THE INTERACTION OF MATTER AND RADIATION IN A HOT-MODEL UNIVERSE*

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Abstract. In this paper we continue the investigation initiated by Weymann as to the reason why the spectrum of the residual radiation deviates from a Planck curve. We shall consider the distortions of the spectrum resultıng from radiation during the recombination of a primeval plasma. Analytical expressions are obtained for the deviation from an equilibrium spectrum due to Compton scattering by hot electrons. On the basis of the observational data it is concluded that a period of neutral hydrogen in the evolution of the universe is unavoidable. It is shown that any injection of energy at $t > 10^{10}$ sec (red shift $z < 10^5$) leads to deviation from an equilibrium spectrum.

1. Introduction

More than 3 years have already passed since the discovery of residual radiation. In this time dozens of experimental papers have already appeared, direct measurements have been carried out, and upper limits to the temperature of the residual radiation have been obtained in the wavelength region between 50 cm to 5.6×10^{-2} cm. The combined results of this work are shown in Figure 1. The temperature of the radiation, according to the present results at 3.2, 1.58 and 0.86 cm is evaluated as $2.68 \begin{array}{l} +0.09 \\ -0.14 \end{array}$ K (Wilkinson, 1967; Stokes *et al.*, 1967).

Values of the excitation temperatures of the rotational levels of the CN, CH and CH^+ molecules, which are of twenty years standing, made it possible to put an upper limit on the temperature of the radiation in the microwave region (Thaddeus and Clauser, 1966) and to evaluate it at $\lambda = 0.263$ cm (Thaddeus and Clauser, 1967; Field and Hitchcock, 1966). This evaluation, together with new measurements in the millimetre range ($\lambda = 3.3$ mm) have established that the observed spectrum of residual radiation is fundamentally different from the spectrum of black-body radiation. Attempts to explain the observed background in the centimetre range by the integrated radiation of sources having an anomalous discontinuity in the spectrum in the 50–30 cm range are doomed to failure (see for example, Pariiskii, 1968). It seems also that attempts to explain the origin of residual radiation by the interaction of the optical radiation of hypothetical primeval stars with dust are impossible to uphold. In a previous paper (Zeldovich and Novikov, 1967) it was shown that such an explanation meets with significant difficulties.

The background radio emission has already given cosmologists significant information about the isotropy of the expansion of the universe (cf. Partridge and Wilkinson,

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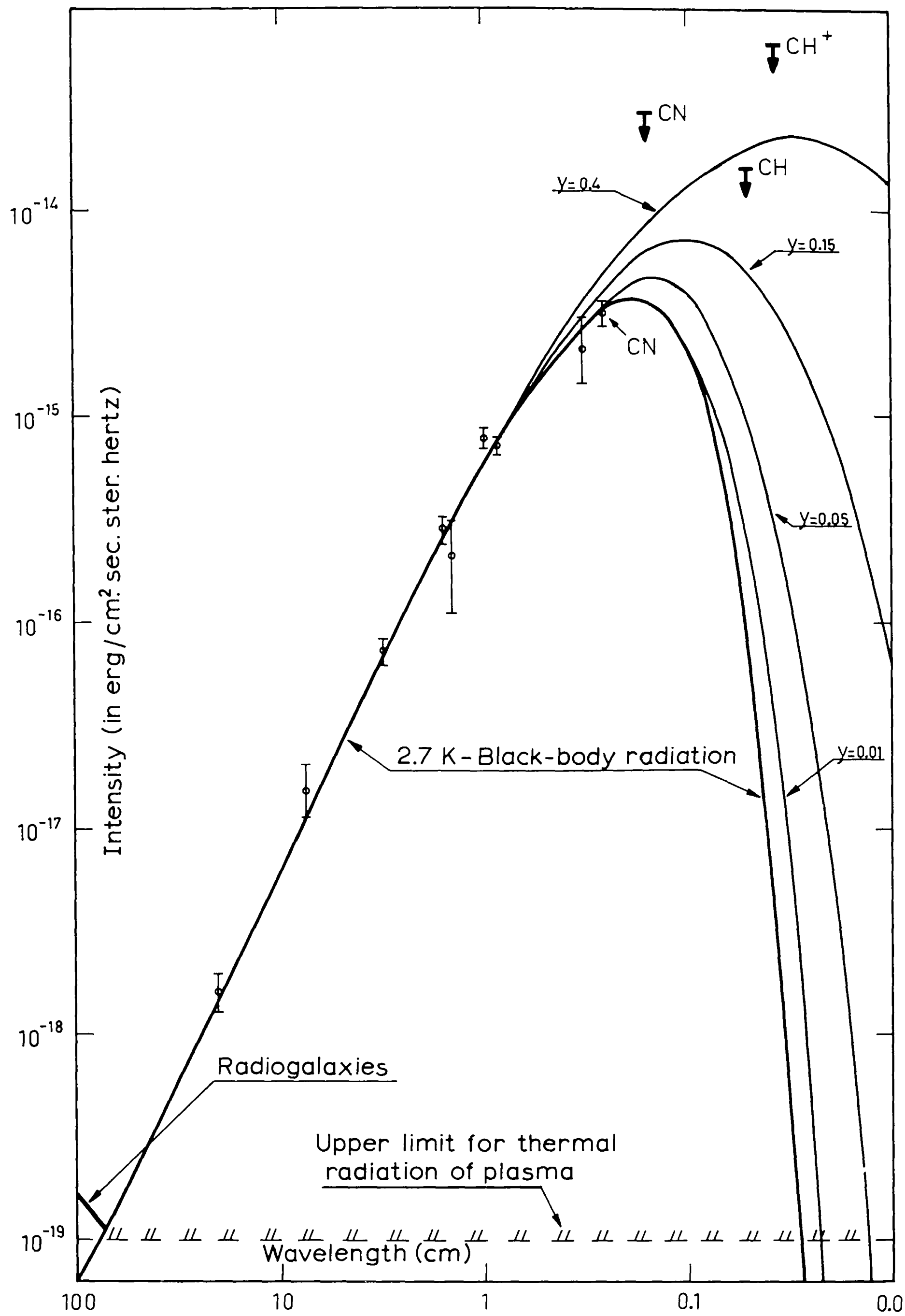

Fig. 1.

