PERTURBATIONS OF A COSMOLOGICAL MODEL AND ANGULAR VARIATIONS OF THE MICROWAVE BACKGROUND

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ABSTRACT

We consider general-relativistic, spatially homogeneous, and isotropic $k = 0$ cosmological models with either pressure zero or pressure one-third the energy density. The equations for general linearized perturbations away from these models are explicitly integrated to obtain density fluctuations, rotational perturbations, and gravitational waves. The equations for light rays in the perturbed models are integrated. The models are used to estimate the anisotropy of the microwave radiation, assuming this radiation is cosmological It is estimated that density fluctuations now of order 10 per cent with characteristic lengths now of order 1000 Mpc would cause anisotropies of order 1 per cent in the observed microwave temperature due to the gravitational redshift and other general-relativistic effects. The $p = 0$ models are compared in detail with corresponding Newtonian models The perturbed Newtonian models do not contain gravitational waves, but the density perturbations and rotational perturbations are surprisingly similar.

I. INTRODUCTION

The actual Universe is quite lumpy, but the usual cosmological models assume a uniform distribution of matter (McVittie 1956; Heckmann and Schücking 1959; Bondi 1960; this group of authors is referred to hereinafter as "Group 1"). One simple method for making somewhat more realistic cosmological models is to consider linear perturbations away from spatially homogeneous isotropic models (Lifshitz 1946; Bonnor 1957; Lifshitz and Khalatnikov 1963; Irvine 1965; Peebles 1965; Hawking 1966; Silk 1966; this group of authors is referred to hereinafter as "Group 2"). In this paper we shall integrate the equations governing perturbations of an expanding Friedmann model. The background model has the spatial curvature parameter, $k = 0$, and pressure, p, either zero or $\rho/3$, where ρ is the density. The corresponding values of the deceleration parameter (see Group 1), q_0, of the background model are $+\frac{1}{2}$ for $p = 0$ and $+1$ for $p = \rho/3$. After finding the perturbations we shall integrate the lightlike geodesics of the perturbed model. We shall then use our model to estimate the temperature variations in angle induced by the gravitational effects of the perturbations on the microwave background radiation.

Because we assume $k = 0$, our calculations are less general than those given previously (see Group 2). The advantage is that in our case all the perturbation equations can be explicitly integrated in terms of elementary functions. The value $q_0 = +\frac{1}{2}$ is consistent with current observations (Sandage 1965), although not demanded by them.

The main mathematical result of this paper is the theorem of § II*c*.

II. INTEGRATION OF THE PERTURBATION EQUATIONS

a) *Unperturbed Models*

We shall use the units $c = 8\pi G = 1$ throughout. Latin indices run from 1 to 4; Greek indices from 1 to 3; the signature of the metric g_{ab} is taken as $-, -, -, +$; and the Minkowski metric is written as

$$\eta_{ab} = \eta^{ab} = \text{diagonal}\ (-1, -1, -1, +1)\ . \quad (1)$$

The signs of the Riemann and Ricci tensors are fixed by

$$v_{a;b;c} - v_{a;c;b} = v_i R^i{}_{abc},\ R^i{}_{aib} = -R_{ab}\ . \quad (2)$$

The Einstein field equations for a perfect fluid with density ρ, pressure p, and average world velocity u^a are

$$G^a{}_b = -(\rho + p)u^a u_b + p\delta^a{}_b, \; (u^a u_a = 1) \,. \tag{3}$$

The unperturbed $k = 0$ Friedmann-Tolman models are (Group 1)

$$ds^2 = a^2(\eta)[d\eta^2 - dx^2 - dy^2 - dz^2] = a^2(\eta)\eta_{ab}dx^a dx^b \,. \tag{4}$$

Here we use spatial coordinates $x^\mu = (x,y,z) = \mathbf{x}$ and we choose $\eta = x^4$. Let H be the Hubble parameter and H_R the Hubble parameter now; then $H = a'/a^2$, where the primes denote η-derivatives. The function $a(\eta)$, the pressure p, the density ρ, the cosmological proper time t, and the present value η_R of η are, for $p = 0$ or $\rho/3$, respectively, given by (Group 1)

$$p = 0 \,, \qquad a(\eta) = \frac{2\eta^2}{H_R} \,, \qquad \rho = \frac{3H^2{}_R}{\eta^6} \,, \qquad t = \frac{2\eta^3}{3H_R} \,, \qquad \eta_R = 1 \,, \tag{5}$$

$$p = \tfrac{1}{3}\rho \,, \qquad a(\eta) = \frac{\eta}{H_R} \,, \qquad \rho = \frac{3H^2{}_R}{\eta^4} \,, \qquad t = \frac{\eta^2}{2H_R} \,, \qquad \eta_R = 1 \,. \tag{6}$$

Thus we can regard η as a dimensionless variable that replaces the proper time and has value unity now. The variables x^μ are also dimensionless. The coordinates in which equation (4) holds are fixed uniquely up to the rigid rotations and translations of Euclidean 3-space, as in equation (23) below.

b) Field Equations for the Perturbations

In considering perturbations we shall continue to assume a perfect fluid with $p = 0$ or $p = \rho/3$, respectively. We emphasize that for $p = \rho/3$ this assumption is quite nontrivial because it involves neglecting transport processes.

We shall find it convenient to write the perturbations in the form

$$ds^2 = a^2(\eta)[\eta_{ab} + h_{ab}]dx^a dx^b \,. \tag{7}$$

Here $a^2(\eta)$ is to have the same functional form (5) or (6) that it does in the unperturbed models; $h_{ab}(x^d)$ is the small perturbation. Moreover, without loss of generality we can insist that the coordinates x^μ are (Lagrangian-type) comoving coordinates and that $d\eta$ is related to comoving proper time interval dt by the unperturbed equations (5) or (6), respectively. Two well-known (Ehlers 1961) formal characterizations of these coordinate conventions are

$$u^a = \frac{\delta^a{}_4}{a(\eta)} \Leftrightarrow G^\mu{}_4 = 0 \,, \qquad h_{44} = 0 \,. \tag{8}$$

