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EFFECTS OF SELF-ABSORPTION AND INTERNAL DUST ON HYDROGEN-LINE INTENSITIES IN GASEOUS NEBULAE

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ABSTRACT

A general method for computing hydrogenic recombination decrements in gaseous nebulae is described. The effects of any combination of partial Lyman-line leakage (for cases intermediate between case A and case B), Balmer-line self-absorption, and internal dust can be calculated from known case A decrements. In particular, for a homogeneous spherical nebula, we have calculated the combined effects of Balmer-line self-absorption and partial Lyman-line leakage in terms of optical depths in Hα and Lyman α. The relevance of these calculations for real nebulae is discussed. Finally, the influence of small graphite grains and other particles within the nebula on the Balmer decrement is discussed.

I. INTRODUCTION

The theory of hydrogenic recombination is well understood and has been studied by a large number of authors. Recent contributions by Pengelly (1964) and Clarke (Aller and Liller 1968) have considered many n- and l-states in solving the cascade problem. Yet, in spite of these efforts, photoelectric observations of the lower Balmer and Paschen lines in planetary nebulae (Osterbrock, Capriotti, and Bautz 1963; O'Dell 1963) often result in decrements which do not include the theoretical decrements within the expected margin of error (Osterbrock 1964). Capriotti (1964*a*, *b*) has studied the effects on the Balmer decrement of self-absorption of Balmer photons from the $2s$ and $2p$ levels under the conditions of case B. In a later paper Capriotti (1966) calculated the Balmer decrements for nebulae in which there is some leakage of Lyman-line radiation, i.e., for situations intermediate between cases A and B.

The purpose of the present work is to study the combined effects of Lyman-line leakage, Balmer self-absorption, and dust grains on the Lyman, Balmer, and Paschen decrements. Our results for the effect of Lyman-line leakage alone are qualitatively similar to those of Capriotti (1966). The combined effects of Balmer self-absorption from the $2s$ level and partial Lyman leakage are presented in § III. These results demonstrate that rather exotic decrements can be obtained for some nebulae, but it is shown in § IV that these results cannot apply to the majority of planetary nebulae. Finally, we have calculated the effects of various kinds of dust on the relative intensities of the recombination lines. We discuss these effects in § V. It is certain that dust exists within some H II regions (Krishna Swamy and O'Dell 1967). There is now evidence that planetary nebulae may also contain significant amounts of dust (Gillett, Low, and Stein 1967; Krishna Swamy and O'Dell 1968). Throughout the present paper, we have restricted the model nebulae to homogeneous, static, isothermal ($T_e = 10000°$ K) spheres of pure hydrogen. When dust is included in the nebula, it is assumed to be uniformly distributed. In the next section we discuss the computational technique which we have employed. Although results for Balmer self-absorption from the $2p$ level are not discussed in this paper, we include this possibility in the following section to show the generality of the adopted formalism.

II. CALCULATIONAL PROCEDURE

Capriotti (1966), who studied the transition between case A and case B in the hydrogen-recombination spectrum, used a formalism which involved the mean escape probability per scattering (Capriotti 1965). This reduces the difficulty of calculation considerably by circumventing the difficult radiative-transfer problem for the line radiation without altering the essentials of the results. We adopt a similar procedure here. We therefore assume that the line emission and absorption rates are independent of position in the nebula. If τ is the optical depth (measured from the center to the edge of the nebula) for a certain frequency within an emission line, then the mean escape probability is found by averaging the escape probability over direction and volume. The result for a homogeneous sphere is

$$\epsilon(\tau) = \frac{3}{4\tau}\left[1 - \frac{1}{2\tau^2} + \left(\frac{1}{\tau} + \frac{1}{2\tau^2}\right)e^{-2\tau}\right] .$$

The total optical depth,

$$\tau(x) = \tau_d + \tau_0 \exp(-x^2) ,$$

includes optical depths due to dust as well as self-absorption. Here $x = (\nu - \nu_0)/\Delta\nu_D$ is a dimensionless frequency, where $\Delta\nu_D = (2kT/M\lambda_0^2)^{1/2}$ is the Doppler width and M is the mass of atomic hydrogen. The optical depth for self-absorption at the line center is τ_0. Under the assumption of complete redistribution, the probability of creating or re-emitting a photon at some x is then $2\pi^{-1/2} \exp(-x^2)$. The mean escape probability for the entire line is then

$$E = \frac{2}{\sqrt{\pi}} \int_0^\infty \epsilon[\tau(x)] \exp(-x^2) dx = \frac{1}{\tau_0\sqrt{\pi}} \int_{\tau_d}^{\tau_0+\tau_d} \frac{\epsilon(\tau) d\tau}{[\ln \tau_0 - \ln(\tau - \tau_d)]^{1/2}} . \quad (1)$$