1967) and about the early stages of its evolution, supporting the hot universe model (cf. Dicke *et al.*, 1965).

An examination of the physical processes which lead to the formation of the spectrum of residual radiation shows that practically any energy produced during the expansion of the universe up to the time $t \sim 10^{10}$ sec will be transmuted in full; the plasma and the radiation will at all times be in thermodynamic equilibrium, the spectrum of the radiation will be Planckian. In the interval $10^{10} < t < 3 \times 10^{17}$ sec the interaction of radiation and electrons can no longer transform a non-Planckian spectrum to a Planckian one. From this it follows that the residual radiation presently observed was produced before the time $t \sim 10^{10}$ sec, corresponding to a redshift of $z \sim 10^5$.

In the following we shall accept the existence of full equilibrium for $z > 10^5$. This equilibrium is preserved up to $z \sim 1500$. The point is that, in agreement with the hot universe model, in the early stages of expansion the fully ionized plasma existed in equilibrium with the radiation. Later cooling of the plasma must have led to the recombination of hydrogen accompanied by the distortion of the residual radiation spectrum in its 'blue' region (at $h\nu/\kappa T_r > 30$, that is at $\lambda < 0.02$ cm for a present-day observer). We shall introduce the results of the calculations carried out by Zeldovich *et al.* (1968).

In the process of further expansion of a strictly homogeneous model, the hydrogen would necessarily remain neutral. However, we know that in fact matter combines to form stars, galaxies and clusters of galaxies – in other words, it is known that a significant inhomogeneity plays a part. It seems that such inhomogeneity leads to the ionization of the gas possibly even at a much earlier stage than that in which present-day structure was formed.

Observations in our neighbourhood up to the region corresponding to a redshift of $z \sim 2$ have not disclosed significant densities of neutral hydrogen. At the same time, they do not rule out the existence of a hot ionized metagalactic gas*. This supports the hypothesis regarding the ionization of the gas. The existence of the metagalactic gas is compatible with the hot universe model if we postulate a second heating and ionization of the gas, which no longer exists in equilibrium with the residual radiation afterwards.

A very serious qualitative question now arises: Is it possible to maintain that there existed a stage of neutral hydrogen between the period of equilibrium ionized gas at $z \sim 1500$ and the non-equilibrium ionized gas postulated at the present time? At what moment, on the average (at what z) did the second ionization take place? Weymann (1966) has noted that the radiation of the gas after ionization together with Compton scattering of the quanta from electrons should lead to significant distortions of the residual radiation spectrum and should convey information about the time of heating of the gas.

We have been able to find an analytical expression for the distortion of the spectrum as a result of Compton effect and to prove on the basis of existing measurements that a neutral hydrogen period is unavoidable in the evolution of the universe. A part of the results presented here has been previously published by Zeldovich *et al.* (1968) and Sunyaev (1968).

* Moreover, the most recent results from observations of the soft X-ray background are interpreted as revealing a metagalactic gas with a temperature in the range 10^5–10^6 K and with a density of the order of the critical density $\varrho_{\mathrm{crit.}} = 3\, H_0^2/8\pi G = 2 \times 10^{-29}$ g/cm^3 (cf. Henry *et al.*, 1968; Veinstein and Sunyaev, 1968). It is true, as is pointed out by one of the authors (R.S.), that the existence of very extended bridges between galaxies and the distribution of neutral hydrogen on the periphery of galaxies casts doubt upon such an interpretation (Sunyaev, 1969).

2. The Distortion of the Spectrum during the Recombination of Hydrogen

In the hot universe model it is proposed that in the early stages of expansion, the fully ionized plasma is found in equilibrium with radiation. Cooling during expansion leads to recombination. In accordance with Saha's equation, a state of ionization of 50%: $e=p=H$, is achieved at $T_r \sim 4000$ K at a period corresponding to a red shift $z \sim 1500$.

The density of quanta n_γ in the universe is much larger than the density of ions, electrons and atoms

$$n_\gamma/(p + H) = 0.244/\Omega n_{\text{crit.}} (\kappa T_r/hc)^3 = 4.15 \times 10^7 \Omega^{-1},$$

where

$$\Omega = \varrho/\varrho_{\text{crit.}} = n/n_{\text{crit}}.$$

However, the temperature at which the recombination effectively proceeds is significantly less than the temperature of ionization and excitation of the first level in a hydrogen atom

$$\kappa T_r \simeq I/40 \simeq h\nu_\alpha/30,$$

where ν_α is the frequency of a Lα quantum. Therefore, in the period of recombination, the density of energy quanta with $\nu > \nu_\alpha$ comprises a small part of the general quantum density, and represents a density less than that of ions and atoms by a factor of about 200.

V. G. Kurt has noted that if every recombination were accompanied by the emission of an energetic quantum, then the density of these quanta would rise sharply in comparison with that under the Planck spectrum. Since all wavelenghts have increased by a factor of 1500 from the time of recombination, it would be legitimate to expect an anomaly in the residual spectrum for wavelengths $\lambda \sim 10^{-2}$ cm.