We have chosen the coordinate conventions (8) because they have a direct meaning independent of any approximation scheme. In linear approximation we are then left with a restricted set of allowed "gauge transformations"

$$[x]^a = x^a - \xi^a(x^b) \,. \tag{9}$$

Here ξ^a is small in the same sense that h_{ab} is. A short calculation shows that in linear approximation the conventions (8) restrict the allowed form of ξ^a by either of the two equivalent conditions:

$$u^a{}_{,b}\xi^b - \xi^a{}_{,b}u^b = 0 \Leftrightarrow \xi^4 = \frac{b(x^\mu)}{a(\eta)} \,, \qquad \xi^\mu = c^\mu(x^\beta) \,, \tag{10}$$

where b and c^μ are arbitrary functions of the spatial coordinates x^β alone. The functional change induced in h_{ab} by the transformations (9) and (10) is the Lie derivative of g_{ab} with respect to ξ^a, namely,

$$[h]_{\mu\beta} = h_{\mu\beta} + c_{\mu,\beta} + c_{\beta,\mu} + 2\frac{a'}{a^2}\, b\,\eta_{\mu\beta}\,;$$
$$[h]_{\mu 4} = \frac{b_{,\mu}}{a} + h_{\mu 4}\,; \qquad [h]_{44} = h_{44} = 0\,. \tag{11}$$

In equation (11), and throughout this paper, all indices on h_{ab}, c_μ, and other small quantities are raised and lowered with the Minkowski metric η_{ab}, η^{ab}; thus $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}, c_\mu = \eta_{\mu\beta}c^\beta = -\delta_{\mu\beta}c^\beta$, etc. The conformal tensor, fluid shear tensor, and fluid vorticity tensor are gauge-invariant because their unperturbed values vanish (Hawking 1966; Sachs 1964).

We must now work out the Einstein tensor of the perturbed metric. One can proceed by force, computing the contravariant metric tensor, Christoffel symbols, Riemann tensor, and Einstein tensor of equation (7) while systematically throwing away all terms quadratic or higher in h_{ab}. A much faster, though conceptually more complicated, technique is to consider first the conformal metric $d\bar{s}^2 = (\eta_{ab} + h_{ab})dx^a dx^b$ and then use standard conformal methods (Jordan, Ehlers, and Kundt 1960). In any case, we find up to first order

$$G^a{}_b = {}_oG^a{}_b + \delta G^a{}_b\,, \tag{12}$$

with the unperturbed ${}_oG^a{}_b$ given by

$${}_oG^a{}_b = -F(\eta)\,\delta^a{}_4\delta^4{}_b - G(\eta)\,\delta^a{}_b\,, \qquad F = \frac{4a'^2}{a^4} - \frac{2a''}{a^3}\,, \qquad G = \frac{2a''}{a^3} - \frac{a'^2}{a^4}\,.$$

The reader should note that we have defined $\delta G^a{}_b$ as the first-order correction to the mixed form $G^a{}_b$ of the Einstein tensor and *not*, for example, as the correction to $\eta^{ab}G_{bc}$. One gets for $\delta G^a{}_b$

$$\delta G^a{}_b = \frac{\chi^a{}_b}{2a^2} + F(\eta)\,\delta^4{}_b h^{a4} - \frac{a'}{a^3}(h^4{}_b{}^{,a} + h^{4a}{}_{,b} - h^a{}_b{}') + \frac{a'}{a^3}(2h^{4\mu}{}_{,\mu} - h')\,\delta^a{}_b\,. \tag{13}$$

Here all indices on the right are, as mentioned above, raised and lowered and with η_{ab}, $h = h^i{}_i = h^\mu{}_\mu = \eta^{ab}h_{ab} = \eta^{\mu\beta}h_{\mu\beta}$, and $\chi^a{}_b$ is the familiar (Bergmann 1942) expression:

$$\chi^a{}_b = (h^a{}_b - h\delta^a{}_b)^{,i}{}_i + h^{,a}{}_b + h^{cd}{}_{,cd}\delta^a{}_b - h_{bd}{}^{,ad} - h^{ac}{}_{,bc}\,. \tag{14}$$

Because of equations (3) and (8) our linearized field equations are

$$\delta G^4{}_4 = -\delta\rho,\ \delta G^\mu{}_4 = 0,\ \delta G^\mu{}_\beta = \delta^\mu{}_\beta\delta p\,, \tag{15}$$

where $\delta p = 0$ or $\delta p = \delta\rho/3$ for $p = 0$ or $\rho/3$, respectively, and $\delta G^a{}_b$ is given by equation (13).

c) Solutions for the Perturbations

It turns out that when we assume suitable regularity conditions on $h_{\mu 4}$ and $h_{\mu\beta}$ we can find the general solution of equation (15). The method is to take first spatial Fourier transforms of $h_{\mu 4}$ and $h_{\mu\beta}$:

$$h_{\mu 4} = \int d^3k\, \mathfrak{h}_{\mu 4}(\boldsymbol{k},\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,,$$
$$h_{\mu\beta} = \int d^3k\, \mathfrak{h}_{\mu\beta}(\boldsymbol{k},\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{16}$$

Then one solves the field equations (15) and transforms back to position space. The regularity conditions we shall use are: (i) $\mathfrak{h}_{\mu 4}(\boldsymbol{k},\eta)$ and $\mathfrak{h}_{\mu\beta}(\boldsymbol{k},\eta)$ are generalized functions

(Lighthill 1958) that coincide with continuous ordinary functions near $\boldsymbol{k} = 0$; (ii) the quantities $\mathfrak{h}_{\mu\beta}k^{\mu}k^{\beta}$, $\mathfrak{h}_{\mu\beta}k^{\beta}$, $\mathfrak{h}^{\mu}{}_{\mu}$, and $\mathfrak{h}_{\mu 4}k^{\mu}$ shall admit representations

$$\begin{aligned} \mathfrak{h}_{\mu\beta}k^{\mu}k^{\beta} &= k^4 f(\boldsymbol{k},\eta), \qquad \mathfrak{h}_{\mu\beta}k^{\beta} = k^2 g_{\mu}(\boldsymbol{k},\eta)\,, \\ \mathfrak{h}^{\mu}{}_{\mu} &= k^4 j(\boldsymbol{k},\eta), \qquad \mathfrak{h}_{\mu 4}k^{\mu} = -ik^2 m(\boldsymbol{k},\eta)\,, \\ k^2 &= \boldsymbol{k}\cdot\boldsymbol{k} = -k_{\mu}k^{\mu}\,. \end{aligned} \tag{17}$$