This integral must be integrated numerically, but we find that the result for $\tau_d = 0$, $\tau_0 \leqq 40$ can be fitted rather well with $E(\tau_0) = 1.72/(1.72 + \tau_0)$. When the effects of dust are included, one must also consider the mean probability P that the photon is reabsorbed and the mean probability S that it interacts with a dust particle and is either scattered or absorbed. The integral expressions for P and S are analogous to equation (1) with $\epsilon(\tau)$ replaced with $[1 - \epsilon(\tau)](1 - \tau_d/\tau)$ and $[1 - \epsilon(\tau)]\tau_d/\tau$, respectively.

For a given dust grain, the optical depth for extinction is proportional to Q_{ext} for each emission line. Therefore, if τ_d is specified for Hα, and the optical depths $\tau_{2\alpha} = \tau_0$(Hα) and $\tau_{1\alpha} = \tau_0$(Ly-α) are chosen, then the mean escape, reabsorption, and dust-extinction probabilities can be calculated for the entire Lyman, Balmer, and Paschen series. Furthermore, if the albedo of the dust is known for each line and the population ratio of the $2s$ and $2p$ levels is specified, then the effects of self-absorption and scattering on the hydrogen-recombination decrements can be calculated from the known case A decrement. For this purpose we have used Pengelly's (1964) case A decrements for the three lowest series of hydrogen. The probabilities P and E defined above refer to only one interaction with the dust. They must be corrected by multiplication by a geometric series which represents multiple reflections off dust particles, viz.,

$$E' = E/(1 - \gamma S) \quad \text{and} \quad P' = P/(1 - \gamma S) ,$$

where γ is the albedo. Aside from the initial effect on the escape and reabsorption probabilities, this is the only modification required to compensate for the presence of dust. We have tacitly incorporated the assumption that a photon is not shifted in frequency upon reflection by the dust.

Following Capriotti, we make the following definitions:

$PL(n,m)$ = probability that Lyman-m emission results from a self-absorption of Lyman n or Balmer n from the $2s$ level.

$PB(n,m)$ = probability that Balmer-m emission results from a self-absorption of Lyman n or Balmer n from the $2s$ level.

$PLP(n,m)$ = probability that Lyman-m emission results from a self-absorption of Balmer n from the $2p$ level.

$PBP(n,m)$ = probability that Balmer-m emission results from a self-absorption of Balmer n from the $2p$ level.

$F2S(n)$ = fraction of Balmer-n self-absorptions which arises out of the $2s$ level.

Clearly

$$F2S(n) = \frac{3A(np,2s)}{3A(np,2s) + [A(ns,2p) + 5A(nd,2p)]N_{2p}/3N_{2s}} ,$$

and so

$$\langle PL(n,m)\rangle = F2S(n)PL(n,m) + [1 - F2S(n)]PLP(n,m) ,$$

together with an analogous expression for $\langle PB(n,m)\rangle$, represent mean conversion probabilities. $PL(n,m)$, $PB(n,m)$, and $PBP(n,m)$ have been computed by Capriotti (1964*a*, *b*).

Several additional quantities must also be defined. Quantities having subscripts, 1, 2 and 3 are defined for Lyman, Balmer, and Paschen photons, respectively.

$$A_1(n) = P_1'(n)/[1 - P_1'(n)\,PL(n,n)]$$

is the mean number of times that a Lyman-n photon is absorbed before either converting to a lower line, escaping, or being absorbed by dust.

$$A_2(n) = P_2'(n)/[1 - P_2'(n)\langle PB(n,n)\rangle]$$

is the similar quantity for Balmer n. On the other hand, Lyman n can convert to Balmer n and then back to Lyman n so that the total mean number of scatterings of Lyman n is enhanced to

$$Q_1(n) = A_1(n) + A_1(n)PB(n,n)A_2(n)\langle PL(n,n)\rangle Q_1(n)$$
$$= \frac{A_1(n)}{1 - A_1(n)PB(n,n)A_2(n)\langle PL(n,n)\rangle} .$$

When $2p$ self-absorption is negligible, $\langle PL(n,n)\rangle = PL(n,n)$. The quantity $A_1(n)PB(n,n)A_2(n)\langle PL(n,n)\rangle$ represents the probability of converting Lyman n to Balmer n and back again before the excitation is lost. The total mean number of scatterings of Balmer n is similarly

$$Q_2(n) = \frac{A_2(n)}{1 - A_2(n)\langle PL(n,n)\rangle A_1(n)PB(n,n)} = \frac{A_2(n)}{A_1(n)} Q_1(n) .$$

The net probability that a Lyman-n line escapes after considering reflections from dust, loss by conversion to lower lines, and conversion to Balmer n followed by Balmer self-absorption is

$$\{1 + Q_1(n)[PL(n,n) + PB(n,n)A_2(n)\langle PL(n,n)\rangle]\}E_1'(n) ,$$

which can be shown to equal $Q_1(n)E_1'(n)/P_1'(n)$. As expected intuitively, this is simply the mean number of absorptions divided by the absorption probability per formation, multiplied by the escape probability per formation.

