\Gamma_1$ disappears from equation (45). This leads to a slightly more complex treatment of the self-consistent model than is given here. The results for this model given here are valid if the diffusion of the cosmic rays is somewhat faster than V_A, and also for comparable diffusion velocities the solution for N is still valid, but for intermediate diffusion rates a more detailed treatment is needed. This will be given in a subsequent paper.

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