This question has been investigated in detail by Zeldovich *et al.* (1968).* The hypothesis of production of an energetic quantum at each recombination is not accurate in the given situation, since there is absorption of Lα and photons of higher energy. As a result of this there occurs a marked slowing of the recombination of electrons and protons, since a super-equilibrium concentration of photons with $\nu > \nu_\alpha$ engenders a super equilibrium concentration of excited hydrogen atoms, which are easily ionized by soft photons ($I_2 = \frac{1}{4}I = 3.4$ eV).

A decrease in the concentration of protons and electrons existing in thermodynamic equilibrium with excited states occurs principally by the two-quantum decay of the metastable 2s level of the hydrogen atom. The production of energetic quanta in the two-quantum decay of the 2s level should distort the appearance of the spectrum of the observed residual radiation in the microwave range. Zeldovich *et al.* (1968)

* The results of this work were discussed by Novikov and Zeldovich (1967), and Shklovskii (1967). The results of Peebles (1968) are in good agreement with ours.

carried out a calculation of the distortion of the spectrum for the case $T_r = 3$ K. In Figure 2 curves showing the distortion of the spectrum for $T_r = 2.7$ K are presented. A marked deviation $\Delta J/J \sim 30\%$ may already be expected at $\lambda \sim 230\ \mu$. For $\lambda < 230\ \mu$ the recombination radiation always markedly outweighs the equilibrium radiation. Clearly the radiation of radio-galaxies and stars in the infra-red is markedly less.

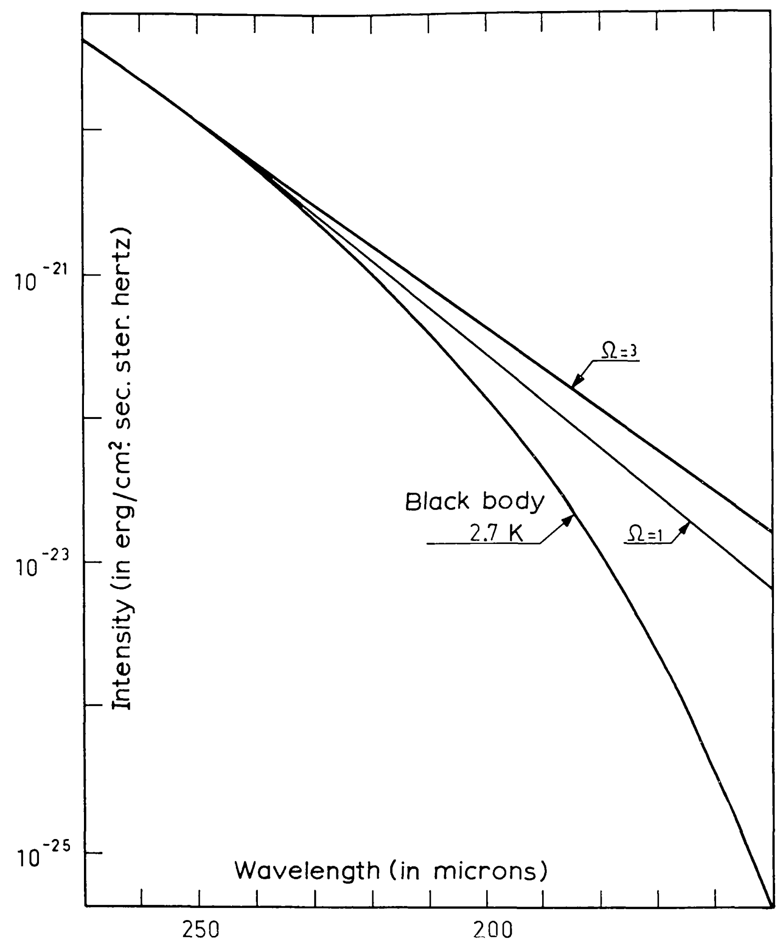

Fig. 2.

Estimates of the thermal radiation of the metagalactic plasma and of the distortion due to Compton effect from thermal electrons show that if the heating of the gas occurred at $z < 20/\Omega$, then in the region $\lambda \sim 240\ \mu$ these would be considerably weaker than the recombination radiation. Only the radiation of dust in galaxies (existing theoretical evaluations of the intensity of this radiation are extremely uncertain) and the radiation of infra-red objects may prove to be stronger.

Measurements in the wavelength region $\lambda \sim 200\ \mu$ are very interesting since, first of all, they can give information about the period $z \sim 1000$, and, secondly, they enable the evaluation of the density of matter in the universe. As is seen in Figure 2, the flux of recombination radiation essentially depends on the density of matter.

3. Bremsstrahlung Radiation

It is known that if ionization is determined by electron impacts, then up to a temperature of $T_e \sim 10^4$ K a hydrogen plasma will remain practically neutral, while at greater temperatures, the degree of ionization quickly increases. This fact, together with the observed low brightness temperature of the thermal radiation of the gas (see below) testifies to its low optical depth up to the period corresponding to a red shift $z \sim 10^4$. For this reason, in the following evaluation of the thermal radiation of the metagalactic plasma we shall not take into account self-absorption, that is, we shall use the formula for the radiation of a thin sheet.