Here f, g_{β}, j, and m are to be generalized functions that coincide (Lighthill 1958) with continuous ordinary functions in a neighborhood of $\boldsymbol{k} = 0$. We call equations (17) "moment conditions"; they are rather weak conditions on $h_{\mu 4}(\boldsymbol{x},\eta)$ and $h_{\mu\beta}(\boldsymbol{x},\eta)$.[1] Our solutions will, of course, contain some arbitrary functions of three variables, which are determined by initial conditions on the gravitational waves, density perturbations, and other perturbations that make up the most general perturbations. For $p = 0$ we need (i) two arbitrary "scalar"[2] functions A and B of x^{β} alone, which correspond to potentials for density perturbations; (ii) a "vector" function C_{μ} of x^{β} alone, restricted by the transversality condition

$$C^{\mu}{}_{,\mu} = 0 \tag{18}$$

which will presently be related to the perturbed rotation tensor; and (iii) an arbitrary transverse-transverse trace-free "tensor" solution $D_{\mu\beta}(x^{\alpha},\eta)$ of the flat-space d'Alembert equation, e.g.,

$$D_{\mu\beta} = D_{\beta\mu}, \qquad D_{\mu\beta}{}^{,\beta} = 0\,, \qquad D^{\mu}{}_{\mu} = 0\,, \qquad \left(\frac{\partial^2}{\partial\eta^2} - \nabla^2\right) D_{\mu\beta} = 0\,, \tag{19}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{20}$$

Giving $D_{\mu\beta}$ is equivalent to giving four functions of x^{μ} alone. For the case $p = \rho/3$, $\delta p = \delta\rho/3$, we need a vector C_{μ} and a tensor $D_{\mu\beta}$ as above and also a scalar solution $E(x^{\mu},\eta)$ of the flat-space density-wave equation for sound with speed $1/\sqrt{3}$, namely

$$\left(3\frac{\partial^2}{\partial\eta^2} - \nabla^2\right) E = 0\,. \tag{21}$$

Since the calculations are rather long-winded while the results are simple, we shall state the results in the form of a theorem:

Solutions of the perturbed field equations (*15*) *are*

i) $p = 0, \quad \delta p = 0\,,$

$$\begin{aligned} h_{\mu\beta} &= \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}D_{\mu\beta}\right) - 2\left(\frac{8}{\eta^3} - \frac{\nabla^2}{\eta}\right)(C_{\mu,\beta} + C_{\beta,\mu}) + \frac{A_{,\mu\beta}}{\eta^3} + \eta_{\mu\beta}B - \frac{\eta^2}{10}B_{,\mu\beta}\,, \\ h_{\mu 4} &= -2\nabla^2 C_{\mu}/\eta^2\,, \qquad \delta\rho = \frac{H^2{}_R}{4}\nabla^2\left(\frac{6A}{\eta^9} - \frac{3B}{5\eta^4}\right), \end{aligned} \tag{22}$$

[1] For example, in one dimension if

$$\mathfrak{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx\,,$$

then $\mathfrak{f}(k)/k$ is finite at $k = 0$ when

$$\int_{-\infty}^{\infty} dx \int_0^x f(x')\, dx'$$

is finite, etc. The moment conditions are sufficient, but not necessary for our calculations They can always be imposed by altering h_{ab} at locations outside the observable portion of the universe.

[2] "Scalar," "vector," and "tensor" here refer to the transformation properties under the transformations (23).

ii) $p = \rho/3$, $\delta p = \delta\rho/3$,

$$h_{\mu\beta} = \frac{D_{\mu\beta}}{\eta} - \left(\eta\nabla^2 + \frac{8}{\eta}\right)(C_{\mu,\beta} + C_{\beta,\mu}) + \frac{\eta^2}{2}\frac{\partial}{\partial\eta}\left(\frac{E_{,\mu\beta}}{\eta^2}\right) - \frac{\eta_{\mu\beta}}{\eta^2}\frac{\partial E}{\partial\eta},$$

$$h_{\mu 4} = -\nabla^2 C_\mu + \frac{\eta^2}{4}\frac{\partial}{\partial\eta}\left(\eta^{-2}\frac{\partial E_{,\mu}}{\partial\eta}\right), \qquad \delta\rho = \frac{3H^2{}_R}{\eta^4}\frac{\partial}{\partial\eta}\left[\eta^2\frac{\partial}{\partial\eta}\left(\frac{\partial E/\partial\eta}{\eta^2}\right)\right].$$

Moreover, every solution obeying the moment conditions has the form (22) up to a gauge transformation. Finally, the gauge and coordinate frame in which equations (7) and (22) hold are fixed uniquely up to the transformations

$$x' = 0x + \varepsilon;\ 00^T = I;\ 0,\ \varepsilon = \text{const.} \tag{23}$$

Proof: To see that equations (22) form a solution, we merely substitute into the field equations (15); the result is an indentity. Next, to see that every solution of equations (15) which obeys the moment conditions has the form (22) in some gauge frame we proceed as follows. We Fourier-transform $h_{\mu 4}$, $h_{\mu\beta}$, $\delta G^\mu{}_\beta$, $\delta G^\mu{}_4$, and $\delta G^4{}_4$ as in equation (16). We are then left with coupled ordinary differential equations with independent variable η. We next split $\mathfrak{h}_{\mu 4}$ and $\delta\mathfrak{G}^\mu{}_4$ into longitudinal and transverse parts, for example,

$$\mathfrak{h}_{\mu 4}(k,\eta) = n_\mu + imk_\mu, \qquad n_\mu k^\mu = 0\,. \tag{24}$$

The moment conditions (17) guarantee the uniqueness of this splitting if we demand that $m\ (0,\eta)$ be finite. We similarly split $\mathfrak{h}_{\mu\beta}$ and $\delta\mathfrak{G}^\mu{}_\beta$ into traces, longitudinal-longitudinal parts, longitudinal-transverse parts, and trace-free transverse-transverse parts, again using the moment conditions. The system of ordinary differential equations then decouples into sets whose solutions are either powers of η or spherical Bessel functions of low order. After solving, we transform back to position space. The result is the solution (22) up to terms of the form (11). The extra terms can, of course, be eliminated by a gauge transformation. Thus we obtain solution (22). Since the details of the calculation are both tedious and straightforward, we omit them; the reader who wishes to reproduce the calculation will find some auxiliary equations in Appendix I. Finally, we can ask what gauge transformations are still allowed after we have not only made the restrictions (10) but also demanded that the solutions take the particular form (22). By assuming the moment conditions, we find from equations (11) and (22) that $c_{\mu,\beta} + c_{\beta,\mu} = 0$, $b = 0$. Consequently $c_\mu = \epsilon_{\mu\beta}x^\beta + \epsilon_\mu$ where $\epsilon_{\mu\beta} = -\epsilon_{\beta\mu} = \text{const.}$, $\epsilon_\mu = \text{const.}$