The net probability that a Lyman-n line produces a Balmer-n line which subsequently escapes the nebula is

$$
\begin{aligned}
A_1(n)PB(n,n)Q_2(n)E_2'(n)/P_2'(n) &= A_1(n)PB(n,n)[1 + Q_2(n)\langle PB(n,n)\rangle \\
&\quad + Q_2(n)\langle PL(n,n)\rangle A_1(n)PB(n,n)]E_2'(n) \\
&= Q_1(n)PB(n,n)[1 + A_2(n)\langle PB(n,n)\rangle]E_2'(n)\ .
\end{aligned}
$$

The last expression can be understood as the total mean number of absorptions of Lyman n multiplied by the probability per absorption of conversion to Balmer n and the mean number of formations of Balmer n, assuming no conversions back through Lyman n, since these are all accounted for in $Q_1(n)$. This is then multiplied by the net escape probability. The first expression has an alternate physical description. It represents the probability that Lyman n eventually converts to Balmer n, multiplied by the total mean number of formations of Balmer n (assuming that it has formed once), and finally by the total escape probability.

One can similarly define the mean probability that Lyman n forms a lower Lyman or Balmer line through cascade and the probability that a Balmer-n photon eventually escapes as Lyman n, Balmer n, or cascades to one of the lower Lyman or Balmer lines. After each cascade, the new photons formed must also be subjected to escape and self-absorption probabilities so that the eventual fate of any given excitation is affected at each intermediate stage of the radiative cascade at which a Lyman or Balmer line may be formed.

In order to take all these processes into account in a relatively clean fashion, we define the following quantities:

$$
\begin{aligned}
CB_1(n,n) &= Q_1(n)PB(n,n)[1 + A_2(n)\langle PB(n,n)\rangle]\ , \\
CB_2(n,n) &= Q_2(n)[\langle PB(n,n)\rangle + \langle PL(n,n)\rangle A_1(n)PB(n,n)] + 1 = Q_2(n)/P_2'(n) \\
CB_1(n,m) &= Q_1(n)[TB(n,m) + PB(n,n)A_2(n)\langle TB(n,m)\rangle]\ , \\
CB_2(n,m) &= Q_2(n)[\langle TB(n,m)\rangle + \langle PL(n,n)\rangle A_1(n)TB(n,m)]\ , \\
TB(n,m) &= PB(n,m) + PL(n,m)A_1(m)PB(m,m) + \sum_{n>i>m}[PL(n,i)CB_1(i,m) \\
&\quad + PB(n,i)CB_2(i,m)]\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle TB(n,m)\rangle &= \langle PB(n,m)\rangle + \langle PL(n,m)\rangle A_1(m)PB(m,m) \\
&\quad + \sum_{n>i>m}[\langle PL(n,i)\rangle CB_1(i,m) + \langle PB(n,i)\rangle CB_2(i,m)]\ .
\end{aligned}
$$

Then the contribution to Balmer m from cascades from level $n > m$ is

$$H(n,m) = [CB_1(n,m)L_A(n) + CB_2(n,m)B_A(n)]\frac{Q_2(m)}{P_2'(m)}E_2'(m)\ ,$$

or

$$H(m,m) = [CB_1(m,m)L_A(m) + CB_2(m,m)B_A(m)]E_2'(m)$$

if $n = m$. $L_A(n)$ and $B_A(n)$ are the case A photon intensities of the Lyman and Balmer lines. The new Balmer decrement by number is then

$$B(m) = \sum_{n\geq m} H(n,m)\ .$$

Similar expressions for the Paschen decrement can be derived and we have included them in the calculations. We find that $H(m,m) \gg H(n,m)$ for $n > m$ follows from the numerical values of the conversion probabilities. In the calculations described below, we considered only those levels for which $n \leq 12$.