The volume emissivity coefficient for free-free transitions (cf. Allen, 1963) is

$$E_{\mathrm{ff}}(\nu) = 5.44 \times 10^{-39} \, g T_e^{-1/2} e^{-h\nu/\kappa T_e} n_e^2 \, \frac{\mathrm{erg}}{\mathrm{cm}^3 \ \mathrm{sec\ ster\ Hz}}, \tag{1}$$

where the Gaunt factor for $h\nu \gg \kappa T_e$ is $g = 1$ and for $h\nu \ll \kappa T_e$

$$g = \frac{\sqrt{3}}{\pi}\left(\ln \frac{4\kappa T}{h\nu} - 0.577\right)$$

according to Karzas and Latter (1961).

Our basic interest will be in the case $h\nu \ll \kappa T_e$ when the spectrum of radio-emission of the metagalactic plasma is flat, facilitating its identification

$$E_{\mathrm{ff}}(\nu) = 5.44 \times 10^{-39} \, g T_e^{-1/2} n_e^2 \, \frac{\mathrm{erg}}{\mathrm{cm}^3 \ \mathrm{sec\ ster\ Hz}}. \tag{2}$$

Since the spectrum of residual radiation has a maximum in the wavelength range $\lambda \sim 0.1$ cm, it follows that the traces of thermal radio emission of the plasma should be sought in the long wavelength $\lambda \sim 50$ cm and short wavelength $\lambda \sim 0.03$ cm regions.

In the wavelength region $\lambda \sim 50$–100 cm the contribution of the combined radio emission of discrete sources, galaxies, radiogalaxies and quasars to the radio background of the universe is comparable to the contribution due to residual radiation. According to Bridle (1967) the brightness temperature of the extragalactic component of the radio background at 170 cm amounts to 30 ± 7 K. Its spectral index is 0.7 ($T_b \sim 3$ K at $\lambda = 75$ cm).

The contribution of radiation from our Galaxy to the radio-background strongly depends on direction and is comparable to the contribution due to residual radiation up to $\lambda \sim 40$–60 cm. The measurements by Howell and Shakeshaft (1967) at the wavelengths $\lambda = 75$ cm and $\lambda = 50$ cm and for $\lambda = 15$, 20, and 30 cm by Peljushenko *et al.* (1969) made it possible to set an upper limit to the intensity of thermal radio emission of the metagalactic plasma

$$J_\nu < 10^{-19} \, \frac{\mathrm{erg}}{\mathrm{cm}^2 \ \mathrm{sec\ ster\ Hz}} \quad \left(T_e(50\ \mathrm{cm}) < 1\ \mathrm{K}\right).$$

4. The Distortion of the Spectrum of Residual Radiation as a Result of Compton Scattering from Thermal Electrons

This equation has been examined in detail by Weymann (1966) who calculated numerically several variants of the distorted spectrum. We were able to obtain an analytical expression for the distortion of the spectrum due to Compton effect of thermal electrons (the derivation is presented in the appendix). In the derivation we used formulae obtained earlier by Kompaneetz (1956) and by Weymann (1965).

For the time change of the occupation index

$$n(\nu, T) = \left(\frac{8\pi h\nu^3}{c^3}\right)^{-1} \frac{\mathrm{d}E}{\mathrm{d}\nu}$$

in the non-relativistic approximation $\kappa T_e \ll m_e c^2$ and $\kappa T_r \ll m_e c^2$ (in this case the change of frequency of a quantum in each scattering process is due to the thermal velocity of the electron, that is, as a result of the Doppler effect) both Kompaneetz and Weymann give the formula

$$\frac{\partial n}{\partial t} = \frac{\kappa T_e}{m_e c^2} \frac{n_e \sigma_e}{c} (x^1)^{-2} \frac{\partial}{\partial x^1} (x^1)^4 (\partial n/\partial x^1 + n + n^2), \tag{3}$$

where T_e and T_r are the electron and radiation temperatures, n_e is the electron density, σ is the Thomson cross-section, and $x^1 = h\nu/\kappa T_e$. In the case $T_e \gg T_r$ this equation simplifies to the form

$$\frac{\partial n}{\partial t} = \frac{\kappa T_e}{m_e c^2} \frac{n_e \sigma_e}{c} x^{-2} \frac{\partial}{\partial x} x^4 \frac{\partial n}{\partial x}. \tag{4}$$

It is useful to change to the variable $x^1 = h\nu/\kappa T_r$ since during the expansion of the universe the frequency and temperature of the radiation change according to the same rule $\nu \sim (1+z)$ and $T_r \sim (1+z)$. Now it is possible to introduce, following Kompaneetz, the dimensionless variable

$$y = -\int_{t_0}^{t} \frac{\kappa T_e}{m_e c^2} n_e \sigma_0 c \mathrm{d}t = \int_0^{\tau} \frac{\kappa T_e}{m_e c^2} \mathrm{d}\tau, \tag{5}$$

where $\tau = \int \sigma_0 n_e \, \mathrm{d}l$ is the optical depth due to Thompson scattering. Going from t and n_e to the red shift z and the dimensionless density $\Omega = \varrho/\varrho_{\mathrm{crit.}}$ we have

$$\frac{\partial n}{\partial y} = x^{-2} \frac{\partial}{\partial x} x^4 \cdot \frac{\partial n}{\partial x} \tag{4'}$$

$$y = \Omega n_{\mathrm{crit.}} \sigma_0 c H_0^{-1} \int_0^{z_{\max}} \frac{\kappa T_e(z)}{m_e c^2} \cdot \frac{(1+z)}{\sqrt{(1+\Omega z)}} \mathrm{d}z, \tag{5'}$$

where H_0 is the present value of Hubble's constant; it is taken that

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{c H_0^{-1}}{(1+z)^2 \sqrt{(1+\Omega z)}}.$$