These transformations are simply infinitesimal versions of the zero-order transformations (23) and can therefore be included in the zero-order transformations (23) without loss of generality. The net effect is that no gauge transformations whatsoever are left and the only coordinate freedom is the zero-order group of motions (23). *Q.E.D.*

d) Interpretations

The gravitational waves with generating function $D_{\mu\beta}$ have the expected two degrees of freedom, since the restrictions (20) are those for the rest-mass zero, spin-two representations of the Poincaré group (Ehlers 1965*a*). To see how gravitational waves are redshifted, we may consider a plane wave (say, for $p = 0$):

$$h_{\mu\beta} = \frac{{}_oD_{\mu\beta}}{\eta}\frac{\partial}{\partial\eta}\left[\frac{e^{i(k\cdot x + k\eta)}}{\eta}\right], \qquad {}_oD_{\mu\beta} = \text{const.}, \qquad {}_oD_{\mu\beta}k^\beta = 0\,, \qquad {}_oD_\mu{}^\mu = 0\,; \tag{25}$$

suppose that $k\eta \gg 1$. Then the phase ϕ of the wave, as seen by an observer moving with the fluid, is effectively determined by the factor $e^{ik\eta}$. Then $d\phi/dt = ikd\eta/dt = ika^{-1}(\eta)$.

Thus a wave emitted at η_E and received at η_R is redshifted by the amount $z + 1 = a(\eta_R)/a(\eta_E)$, just as an electromagnetic wave is (Group 1). For $k\eta \gg 1$ the other time-varying factors in equation (25) are amplitude modulations.

We mention without proof that calculating the contribution of plane waves (25) to the conformal tensor (Pirani 1965) shows three things: (i) the contribution is not Petrov type N as follows independently from Szerkes (1966); (ii) for large η the dominant term in the contribution is Petrov type N; (iii) in any case, *comoving* observers who measure the relative accelerations of neighboring test particles see the typical transverse pattern of gravitational plane waves. We shall henceforth ignore the gravitational waves and concentrate on the other terms in solution (22). In Appendix II we show for $p = 0$ that all the remaining terms in solution (22) have very direct analogues in the Newtonian theory. To analyze the C_μ terms we introduce the rotation (vorticity) tensor ω_{ab}, defined by Ehlers (1961)

$$\omega_{ab} = \tfrac{1}{2}h^c{}_a h^d{}_b(u_{c;d} - u_{d;c}) , \tag{26}$$

where $h^a{}_b = \delta^a{}_b - u^a u_b$ is the projection operator. In our case the zero-order contribution to the vorticity tensor vanishes and the first-order contribution comes out $\omega_{\mu 4} = 0$ and

$$\omega_{\mu\beta} = \frac{1}{H_R}\nabla^2(C_{\beta,\mu} - C_{\mu,\beta}) \qquad (p = 0),$$
$$\omega_{\mu\beta} = \frac{\eta}{2H_R}\nabla^2(C_{\beta,\mu} - C_{\mu,\beta}) \qquad (p = \rho/3). \tag{27}$$

Since $C_{\mu,}{}^{\mu} = 0$, equation (27) and the moment conditions show that the rotation tensor at any fixed η and C_μ uniquely determine each other. In this linear approximation the rotation tensor is not coupled to the density fluctuations $\delta\rho$, as we see from solution (22).

Finally we consider the terms responsible for the density fluctuations. When $p = \delta p = 0$, there are two kinds of terms, corresponding to A and B, in both of which $\delta\rho$ decreases; $\delta\rho/\rho$ decreases or increases, respectively. In the latter case the relative increase takes place on the same kind of time scale as the time scale of the background. There are two rather tenuous bits of evidence to suggest that the density fluctuations we actually observe are of the relatively increasing type: (i) our own supercluster seems to be expanding less rapidly than the background; (ii) most galaxies seem to occur in clusters, whereas one might expect that in density fluctuations for which $\delta\rho/\rho$ decreases galaxies would be flung out individually (de Vaucouleurs 1959). It should be emphasized that the linear approximation we are using is quite accurate for calculating the field of a given lump but very inadequate for describing the internal dynamics of small lumps. For example, the internal dynamics of our Galaxy at present is governed by gravitational self-interactions and by anisotropic "pressures" that correspond to a suitable solution of the Boltzmann equation for stars; both of these effects are ignored in our treatment so that there is no use trying to analyze the present structure of our Galaxy with our model. On the other hand, suppose one has as given the essential parameters for our Galaxy—mass, size, angular momentum, etc.; then one can in the present model get the external field of the Galaxy accurate to about one part in 10^7 ($GM/Rc^2 \approx 10^{-7}$). The situation is wholly analogous to that in linearized theory (Fock 1959). At characteristic lengths $L \approx 10^{-2}/H_R \approx 10^8$ lt-yr we start to see lumps so loosely bound that the present approximation may give a reasonably good picture even of the internal dynamics. The effect of small, tightly bound lumps on light rays has been analyzed often; two recent treatments are those of Bertotti (1966) and Gunn (1966).

For $p = \rho/3$, $\delta p = \delta\rho/3$, and $k\eta \gg \sqrt{3}$ the density perturbations, governed by E in solution (22), are simply density waves with the characteristic sound velocity $v^2 =$

$dp/d\rho = \frac{1}{3}$. This fact is most easily seen by looking at the Fourier transform of equations (22) (ii); the relevant term is

$$\delta\, \mathfrak{r} = \text{const.}\ \eta^{-4} \frac{\partial}{\partial \eta} \left\{ \eta^2 \frac{\partial}{\partial \eta} \left[\frac{\exp i(\boldsymbol{k}\cdot \boldsymbol{x} + k\eta/\sqrt{3})}{\eta^2} \right] \right\}, \tag{28}$$

where $\delta\mathfrak{r}$ is the Fourier transform of $\delta\rho$.