III. COMPUTATIONAL RESULTS

Figures 1 and 2 illustrate the effects of various combinations of $\tau_{1\alpha}$ and $\tau_{2\alpha}$ on the intensity ratios of the lowest Balmer lines. The small loop which appears in both figures represents the path taken by the decrement if $\tau_{2\alpha} = 0$ and $\tau_{1\alpha}$ varies from zero for case A (*filled circle*) to 10^7, essentially case B (*open circle*). Such a loop was found by Capriotti (1966), who also performed similar calculations with escape probabilities characteristic of expanding nebulae. The curves which radiate out from this loop represent the effect of increasing $\tau_{2\alpha}$, the optical depth of Hα. In both figures, the Balmer self-absorption curves are marked (with full bars) at points where $\tau_{2\alpha} = 5$ and 10. The curves for

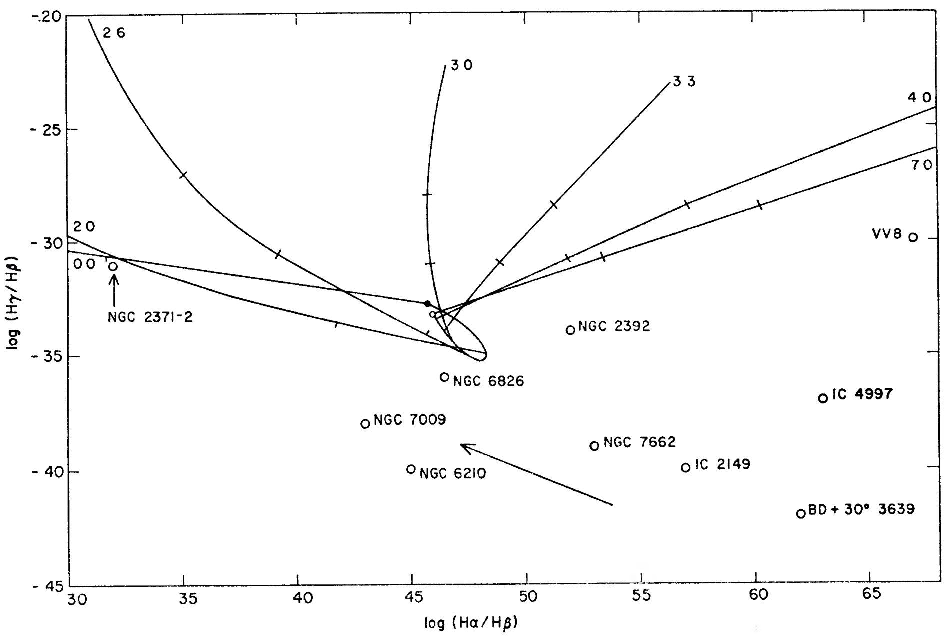

FIG. 1.—Balmer-line intensity ratios (Hα/Hβ) and (Hγ/Hβ) as functions of $\tau_{1\alpha}$ and $\tau_{2\alpha}$. Values of log $\tau_{1\alpha}$ label each Balmer self-absorption curve.

$\tau_{1\alpha} = 0.0$, 2.0, and 2.6 in Figure 1 are also marked by half-bars where $\tau_{2\alpha} = 1.0$. The curve for log $\tau_{1\alpha} = 7$ agrees well with the 2*s* self-absorption curve calculated by Capriotti (1964*b*). For case B, as $\tau_{2\alpha}$ increases, the first effect on the decrement occurs when Hβ converts into Hα, and the intensity ratio I(Hα)/I(Hβ) therefore increases. In the opposite (and absurd) extreme, increasing $\tau_{2\alpha}$ with $\tau_{1\alpha} = 0$ (case A) will cause Hα to convert to Lyman β and escape, so I(Hα)/I(Hβ) will decrease. This trajectory and various intermediate possibilities are shown in Figure 1. If self-absorption took place from the 2*p* level, this effect would not occur. In Figure 2 the Balmer self-absorption curves are roughly parallel, as expected. For comparison, we have included in both figures mean values of the observed decrements for several planetary nebulae given by Osterbrock *et al.* (1963) and O'Dell (1963). The unreddening trajectory, based on the data of Whitford (1958), is shown with an arrow. We have made a variety of checks on the calculations. The Balmer and Paschen decrements and absolute line intensities for log

$\tau_{1\alpha} = 7$ differ by less than 1 per cent from the case B decrement given by Pengelly for all of the lowest twelve levels. The one exception is our value of $I(\mathrm{P}\alpha)/I(\mathrm{H}\beta)$ which is 1.8 per cent higher than Pengelly's. Approximate hand calculations also bear out the general trend of our results.

In Table 1 we list the values for two ratios of Paschen and Balmer lines with the same upper state. Care must be taken in using these ratios to obtain the reddening since they are functions of $\tau_{1\alpha}$, and, to a lesser extent, $\tau_{2\alpha}$.