The Equation (5′) is easily solved for small values of the parameter y when the deviations from a Planck curve are small, by inserting in the right-hand side the unperturbed Planck function $n_0(x)=1/(e^x-1)$. Then, as shown in the Appendix, the relative distortion of the spectrum is expressed by the equation

$$\frac{\Delta n}{n_0}=\frac{\Delta J}{J_0}=xy\,\frac{e^x}{e^x-1}\left\{\frac{x}{\tanh(x/2)}-4\right\}. \tag{6}$$

The physical meaning of the degree of dependence of the distortion upon frequency is clear; conservation of the overall number of quanta is accompanied by an increase in the mean energy of a photon; at the same time the flux in the Rayleigh-Jeans domain decreases ($x<3.83$) and increases in the microwave domain. But we know neither the original nor the distorted flux. We determine the temperature of the residual radiation by measurements in various parts of the spectrum; with greatest accuracy in the Rayleigh-Jeans domain. It is useful to transfer our attention to an expression into which enter only measured values of temperature. Taking into account the decrease of temperature in the Rayleigh-Jeans region, we obtain (see Appendix)

$$\frac{T(x)-T_{\mathrm{RJ}}}{T_{\mathrm{RJ}}}=y\left\{\frac{x}{\tanh(x/2)}-2\right\};\quad \frac{J(x)-J(x,T_{\mathrm{RJ}})}{J(x,T_{\mathrm{RJ}})}=\frac{xye^x}{e^x-1}\left\{\frac{x}{\tanh(x/2)}-2\right\}. \tag{7}$$

The amount of distortion, as expected, increases with increasing frequency. In the microwave domain for large x, where even $y\ll1$ may lead to marked deviations from the Planck curve, as is shown in the Appendix

$$\frac{J(x)}{J(x,T_{\mathrm{RJ}})}=\frac{1}{\sqrt{(1-\eta_0)}}\exp\left(x+\frac{(\eta_0-\eta_0^2/2)}{2y}\right)$$

where

$$-\eta_0\,e^{-\eta_0}=2xye^y. \tag{8}$$

This formula is accurate for $y<1$. Finally, for a given y, the following relation holds true for the energy density of the radiation

$$E=\frac{4\pi}{c}\int_0^\infty J_\nu \mathrm{d}x=\sigma T_{\mathrm{RJ}}^4 e^{12y}. \tag{9}$$

Part of the exponential index, $8y$, is connected with the fact that the observed temperature in the Rayleigh-Jeans domain is lower than the original temperature $T_{\mathrm{RJ}}=T_0e^{-2y}$; and the second part, $4y$, owes its existence to the increase in the mean photon energy in the Compton process $E=E_0e^{4y}=\sigma T_0^4e^{4y}$. The lowest values of y from present observations are obtained from the Equations (7) and (8) using the given measurements of the excitation temperature of the levels of the molecule CN and the direct measurements of the flux at the wavelength $\lambda=0.33$ cm. The latest measurements by Bortolot *et al.* (1969) gave $T_r=2.83\pm0.15$ K at $\lambda=0.263$ cm and from the

Princeton measurements it follows that, at $\lambda=0.33$ cm, $T_r=2.46\pm0.44$ K. These results agree within measurement errors with the measurements in the Rayleigh-Jeans region $T_{RJ}=2.68\begin{array}{c}+0.09\\-0.14\end{array}$ K; therefore, it cannot be ruled out that $y=0$. However, within these errors the results do not exclude values of $y<0.15$ either, which can lead to a marked distortion of the spectrum in the microwave domain and to an enormous ($e^{12y}\sim6$) increase in the energy density of radiation

$$0.25\ \mathrm{eV/cm^3} \leqslant E < 1.5\ \mathrm{eV/cm^3}.$$

In this way, the lack of exact measurements of the residual radiation in the region of the maximum leads to a large indeterminacy in the most important quantity E. At the present time, the calculation of the X-ray emission and thermal balance of the metagalactic gas, both linked to the inverse Compton effect, are based upon the value of $E=0.25$ eV/cm^3.

5. The Inevitability of a Period of Neutral Hydrogen in the Evolution of the Universe

It has already been mentioned in the second section in the case of $z\sim1500$, that an agreement with the hot universe model required that a recombination of the hydrogen should take place. In further expansion the hydrogen was to remain neutral. However, observations in the 21-cm line and measurements of Lα absorption in the spectra of distant quasars indicate the absence of neutral metagalactic hydrogen.

Let us make a rough estimate of the z_{max} at which the second heating took place. Previously, we showed that the heating of the gas could not have taken place earlier than $z\sim300$ (Sunyaev, 1968). In fact, the existence of an experimental limit to the radio-emission of the gas (see Section 3)

$$\int \frac{E_{ff}\, \mathrm{d}l}{(1+z)^3} < J_\nu, \tag{10}$$

and the limit to the energy losses of the gas by the most efficient inverse Compton-effect of thermal electrons on the residual radiation

$$\int L^- \,\mathrm{d}t < w \tag{11}$$

enable us to set boundaries to the period of existence of the non-equilibrium ionization of the gas. In Equation (11) in agreement with Weymann (1965)

$$L = 4\sigma_0 \frac{\kappa(T_e - T_r)}{m_e m_p c}\, \sigma T_r^4 = 3\times10^{-12}(1+z)^4\, T_e(z)\, \frac{\mathrm{erg}}{\mathrm{g\ sec}}, \tag{12}$$

and E_{ff} in (10) is replaced by (2) in which the Gaunt factor is taken as $g=10$, since the measurements are carried out in decimetre region with a gas temperature of the order 10^5–10^6 K. The inequalities (10) and (11) give us a system of two functionals

$$\Omega^2 \int_0^{z_{\max}} \frac{1+z}{\sqrt{(1+\Omega z)}} T_e^{-1/2}(z)\, dz < 1.8 \times 10^{19} J_\nu , \tag{10'}$$

$$\int_0^{z_{\max}} \frac{(1+z)^2}{\sqrt{(1+\Omega z)}} T_e(z)\, dz < 10^{-6} w . \tag{11'}$$