The factors in η are merely slowly varying modulations when $k\eta \gg \sqrt{3}$. Because the relation between η and t is universal, the waves are redshifted in the same way that gravitational and electromagnetic waves are; similarly the charcteristic length of the density wave is $L \approx a(\eta)/k$ with k constant, and this length grows at a corresponding rate. For long wavelengths, $k\eta \ll \sqrt{3}$, the dominant time dependence in equation (28) is carried by the factors $1/\eta^6$, etc. In that case, the density perturbation $\delta\rho$ decreases, but the time scale for the decrease of $\delta\rho$ is of the same order of magnitude as the time scale for the decrease of ρ. Specifically, in a given interval $\Delta\eta$ we have $\Delta(\delta\rho)/\delta\rho \simeq \frac{3}{2}\Delta\rho/\rho$. This transition from a time dependence governed by $e^{i(k\eta/\sqrt{3})}$ for k large to a time dependence on the same time scale as that of the background for k small is sometimes called a Jeans instability (Bonnor 1957; Peebles 1965); "instability" is not the best word; gravitational and electromagnetic waves show the same kind of behavior.

The above methods and results are similar to those of Lifshitz (Group 2). His background models are less restricted, but our solutions are more explicit.

e) Lightlike Geodesics and Redshifts

The models considered here have the very convenient property that one can integrate the equations for lightlike geodesics in the perturbed metric. These lightlike geodesics are the key elements which relate formal equations like (22) to astronomical observations. We shall now perform the integration. In this subsection we shall use only the form (7) of the metric and the "comoving" coordinate conventions (8); the more explicit form (22) of the metric is not needed in this subsection, nor is the special gauge in which (22) holds relevant.

The geodesic equations can be integrated by force, but it is a little simpler to use conformal techniques. Suppose two metrics ds^2 and $d\bar{s}^2$ are related by a conformal transformation

$$ds^2 = a^2(x^a)d\bar{s}^2 . \tag{29}$$

Then the lightlike geodesics of ds^2 coincide with those of $d\bar{s}^2$. However the preferred (affine) parameters do not coincide. More specifically, suppose we are given a lightlike geodesic $x^a(v) = x^a(w)$, where v and w are affine parameters for ds^2 and $d\bar{s}^2$, respectively. Let $k^a = dx^a/dv$ and $\bar{k}^a = dx^a/dw$ be the respective tangents. Then the relation between v and w can be written in any of the three forms

$$\bar{k}^a = a^2(x^b)k^a \Leftrightarrow \bar{k}_a = k_a \Leftrightarrow dv = a^2 dw . \tag{30}$$

Let us now apply these results to the metric (7), (8) with $a^2 = a^2(\eta)$ and ds^2 the physical metric. We shall first find $\bar{k}^a$, $x^a(w)$ and then use equation (30). For $d\bar{s}^2$ the geodesic equations are

$$\delta \int \left(\frac{d\bar{s}}{dw}\right)^2 dw = 0 \Leftrightarrow \frac{d}{dw}\left(\eta_{ab}\frac{dx^b}{dw} + h_{ab}\frac{dx^b}{dw}\right) = \tfrac{1}{2} h_{bc,a}\frac{dx^b}{dw}\frac{dx^c}{dw} . \tag{31}$$

To zero order we get for ${}_0x^a(w)$

$$\frac{d^2}{dw^2}\, {}_0x^a = 0 , \tag{32}$$

Suppose a light signal is emitted at event (x^μ,η_E) and received at $(0,\eta_R)$, where we can set the spatial coordinate of the reception event to zero without essential loss of generality. Then equation (32) has the solution

$$ {}_o\eta = \eta_R - w\,, \; {}_ox^\beta = e^\beta w\,, \; e^\beta e_\beta = -1\,, \; e^\beta = \text{const.} \tag{33} $$

where $e_\beta = \eta_{\beta\mu}e^\mu$. In equation (32) we have chosen a specific origin and normalization factor for w without essential loss of generality. The vector $e^\beta = \boldsymbol{e}$ represents to zero order the spatial direction of the light signal as seen by a receiver moving with the fluid. The zero-order tangent ${}_o\bar{k}^a$ is given by

$$ {}_o\bar{k}^a = (e, -1)\,. \tag{34} $$

In the following equations we shall denote by "o" or "(o)" a quantity evaluated at the unperturbed ${}_ox^a(v)$ or ${}_ox^a(w)$; for example,

$$ (h_{a\beta,\nu}e^\beta)_{(o)} = e^\beta \left(\frac{\partial h_{a\beta}}{\partial x^\nu}\right)_{x^e = {}_ox^c(w)}\,. \tag{35} $$

Then the first-order correction ${}_1x^a$ to $x^a(w)$ is, according to equation (31), given by

$$ \frac{d^2{}_1x^a}{dw^2} = \eta^{ac}[\,(\tfrac{1}{2}h_{db,c} - h_{bc,d})\,{}_o\bar{k}^d\,{}_o\bar{k}^b\,]\,. \tag{36} $$

Since the right-hand side of equation (36) is explicitly known when h_{ab} is known, equation (36) can be integrated directly. In the next section we shall need only $d_1\eta/dw$:

$$ \frac{d_1\eta}{dw} = -(h_{\beta 4}e^\beta)_{(o)} + \tfrac{1}{2}\int_0^w \left(\frac{\partial h_{\mu\beta}}{\partial\eta}e^\mu e^\beta - 2\frac{\partial h_{\beta 4}}{\partial\eta}e^\beta\right)_{(o)} dy\,, \tag{37} $$

where all the quantities in the integral are evaluated at the unperturbed ${}_ox^a(y)$.

In equation (37) we have set an integration constant to zero without loss of generality. Equation (37) can be used to calculate redshifts. Let $z = \Delta\lambda/\lambda$ as usual. Then (Kristian and Sachs 1966; Schrödinger 1959) for emitter and receiver moving with the fluid we have

$$ z + 1 = \frac{(k^a u_a)_{w=\eta_R-\eta_E}}{(k^a u_a)_{w=0}} = \frac{a(\eta_R)(\bar{k}^a\bar{u}_a)_{w=\eta_R-\eta_E}}{a(\eta_E)(\bar{k}^a\bar{u}_a)_{w=0}}\,. \tag{38} $$

In equation (38) we have set $\bar{u}^a = au^a \Leftrightarrow \bar{u}_a = a^{-1}u_a$. From expressions (38), (7), (8), and (37) we get for the redshift correct up through first-order terms

$$ z + 1 = \frac{a(\eta_R)}{a(\eta_E)}\left[1 - \tfrac{1}{2}\int_0^{\eta_R-\eta_E}\left(\frac{\partial h_{\mu\beta}}{\partial\eta}e^\mu e^\beta - 2\frac{\partial h_{\beta 4}}{\partial\eta}e^\beta\right)_{(o)} dy\right]. \tag{39} $$

As a check, we note that, since z is a directly observable quantity, equation (39) must be invariant under the gauge transformations (9) and (10). In fact a direct calculation shows that a transformation (9) and (10) leaves the right-hand side of (39) invariant. (Note that η_E and η_R change numerically under a gauge transformation for which $b \neq 0$, and that by gauge invariance we here mean numerical, *not* functional, invariance.)