IV. CORRESPONDENCE WITH REAL NEBULAE

We now estimate values of $\tau_{1\alpha}$ and $\tau_{2\alpha}$ that can be expected in various astronomical situations. The optical depth of Lyman α at line center is simply proportional to the optical depth in the Lyman continuum, i.e., $\tau_{1\alpha} = (\kappa_{1\alpha}/\kappa_c)\tau_c$, where $\kappa_c = 6.32 \times 10^{-18}$

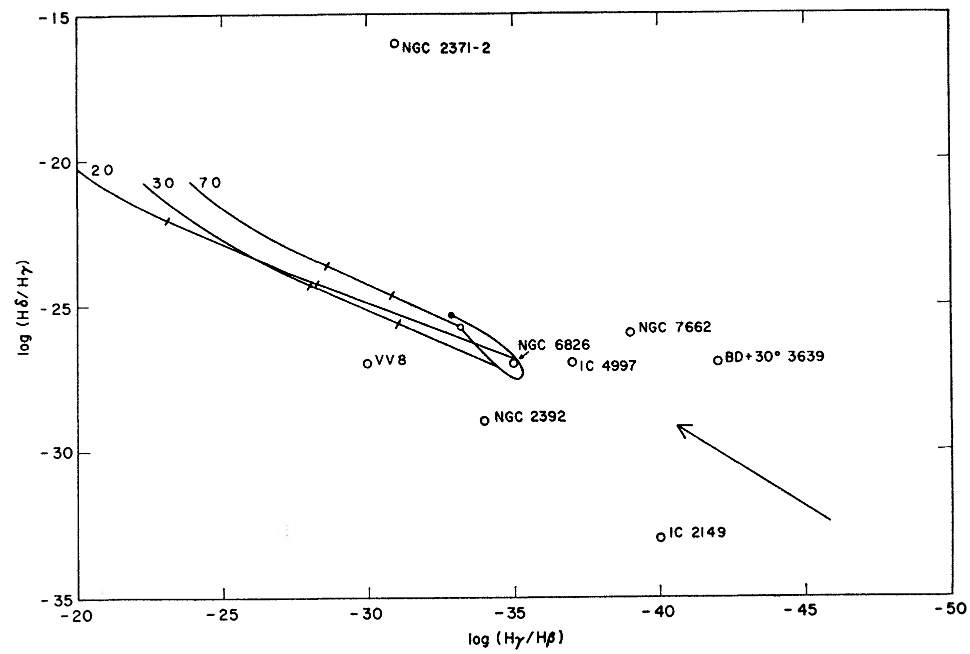

FIG. 2.—Balmer-line intensity ratios (Hγ/Hβ) and (Hδ/Hγ) as functions of $\tau_{1\alpha}$ and $\tau_{2\alpha}$. Values of log $\tau_{1\alpha}$ label each Balmer self-absorption curve.

TABLE 1

INTENSITY RATIOS P9/H9* AND P6/H6

$\tau_{2\alpha}$	LOG $\tau_{1\alpha}$								
	0 0	2 0	2 6	2 8	3.0	3 3	3 7	4 0	7 0
0	{0 441	0 435	0 421	0 413	0 403	0 388	0 372	0 363	0 352
	450	.432	403	.392	383	371	362	.358	353
1	{ 442	437	422	414	.404	389	372	.364	352
	0 457	0 438	0 408	0 397	0.387	0 375	0 365	0.361	0 356

* Upper value.

cm^2 is the photo-ionization cross-section at the Lyman limit. The condition for steady-state ionization balance at some point r in the nebula and the equation of transfer for the ionizing radiation combine to give

$$\exp(-t_c)dt_c = \frac{4\pi\beta}{L_*} R^3 N_e^2 \xi^2 d\xi \,, \tag{2}$$

where $\xi = r/R$ and $t_c(\xi)$ is the local value of the optical depth. The diffuse Lyman continuum is assumed to be absorbed on the spot. The recombination coefficient to excited levels is $\beta = 2.60 \times 10^{-13}$ cm^3 sec^{-1} from the calculations of Pengelly (1964). In an isothermal, constant-density, pure-hydrogen nebula, the combined number density of ions and neutrals $N = N_e + N_1$ is constant. However, if $N_1 \ll N_e$ throughout, equation (2) can be directly integrated, giving

$$\tau_c = -\ln\left(1 - \frac{\theta_0}{3}\right) , \tag{3}$$

where $\theta_0 = 4\pi\beta\, R^3\, N_e^2/L_*$ is a dimensionless parameter. For any value of N_1 it is easy to show that τ_c will depend on $\theta_1 = 8\pi N\beta R^2/L_*\kappa_c$ and $\theta_2 = \kappa_c\, NR$, also dimensionless parameters. When $\tau_c \ll \ln\,(2/\theta_1)$, $\theta_0 \approx \theta_1\theta_2/2$ and equation (3) results. For planetary nebulae Capriotti and Kovach (1968) show that τ_c can be as low as 0.05 for which $\tau_{1a} \approx$ 400. The decrements of many planetary nebulae should therefore deviate significantly from the case B decrement and lie along the small loops in Figures 1 and 2.