The bremsstrahlung radiation of an ionized plasma is strong for a low electron temperature, and Compton losses are significant at high temperatures. The system of functionals (10′) and (11′) enables us to find the extremal function

$$T_e(z) = 1.4 \times 10^{-18} (w/J_\nu)^{2/3}\, \Omega^{4/3} z^{-2/3} ,$$

and the highest possible value of the moment of heating $z_{\max} = 1.1 \times 10^5\ (J_\nu{}^4 w^2/\Omega^5)$. In the case of any other dependence $T_e(z)$ for the same values J_ν and w one of the inequalities (10) or (11) leads to lower values of z for the heating. Sunyaev (1968), assuming only nuclear sources of energy and an original chemical composition corresponding to a hot universe model, pointed out the inevitability of a period of neutral hydrogen in the evolution of the universe ($z_{\max} < 300$). Recently, however, the interesting work of Ozernoi and Chernin (1967) appeared in which attention is brought to the possibility of the heating of the gas by residual turbulence and to the accompanying production of energy, markedly exceeding both the nuclear energy and the rest mass energy (mc^2). In such a theory the criteria (11) disappear. Compton effect of electrons heated by turbulence (or by other sources of unlimited – or more accurately – unknown potential) on the residual radiation should distort its spectrum. The presence of an upper limit to the distortion and to the 'effective scattering thickness' (Section 4)

$$y = \sigma_0 \Omega n_{\mathrm{crit.}} c H_0^{-1} \frac{\kappa}{m_e c^2} \int_0^{z_{\max}} \frac{1+z}{\sqrt{(1+\Omega z)}} T_e(z)\, dz < 0.15 , \tag{13}$$

together with the limit to bremsstrahlung radiation of the gas leads to a second system of functionals:

$$\Omega^2 \int_0^{z_{\max}} \frac{1+z}{\sqrt{(1+\Omega z)}} T_e^{-1/2}(z)\, dz < 1.8 \times 10^{19} J_\nu , \tag{10'}$$

$$\Omega \int_0^{z_{\max}} \frac{1+z}{\sqrt{(1+\Omega z)}} T_e(z)\, dz < 9 \times 10^{10} y . \tag{13'}$$

The extremal of this system turns out to be the function

$$T_e(z) = 2.9 \times 10^{-6} (\Omega y/J_\nu)^{2/3} ,$$

(that is, a constant temperature, independent of z) and

$$z_{\max} = 1.3 \times 10^{11} J_\nu^{4/9} y^{2/9} \Omega^{-7/9}.$$

Using the existing limits

$$J_\nu < 10^{-19} \frac{\text{erg}}{\text{cm}^2 \text{ sec ster Hz}}$$

and $y < 0.15$, we obtain $z_{\max} < 300\ \Omega^{-7/9}$ while $T_e = 3.8 \times 10^6\ \Omega^{2/3}$ K. In this way, if the mean density of matter in the universe exceeds $\varrho = 0.12\ \varrho_{\text{crit.}} \simeq 2 \times 10^{-30}$ gm/cm^3 then from the observations follows the inevitability of a period of neutral hydrogen in the universe. During this loss of energy, the coincident outlay of energy to heat the matter has an upper bound $w < 2.5 \times 10^{18}\ \Omega^{-16/9}$ erg/gram.

6. The Formation of the Spectrum of Residual Radiation

Any production of energy during the expansion of the universe, leading to the raising of the temperature of the plasma also distorts the spectrum of the residual radiation. However, there exists a moment t_1 such that the interaction of the mass and radiation in further expansion erases all distortion and makes the spectrum of the radiation closely Planckian. However, heating of the electrons after this moment leads to deviation from the Planck curve. As Weymann and Kompaneetz have shown, Compton scattering of radiation from Maxwellian electrons with $T_e > T_r$ leads to the establishment of a Bose-Einstein distribution with a smaller number of photons than in the Planck distribution

$$n(x) = (e^{\alpha + x} - 1)^{-1}, \quad \alpha \geqslant 0, \tag{14}$$

in which the time taken to approach a quasi-equilibrium distribution (Sunyaev, *op. cit.*) is characterised by the quantity $y' = 4y \cong 1$, where y is given by the formula (5). Under the conditions of an expanding universe with a density of matter $\varrho = \Omega \varrho_{\text{crit.}}$ and with an electron temperature $T_e \simeq T_r(1+z)$, y' takes a value of the order unity for $z \sim 10^4\ \Omega^{-1/5}$.

We notice that for $z_0 = 4 \times 10^4 \Omega$ the energy densities of matter and radiation become equal, and for $z > z_0$ the rate of expansion changes, it proceeds more rapidly, that is, as

$$\mathrm{d}t/\mathrm{d}z \sim z^{-3} \quad \text{instead of} \quad z^{-2.5}.$$

An account of this fact in the case $\Omega \ll 1$ results in an increase in the value of z for which $y' \sim 1$ is achieved.