III. ANGULAR VARIATIONS IN THE MICROWAVE RADIATION

We shall now illustrate how our results are related to observations by an example which has considerable intrinsic interest. We wish to calculate angular variations in the microwave radiation (Dicke, Peebles, Roll, and Wilkinson 1965; Peebles 1965) caused by the following mechanism (*a*) at present there are fluctuations $\delta\rho$ in the mass density; (*b*) these fluctuations contribute to the gravitational field as in equation (22); (*c*) the field causes changes in the redshift as in equation (39); (*d*) if the microwave radiation is cosmological, it shows a corresponding variation of temperature with angle.

The model we shall use is the following highly idealized one: (i) we take the present value of H to be 10^{-10} year^{-1}; (ii) we ignore density variations on scales less than 10^9 lt-yr and assume that at present ($\eta_R = 1$), for some scale $L \approx 10^9$–10^{10} lt-yr, there are density variations of order $\delta\rho/\rho \approx 10$ per cent; (iii) we assume that the appropriate background model is that with $p = 0$, with the microwave radiation giving a negligible contribution to ρ; (iv) we assume that only density perturbations of the relatively increasing type are relevant; (v) we assume that at some $\eta_E \lesssim \frac{1}{2}$ in the gauge frame of equations (22) the microwave radiation as measured by observers moving with the $p = 0$ gas was isothermal with temperature T_E independent of position x^a. We suppose that since η_E no significant Thomson scattering of the microwave background has taken

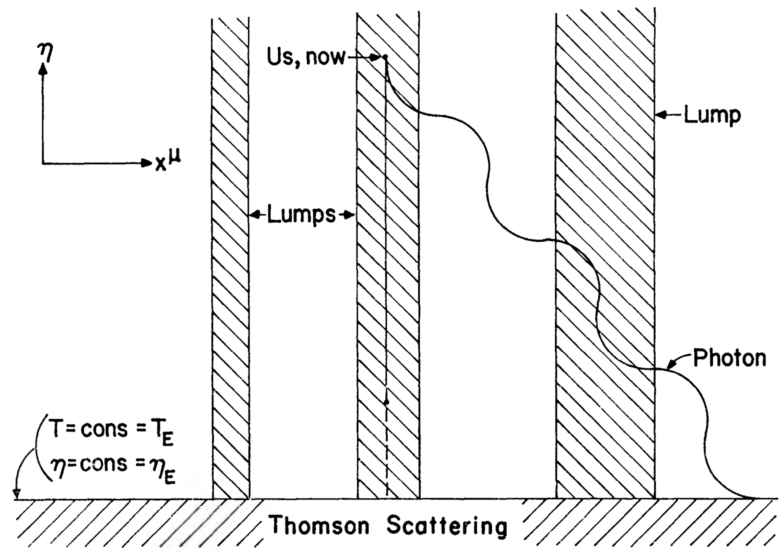

FIG 1 —Space-time diagram for the microwave radiation in the unique gauge frame (22). The lump boundaries have vertical world lines when $A = 0$ in (22) (for dust) because then $\delta\rho = \delta\rho(x^\mu)f(\eta)$ in our comoving frame. This picture in comoving coordinates does *not* accurately represent actual distances, but to zero order, lightlike lines are at 45°.

place (see Figs. 1 and 2). The actual value that we will use is $\eta_E \approx \frac{1}{30}$; this value corresponds to an emission temperature T_E of order 3000° K, since in the background models

$$T_R/T_E = \frac{a(\eta_E)}{a(\eta_R)} = \frac{\eta_E{}^2}{\eta_R{}^2} \approx 10^{-3};$$

however, any value of $\eta_E \lesssim \frac{1}{2}$ would give rather similar results.

Most of the assumptions stated above may be a little on the conservative side. Thus the estimate at which we shall arrive is intended really as a lower limit on the radiation anisotropy. In particular, the assumption (v) of intrinsic uniformity is very questionable. Any intrinsic variations in emission temperature could easily dominate the effects we are analyzing here. In fact, the effects we shall consider are present for any extended source which is of order 10^9–10^{10} lt-yr away; but for, say, galactic groups the effects are swamped by intrinsic variations of the sources. Moreover, the reader should note that assumption (v) is *not* gauge-invariant under the transformations (9)–(11). Assumption (v) becomes meaningful only after we specialize to the (unique) gauge frame in which equations (22) hold (see Fig. 3).

In our model the temperature observed at any one angle, specified by e^μ, is inversely proportional to $z + 1$, where $z = \Delta\lambda/\lambda$ is the redshift between η_E and $\eta_R = 1$ at that angle. This result is proved as follows. In the geometric optics approximation (Kristian and Sachs 1966; Zipoy 1966) we can describe the radiation by the scalar general-relativistic photon-distribution function $F(x^a, p^a)$; here p^a, the photon momentum, is subject to the constraint $p^a p^b g_{ab} = 0$. Since there is no Thomson scattering (or absorption), F obeys the general-relativistic Liouville equation (Lindquist 1965; Ehlers 1965*b*). Imagine some emission event E, and let V^a be that world velocity at the reception event (0,1) obtained by parallel transport of the fluid world velocity from E to (0,1) along the light-like geodesic joining these points. Liouville's theorem implies that an observer at (0,1) moving with world velocity V^a sees the emission temperature T_E in the direction of E.