It is important to know what conditions are necessary for Balmer self-absorption to arise from the $2s$ level rather than the $2p$. In a steady state, the $2p$ level is populated and depopulated at the same rate, i.e.,

$$N_{2p}(A + \tfrac{1}{3}N_e q + \phi_{ps}) = f_2\beta N_e^2 + N_1 \int N_{1a}(\nu)\kappa_{1a}(\nu)\, c d\nu + N_{2s}(N_e q + \phi_{sp}) \,. \tag{4}$$

The left-hand side of equation (4) describes how the density of atoms in the $2p$ level, N_{2p}, is depleted by emission of Lyman α ($A \equiv A(21,10) = 6.26 \times 10^8$ sec^{-1}) and by transitions to the $2s$ level either by direct collisional transition by a passing proton or electron ($q = q(2s \rightarrow 2p) = 5.3 \times 10^{-4}$ cm^3 sec^{-1} at $T_e = 10^4$ °K) or by self-absorption of Balmer-line photons which eventually cascade to the $2s$ level at a total rate ϕ_{ps} sec^{-1}. The $2p$ level is continuously being replenished by the fraction f_2 of the recombinations to excited levels which end on that level. Entries into the $2p$ state are also made by reabsorption of Lyman-α radiation with absorption coefficient $\kappa_{1a}(\nu)$. Finally, entries into the $2p$ level can come by direct collision from the $2s$ level, or by Balmer self-absorption followed by cascade. An analogous equation for the steady-state condition in the $2s$ level is

$$N_{2s}(A_2 + N_e q + \phi_{sp}) = f_1\beta N_e^2 + N_{2p}(\tfrac{1}{3}N_e q + \phi_{ps}) \,, \tag{5}$$

where $A_2 \equiv A(11,10) = 8.23$ sec^{-1}. The number density of Lyman-α photons N_{1a} in steady-state conditions must satisfy

$$\frac{N_1}{Q+1} N_1 \int N_{1a}(\nu)\kappa_{1a}(\nu)\, c d\nu = \beta\Phi N_e^2 \,, \tag{6}$$

where Q is the number of scatterings that a Lyman-α photon experiences in the nebula and N_1 is the density of ground-state atoms. The probability Φ of producing a Lyman-α photon during the initial cascade is given approximately by

$$\Phi = 1 - \frac{f_1 A_2}{A_2 + N_e q + \phi_{sp}} \,. \tag{7}$$

When equations (4) and (6) are combined, it is possible to solve for N_{2p} and N_{2s} without a detailed knowledge of the Lyman-α line profile.

When there is no appreciable amount of Balmer self-absorption, $\phi_{sp} = \phi_{ps} = 0$, and, to a good approximation, $f_1 = \frac{1}{3}$, $f_2 = \frac{2}{3}$ and $\frac{2}{3} \leq \Phi \leq 1$ Furthermore, if $Q \gg 1$, and $Q\Phi N_e q \ll A$ the above equations give simply

$$N_{2s} \approx \frac{\beta N_e^2}{3(A_2 + N_e q)}, \tag{8}$$

and in the same approximation, $N_{2p} \approx 0$. Note that A_2 and $N_e q$ are comparable in many nebulae and that N_{2s} will be constant with radius together with N_e. The condition for self-absorption out of the $2s$ level is therefore $Q \ll A/\Phi N_e q$, and since $\Phi N_e q \lesssim 1$ for most nebulae, it follows that $N_{2s} \gg N_{2p}$ unless $Q \gtrsim 10^8$ where $\tau_{1a} \gtrsim 10^7$ and the Lyman lines are strongly trapped (Osterbrock 1962). Moreover, if $Q < 10^8$, the value of Q does not affect the optical depth of the nebula in the Balmer lines. Of particular interest is the optical depth at the center of the Hα line which is given by

$$\tau_{2a} = \kappa_{2a} N_{2s} R = \frac{\kappa_{2a} \beta R N_e^2}{3(A_2 + N_e q)} = 1.08 \times 10^{-8} \frac{R_{pc} N_e^2}{1 + N_e 6.44 \times 10^{-5}} \tag{9}$$

at $T = 10^4$ ° K. In the last expression the nebular radius is given in parsecs. The optical depths τ_{1a} and τ_{2a} therefore depend on N_e and R in a different manner, and τ_{1a} is also a function of L_*.