The spectrum of the residual radiation observed at the present time is closely Planckian. In order to obtain it from a Bose-Einstein distribution, a creation of photons is unavoidable, since the Compton effect leaves their number unchanged. We neglect the double Compton effect. At sufficiently low frequencies the optical depth due to bremsstrahlung processes is always large and such photons exist in thermodynamic equilibrium with electrons. Following Kompaneetz, it is possible to find the frequency limit x_0, at which the Compton process acts upon a photon more quickly than it (the photon) can be absorbed as,

$$y'/\tau' \sim 1\,, \tag{15}$$

where $\tau' = \int \kappa(\nu)\, n_e^2\, \mathrm{d}l$ is the optical thickness due to bremsstrahlung processes. The absorption coefficient (cf. Allen, 1963)

$$\kappa(\nu) = 3.8 \times 10^8 \frac{g(\nu)}{T_e^{1/2} \nu^3} \left(1 - e^{-h\nu/\kappa T_e}\right).$$

The relation (15) is satisfied for

$$x_0 (\ln 4/x_0)^{-1/2} \approx 80\, z^{-3/4} \Omega^{+1/2}, \quad z > 10^3. \tag{15'}$$

Knowing the overall number of photons in the residual radiation $N = 0.244\ (2\pi\kappa T_r hc/)^3$ it is possible to find the red shift corresponding to the moment t_1 which we are seeking by solving the equation

$$\frac{4\pi\Omega n^2_{\text{crit.}}}{h} \int_0^{z_1} \frac{\int_{z_0}^{\infty} \frac{E_{\text{ff}}(z)}{x}\, \mathrm{d}x}{N(z)} \frac{\mathrm{d}t}{\mathrm{d}z}\, \mathrm{d}z \sim 1; \quad \Omega^2 z_1^{1/2} \ln^2 \frac{z_1 \Omega^{-2/3}}{500} \approx 10^4\,, \tag{16}$$

where x_0 is determined by the relation (15) and $E_{\text{ff}} = \kappa(\nu)\,(B_\nu - J_\nu)$ by formula (1). From (16) $z_1 = 10^5\ \Omega^{-4}$, and $t_1 \simeq 4 \times 10^9\ \Omega^{+8}$ sec. Thus, practically any production of energy during the process of expansion of the universe up to the moment $t_1 \sim 4 \times 10^9$ sec will be fully transformed; the plasma and radiation will at all times be the thermodynamic equilibrium, and the spectrum of the radiation will be Planckian. In the interval $10^{10} < t_1 < 3 \times 10^{17}$ sec the interaction of the radiation with the electrons cannot any longer change a non-Planckian spectrum to a Planckian one.

From this it follows that the residual radiation presently observed was formed up to the moment $t \sim 10^{10}$ sec, corresponding to a red shift $z \sim 10^5$. For $\Omega \ll 1$ the momen tis $t_1 \ll 10^{10}$ sec, however it is clear that the intensive production of electron-positron pairs for $t < 300$ sec rarely increases the optical depth due either to bremsstrahlung or Compton processes.

Appendix

(a) The formula for the distortion of the spectrum by Compton scattering of thermal electrons (for small distortions). If we introduce the variables

$$y = \int_0^{z_{\max}} \frac{\kappa T_e(z)}{m_e c^2} \frac{\mathrm{d}\tau}{\mathrm{d}z}\, \mathrm{d}z\,, \qquad \text{(see Section 4)}$$

(τ is the optical thickness due to Compton effect) and $x = h\nu/\kappa T_r$ (clearly independent of the magnitude of the redshift at which the scattering occurred) it is then easy to obtain, for $T_e \gg T_r$ from the expressions written down by Weymann and Kompaneetz, the following equation for the occupation index

$$n(x, t) = \frac{c^3}{8\pi h\nu^3} \frac{dE}{d\nu}$$

($dE/d\nu$ is the spectral density of the radiation energy), and

$$\frac{\partial n}{\partial y} = x^{-2} \frac{\partial}{\partial x} x^4 \frac{\partial n}{\partial x}. \tag{I}$$

Taking it that the deviations from the Planck function are small, we substitute in the right-hand side of (I) $n_0 = (e^x - 1)^{-1}$. Then

$$\frac{\partial n}{\partial y} = \frac{e^x}{(e^x - 1)^2} x \left\{ \frac{x}{\tanh(x/2)} - 4 \right\}$$

and

$$\frac{\Delta n}{n_0} = \frac{\Delta J}{J_0} = \frac{e^x}{e^x - 1} xy \left\{ \frac{x}{\tanh(x/2)} - 4 \right\}. \tag{II}$$

The distortion in the Rayleigh-Jeans region of the spectrum is found by letting $x \to 0$ to be

$$\frac{\Delta J}{J} \underset{x \to 0}{\to} -2y. \tag{III}$$

From experiments, the temperature in the Rayleigh-Jeans region of the spectrum is known, therefore it is better to present (II) in the form

$$\frac{\Delta n}{n(x, T_{RJ})} = \frac{\Delta J}{J(x, T_{RJ})} = \frac{xye^x}{e^x - 1} \left\{ \frac{x}{\tanh(x/2)} - 2 \right\}. \tag{IV}$$

Since

$$\frac{\Delta J}{J} = \frac{d \ln J}{d \ln T} \cdot \frac{\Delta T}{T},$$

then

$$\frac{\Delta T}{T_{RJ}} = y \left\{ \frac{x}{\tanh(x/2)} - 2 \right\}. \tag{V}$$

(b) The formula for the distortion of the spectrum through the Compton effect of thermal electrons (the asymptote for large x). With the substitution $\xi = \ln x$