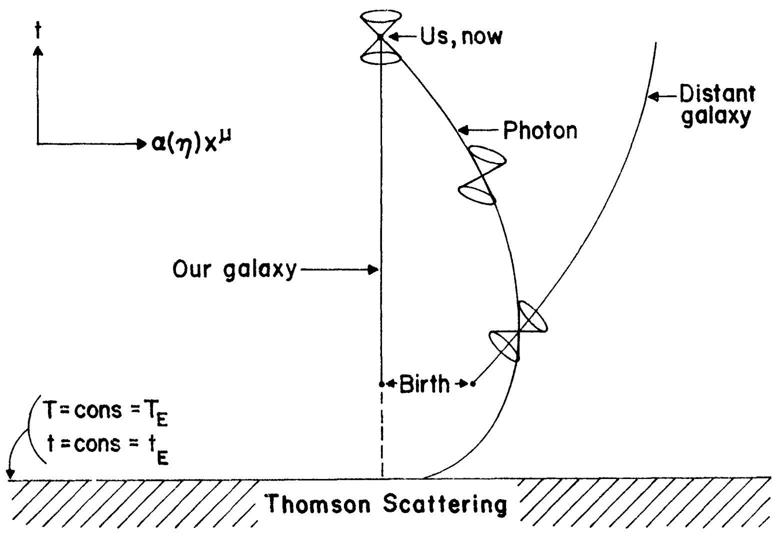

FIG. 2.—Zero-order diagram which schematically represents actual distances more accurately than Fig. 1. The cones are light cones.

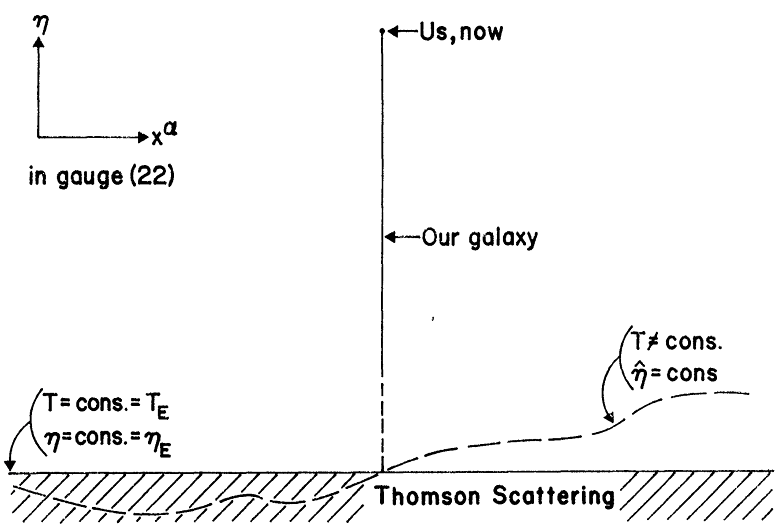

FIG. 3.—Assumption (v) is not invariant under the gauge transformations (11), because T in general varies along the hypersurface $[\eta]$ = const. and the Thomson-scattering cutoff is determined by η_E rather than $[\eta]$ (denoted by $\hat{\eta}$ in figure).

The transformation from an observer at (0,1) with world velocity V^a to an observer at the same event moving with the fluid velocity is the same as in special relativity. Therefore

$$T_R = \frac{T_E}{z+1}, \tag{40}$$

as was to be shown. Note that the proof does not depend on any approximations; equation (40) holds exactly.

From expressions (40) and (39) we have, to first order,

$$T_R = T_E \frac{\eta_E^2}{\eta_R^2}\left(1+\frac{\delta T_R}{T_R}\right), \tag{41}$$

where

$$\frac{\delta T_R}{T_R} = \frac{1}{2}\int_0^{\eta_R-\eta_E}\left(\frac{\partial h_{\mu\beta}}{\partial \eta} e^\mu e^\beta - 2\frac{\partial h_{\beta 4}}{\partial \eta} e^\beta\right)_{(o)} dy\,. \tag{42}$$

According to assumptions (iii) and (iv) above we can evaluate equation (42) using $\delta p = 0$ in solution (22) and setting $A = C_\mu = D_{\mu\beta} = 0$. Then

$$\frac{\delta T_R}{T_R} = \frac{1}{10}\left[(B_{,\mu}e^\mu)_R\eta_R - (B_{,\mu}e^\mu)_E\eta_E + B_R - B_E\right], \tag{43}$$

where R denotes the reception event (0,1) and E the emission event $[e(\eta_R - \eta_E), \eta_E]$.

We shall now analyze each term in equation (43) separately. We shall give some intuitive interpretations; the reader is warned that our interpretations are valid only when we consider the redshifts due to density fluctuations of the relatively increasing kind. If A in (22) is non-zero, the equation corresponding to (43) is more complicated and our heuristic discussion below is not valid.

The angular dependence of the first term, for a coordinate system here and now whose z-axis is aligned with $(\nabla B)_R$, is simply

$$\frac{\delta T_R}{T_R} = (\text{const.})\cos\theta\,, \tag{44}$$

where θ is the usual polar angle. Therefore this first term is essentially a Doppler shift induced by the fact that our fluid velocity here and now does not coincide with that world velocity which would make the received temperature as isotropic as possible. The interpretation of this term as essentially a Doppler shift can also be seen from the Newtonian models of Appendix II.

The second term in equation (43) is essentially a similar Doppler-shift correction for the world velocity of the source; if $\eta_E \approx \frac{1}{30}$ this second term is normally small. Finally the terms

$$\frac{\delta T_R}{T_R} = \frac{1}{10}\{B(0) - B[e^\mu(\eta_R - \eta_E)]\} \tag{45}$$

are rather similar to a standard gravitational redshift since B in equations (22) is rather similar to a Newtonian potential. Note that we should consider the source of the "potential" B to be the fluctuation $\delta\rho$ at the present time $\eta_R = 1$, not at the emission time or intermediate times. The time dependence of $\delta\rho$ and h_{ab} has already been taken into account. We emphasize again that in a generic gravitational field one cannot distinguish gravitational redshifts from Doppler shifts by any standard recipe; thus our division of equation (43) into three parts has only a heuristic significance.