TABLE 2

FRACTIONAL DEPOPULATION OF $2s$ LEVEL BY SELF-ABSORPTION OF Hβ

τ_{2a}	$\tau_{2\beta}$	$E(\beta)$	Case B $g(\beta)$
1	0 1751	0 91	0 01
10 .	1.751	.50	.07
25..	4 38	.28	.14
∞ ..	∞	0 00	0 35

In nebulae which are quite thick to the Balmer lines the factor ϕ_{sp} may become appreciable. When this occurs, the parameters f_1, f_2, and Φ will be complicated functions of the Balmer optical depths and the cascade matrix. Clearly, ϕ_{sp} represents a sum over the self-absorption rates for all the Balmer lines. The lowest Balmer line which can be self-absorbed from the $2s$ level and cascade to the $2p$ is Hβ. Since the contribution to ϕ_{sp} will be highest for Hβ photons, an approximate value of $N_{2s}\phi_{sp}$ can be found by considering Hβ alone. Provided $N_{2s} \gg N_{2p}$, the rate at which the $2s$ level is populated is $\frac{1}{3}\beta N_e^2$. The rate at which it is depopulated due to self-absorption of Hβ is approximately

$$r_{sp}(\beta) = N_{2s}\phi_{sp}(\beta) = N_e^2 \beta_{4,2} \frac{(1 - E)\psi_{sp}}{1 - (1 - E)(1 - \psi_{sp})},$$

where $\beta_{4,2}$ is the effective recombination coefficient for producing Hβ in the nebula (including cascades), E is the escape probability for Hβ and ψ_{sp} is the probability that an absorption of Hβ results in a cascade to the $2p$ level. We have overestimated r_{sp} by assuming pure case B.

In Table 2 we list values of $g(\beta) = 3r_{sp}(\beta)/\beta N_e^2$, the probability that the $2s$ level will be depopulated by self-absorption of Hβ. The first and second columns in Table 2 list values of the central optical depth in Hα and Hβ, the third column gives the corresponding escape probability for Hβ, and the last column tabulates $g(\beta)$ for case B (where $\psi_{sp} = 0.261$, $\beta_{4,2} = 3.03 \times 10^{-14}$ cm^3 sec^{-1}). Even for infinite optical depth, the fractional rate of depopulation of the 2s level due to self-absorption of Hβ is rather small. Values of $g(\beta)$ in Table 2 are underestimated since self-absorption effects of Hγ and higher lines are neglected. Nevertheless, it appears that self-absorption will not appreciably depopulate the 2s level, and it is therefore consistent to consider 2s self-absorption for rather large values of τ_{2a}, at least in principle.

In order to understand the astrophysical significance of equation (9), we examine the conditions which a nebula must satisfy to have unit optical depth in Hα. When the nebular radius is expressed in terms of N_e and the nebular mass, $\mathfrak{M}$, the condition for $\tau_{2a} = 1$ is

$$1 = \frac{\kappa_{2a}\beta}{3A_2}\left(\frac{3\mathfrak{M}}{4\pi M}\right)^{1/3} \frac{N_e^{5/3}}{(1 + N_e q/A_2)} .$$

The solution of this equation for $T = 10^4$ ° K is given in Table 3 for a variety of nebular masses. The third column of Table 3 lists that value of the ionizing-photon luminosity

TABLE 3

CRITICAL NEBULAR CONDITIONS FOR $\tau_{2a} = 1$

$\log (\mathfrak{M}/\mathfrak{M}_\odot)$	N_e (cm^{-3})	L_* (sec^{-1})
−3	4.1×10^6	1.3×10^{48}
−2	1.6×10^6	5.0×10^{48}
−1	5.0×10^5	1.6×10^{49}
0	1.6×10^5	3.1×10^{49}
+1	6.1×10^4	1.9×10^{50}
+2	2.8×10^4	9.9×10^{50}
+3	1.4×10^4	4.4×10^{51}