Equation (I) becomes

$$\frac{\partial n}{\partial y} = 3\frac{\partial n}{\partial \xi} + \frac{\partial^2 n}{\partial \xi^2};$$

and going over to the variables $z=3y+\xi$, y we can reduce the latter equation of heat conduction

$$\frac{\partial n}{\partial y} = \frac{\partial^2 n}{\partial z^2}. \tag{VI}$$

The solution of Equation (I), (VI) with the initial condition $n(x, y=0)$ takes the form of a Planck function, and has the form

$$n(x, y) = \frac{1}{\sqrt{(4\pi y)}} \int_{-\infty}^{+\infty} n(t) \exp\left[-(\ln x + 3y - t)^2/4y\right] \mathrm{d}t.$$

It is easy to find the form of the renormalization factor necessary for the calculations of the distortion in the Rayleigh-Jeans region. For Compton effect in the Rayleigh-Jeans region $n(x)=1/x$, $n(t)=e^{-t}$, it is necessary to set $n(x')=1/x'$,

$$\begin{aligned} n(x, y) &= \frac{1}{\sqrt{(4\pi y)}} \int_{-\infty}^{+\infty} n(t) \exp\left[-(\ln x + 3y - t)^2/4y\right] \mathrm{d}t \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{(4\pi y)}} \exp\left[-y - (\ln x + 3y - t)^2/4y\right] \mathrm{d}t \\ &= \frac{1}{\sqrt{(4\pi y)}} e^{-2y - \ln x} \int_{-\infty}^{+\infty} \exp\left[-(t - \ln x - y)^2/4y\right] \mathrm{d}t = x^{-1} e^{-2y}; \end{aligned}$$

from which an obvious renormalization is (it is interesting to compare with (III))

$$x' = xe^{2y}. \tag{VII}$$

Now

$$n(x', y) = \frac{1}{\sqrt{(4\pi y)}} \int_{-\infty}^{+\infty} n(t) \exp\left[-(\ln x' - t + y)^2/4y\right] \mathrm{d}t,$$

and the substitution $\eta = t - \ln x' - y$ gives

$$n(x', y) = \frac{1}{\sqrt{(4\pi y)}} \int_{-\infty}^{+\infty} \frac{1}{\exp(xe^{\eta + y}) - 1} e^{-\eta^2/4y} \mathrm{d}\eta. \tag{VIII}$$

For $x \gg 1$ it is possible to neglect the quantity unity in the denominator. The function

$f = xe^{\eta+y} + \eta^2/4y$ has a minimum at

$$-\eta_0 e^{-\eta_0} = 2xye^y. \tag{IX}$$

Expanding it in a Taylor series about

$$f = f(\eta_0) + \tfrac{1}{2} f''(\eta_0)(\eta - \eta_0)^2,$$

we obtain

$$n(x', y) = \frac{1}{\sqrt{(4\pi y)}} \cdot \sqrt{\frac{2\pi}{f''(\eta_0)}} \, e^{-f(\eta_0)}.$$

Finally

$$\frac{J(x', y)}{J(x')} = \frac{1}{\sqrt{(1-\eta_0)}} \exp\left(\frac{(\eta_0 - \eta_0^2/2)}{2y} + x'\right), \tag{X}$$

where the parameter $\eta_0(x, y)$ is given by (IX). For $x \gtrsim 1$ the error is decreased by an introduction of the multiplier

$$(\exp\{xe^{\eta_0+y}\} - 1)^{-1}.$$

For $xy < \frac{1}{2}$ the exponent in f may be expanded in the series $e^{\eta+y} = 1 + \eta + y + \frac{1}{2}(\eta + y)^2 + \cdots$. Then for $x > 1$ we obtain

$$\frac{J(x', y)}{J(x')} = \frac{1}{\sqrt{(1 + 2xy)}} \exp\left[xy(x-1)/(1+2xy)\right]. \tag{XI}$$

The asymptotes of (X) and (XI) coincide with the exact solution (IV) for $y \to 0$.

(c) The change in total energy of the residual radiation due to Compton effect of thermal electrons.

Again we revert to the Equation (I)

$$\frac{\partial n}{\partial y} = x^{-2} \frac{\partial}{\partial x} x^4 \frac{\partial n}{\partial x}.$$

Since the expression for the total energy of the radiation takes the form

$$E = DT_4^0 \int nx^3 \, dx,$$

then multiplying both sides of (I) by x^3 and integrating with respect to x we find, following Weymann and Kompaneetz,

$$\frac{1}{DT_4^0} \frac{\partial E}{\partial y} = \frac{\partial}{\partial y} \int_0^\infty x^3 n \, dx = \int_0^\infty x \frac{\partial}{\partial x} x^4 \frac{\partial n}{\partial x} \, dx;$$

but

$$\int_0^\infty x \frac{\partial}{\partial x} x^4 \frac{\partial n}{\partial x} \, dx = x^5 \frac{\partial n}{\partial x} \Big|_0^\infty - \int_0^\infty x^4 \frac{\partial n}{\partial x} \, dx =$$

$$= -x^4 n \Big|_0^\infty + 4 \int_0^\infty x^3 n \, dx = 4 \frac{E}{DT_4^0}.$$

Finally, $E = E_0 e^{4y}$, where

$$E_0 = DT_4^0 \int_0^\infty \frac{x^3}{e^x - 1} \, dx$$

is the energy of Planckian radiation having the same temperature in the Rayleigh-Jeans region. Taking the renormalization $x' = xe^{2y}$, we have

$$E = \sigma T_{RJ}^4 e^{12y}. \tag{XII}$$

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