which is just sufficient to ionize the nebula. These values should be compared with the maximum ionizing-photon luminosities available from known stars. For central stars of planetary nebulae, the largest L_* is about 8×10^{47} sec^{-1}, obtained with a black body having a luminosity of $L \approx 10^4 L_\odot$ and $T_* = 72000$° K. It is clear that no planetary nebula will have a significant optical depth in Hα from the 2s level unless $N_e \gtrsim 10^6$ cm^{-3}. In fact, for a planetary nebula with $\tau_c = 1$ and $\mathfrak{M} = 0.25\,\mathfrak{M}_\odot$, equation (9) gives $\tau_{2a} = 0.05$. Among the well-observed planetaries, only in VV 8 is Balmer self-absorption a possibility.[1] Therefore, while we have shown in Figure 1 that the observations of planetary nebulae can be corrected for interstellar reddening and fall within a region of the diagram which is theoretically possible, it is unfortunate that the corresponding combinations of τ_{1a} and τ_{2a} are not possible for planetary nebulae. On the other hand, very luminous O stars on or near the main sequence (with $L \sim 9 \times 10^5 L_\odot$ and $T_* \sim 40000$° K) have $L_* \sim 5 \times 10^{49}$ sec^{-1} and can ionize nebulae which have a significant amount of Balmer self-absorption, provided $\mathfrak{M} \lesssim 1.0\,\mathfrak{M}_\odot$. Also since τ_{2a} rises rather sharply with increasing N_e, 2s self-absorption may be large at slightly higher densities than those in Table 3.

[1] VV 8 may not be a planetary nebula. O'Dell (1966) has discovered an absorption-line spectrum in this object resembling a G2I star and suggests that it may be a symbiotic star.

V. EFFECTS OF INTERNAL DUST

We have also calculated the effects on the hydrogen-recombination spectrum produced by absorption and scattering of internal dust. Although a great number of papers have compared theoretical properties of various kinds of dust with observations of interstellar extinction, polarization, or the diffuse sky background, rarely are theoretical values both of Q_{ext} and Q_{sca} (or γ) given over a large range of wavelengths. In particular, there is a lack of information regarding these parameters in the ultraviolet, where absorption of Lyman-line radiation is possible. For detailed calculations we are limited to the values of Q_{ext} and γ for pure graphite grains given by Wickramasinghe (1967). We have calculated the effect on the Balmer decrement of graphite grains with radius $a = 0.02\ \mu$ distributed uniformly throughout the nebula. Since the values of Q_{ext} and γ given by Wickramasinghe do not extend into the ultraviolet beyond Lyman α, it is necessary to extrapolate his values to $\lambda \sim 0.09\ \mu$. For this purpose, we have assumed that $Q_{ext} = a + b/\lambda(\mu)$ in the region of the Lyman lines, with $a = 3.93$, $b = -0.154$, $\gamma = 0.3$, and $a = -0.13$, $b = 0.37$, $\gamma = 0.4$. Both these extrapolations give essentially the same results for the Balmer decrement. The results by using the first set of parameters, are listed in Table 4 as a function of τ_d at Hα for two values of $\tau_{1\alpha}$. The trajectory on Figure 1 for both internal reddening curves in Table 4 would be roughly parallel to the Whitford reddening curve. This is due to the fact that both τ_d and γ increase with decreasing λ in the visible. There is the additional effect, however, that the Balmer-line intensities are all reduced by this dust. The factor is listed in Table 4 as Hβ/(Hβ)$_0$ and is noticeably less than unity for both values of $\tau_{1\alpha}$. Although neither extrapolation of the Mie theory for graphite particles results in large changes in the Balmer decrement, the effects of the dust on the Lyman lines could be larger for other grain models. (Unfortunately, the Mie theory would be incorrect if there were appreciable photoelectric interaction of the Lyman lines with the grain; this is quite possible for graphite.) The deviations of NGC 2371, NGC 2372, NGC 7009, and NGC 6210 in Figure 1 might be understood in terms of dust which has a high albedo for Balmer-line radiation and yet has a negligible differential effect on the Lyman lines. In such a situation the decrement should lie along one of the Balmer self-absorption curves in Figure 1. In this case, the stellar continuum might also be reflected by the nebula. For larger graphite particles (with radii $\gtrsim 0.11\ \mu$), Q_{ext} increases with λ across the Balmer lines. For these grains values of log (Hα/Hβ) less than 0.45 might be expected. Also, Balmer-line albedos for these larger graphite grains will be greater than about 0.5, but detailed Mie calculations have not been published. Additional accurate photoelectric measurements of the hydrogen-recombination spectrum in a variety of nebulae will be essential in order to decide among these possibilities.

TABLE 4

EFFECT OF SMALL GRAPHITE GRAINS ON DECREMENT

τ_d	LOG $\tau_{1\alpha}$=3 0			LOG $\tau_{1\alpha}$=4 0		
	log (Hα/Hβ)	log (Hβ/Hγ)	Hβ/(Hβ)$_0$	log (Hα/Hβ)	log (Hβ/Hγ)	Hβ/(Hβ)$_0$
0	0 469	0 345	1 000	0 461	0 334	1 000
.10	488	355	0 866	.477	.343	0 891
.25	511	367	0 715	498	354	0 756
0 50	0 542	0 379	0 544	0.528	0 370	0.593

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