To estimate the order of magnitude of the most interesting term (eq. [45]), imagine that the present density perturbations $\delta\rho$ are sinusoidal with some characteristic amplitude $\delta_0\rho$ and characteristic scale L:

$$\delta\rho = \delta_0\rho\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \; |\boldsymbol{k}| = \frac{a(\eta_R)}{L} = \frac{2}{H_R L}; \tag{46}$$

then from solution (22)

$$B = B_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad B_0 = \tfrac{20}{3}\left(\frac{\delta_0\rho}{k^2 H_R{}^2}\right). \tag{47}$$

Consequently, equations (5), (45), and (46) give

$$\frac{\delta T_R}{T_R} \approx \tfrac{1}{2}\,\frac{\delta_0\rho}{\rho}(H_R L)^2. \tag{48}$$

Suppose the universe contains lumps with scale $HL \approx 0.3$ (e.g., L is about 1000 Mpc) and density fluctuations $\delta\rho/\rho \approx 10$ per cent. Then

$$\frac{\delta T_R}{T_R} \approx 0.5 \text{ per cent}. \tag{49}$$

While we have no convincing direct evidence for or against 10 per cent density fluctuations over scales as large as $HL \approx 0.3$, the fact that much more drastic density fluctuations occur with scales $HL \approx 10^{-3} - 10^{-2}$ (de Vaucouleurs 1961) suggests that (49) is not a severe overestimate. Of course expression (49) is not a surprising result: for the density fluctuations considered, the dimensionless concentration parameter GM/Lc^2 is not negligible and general relativistic effects must come in.

If one uses 10 per cent density fluctuations over scales $HL \approx 0.3$ to evaluate the constant in equation (44), this constant comes out of order 1.5 per cent. Thus the "Doppler shift" term, though less interesting, is a little larger.

A slightly more sophisticated estimate can be obtained if one uses stochastic averages. Suppose (Wax 1954) that at $\eta_R = 1$

$$\delta\rho = H_R{}^2 \int d^3k\, Q(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{50}$$

where $Q(\boldsymbol{k})$ is a random function. Suppose for simplicity

$$\langle Q(\boldsymbol{k}) Q(\boldsymbol{k}')\rangle = S(k)\delta^3(\boldsymbol{k} + \boldsymbol{k}'), \; S(k) \geq 0, \tag{51}$$

where the angular brackets denote an ensemble average. Then

$$B(\boldsymbol{x}) = \tfrac{20}{3}\int \frac{d^3k\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{k^2} Q(\boldsymbol{k}). \tag{52}$$

Consequently, for the angular autocorrelation function $f(\theta)$ of the term (45) we get (Wax 1954)

$$\begin{aligned} f(\theta) &= \tfrac{1}{100}\langle\{B[\boldsymbol{e}(\eta_R - \eta_E)] - B(0)\}\{B[\boldsymbol{e}'(\eta_R - \eta_E)] - B(0)\}\rangle \\ &= \tfrac{4}{9}\int \frac{d^3k\, S(k)}{k^4}\{\exp[i\boldsymbol{k}\cdot(\boldsymbol{e} - \boldsymbol{e}')(\eta_R - \eta_E)] - \exp[i\boldsymbol{k}\cdot\boldsymbol{e}(\eta_R - \eta_E)] \\ &\qquad - \exp[-i\boldsymbol{k}\cdot\boldsymbol{e}'(\eta_R - \eta_E)] + 1\}, \end{aligned} \tag{53}$$

where θ is the Euclidean angle between $\boldsymbol{e}$ and $\boldsymbol{e}'$.

The only term in equation (53) that actually depends on θ is the term

$$g(\theta) = \tfrac{4}{9}\int \frac{d^3k}{k^4} S(k) \exp[i\boldsymbol{k}\cdot(\boldsymbol{e} - \boldsymbol{e}')(\eta_R - \eta_E)] = \frac{16\pi}{9}\int_0^\infty \frac{dk\, S(k)}{k^2}\,\frac{\sin k\chi}{k\chi}, \tag{54}$$

where

$$\chi = 2(\eta_R - \eta_E) \sin \theta/2 \approx 2 \sin (\theta/2) . \tag{55}$$

Suppose $S(k)$ is sharply peaked near some value k_0 of k. Then $[g(0)]^{1/2}$ gives back our former magnitude estimates. From equations (54) and (55) we can estimate the angular resolution needed to detect the effect considered. For $k_0 \ll 1$ the resolution required is of order $2\pi/k_0$ radians of arc.

By an analysis similar to the above one finds that if the present value of $\delta\rho$ comes from density perturbations of the relatively decreasing kind, so that only $A \neq 0$ in equation (22)(i), then $\delta T/T$ is larger than the values given above. Moreover it is possible to imagine that $\delta\rho$ is zero everywhere between us and the emitting event but $B \neq 0$ (or $A \neq 0$). For such terms B and A would have to be solutions of the homogeneous Laplace equation essentially up to the (spatial) particle horizon (Penrose 1964; Rindler 1956). We might visualize such terms as the longitudinal gravitational fields of large masses so distant that the masses are outside our present particle horizon. No a priori upper limits can be set on the size of such terms. Finally, gravitational waves with very long wavelengths could also contribute to $\delta T/T$, and would presumably not be detectable otherwise.

IV. CONCLUSION

We have estimated that anisotropies of order 1 per cent should occur in the microwave radiation if this radiation is cosmological. This figure is a reasonable lower limit provided even rather modest 10 per cent density fluctuations with a scale of $\frac{1}{3}$ the Hubble radius occur at present. Larger variations could arise from intrinsic inhomogeneities in the radiation temperature at the time Thomson scattering became negligible, from the effects discussed at the end of the last section, or from effects to which our perturbation theory here is not applicable, such as non-linear large-scale anisotropies of the universe. Conversely, if isotropy to within 1 per cent or better could be established, this would be a quite powerful null result.

Of course very many other effects, observable in principle, can be obtained from the approach used in this paper. We have not so far found any others that seem particularly promising, though our present ignorance of most of the parameters involved leaves many possibilities open. More interesting seem to be two extensions of the theory developed. First, the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible using the methods of Hawking (1966). One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects. Second, it would be desirable to describe the matter and radiation by the Boltzmann equation (Gilbert 1966) rather than just using fluid dynamics. The mechanism for producing lumps of a certain size and density is at present very obscure. Perhaps, for example, radiation viscosity is an effective mechanism for producing small-scale perturbations and damping large-scale perturbations during the $p = \rho/3$ phase of the universe that general-relativistic cosmologies predict. The fluid dynamical approach is not well suited for discussing transport processes or various non-gravitational instabilities.

Future observations may exclude the homogeneous, isotropic, general-relativistic $k = 0$ models, even as zero-order approximations. At present they are as acceptable as any other models and considerably simpler than most models.

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APPENDIX I

AUXILIARY EQUATIONS

We shall list a few of the equations used in deriving equations (22). We split $\mathfrak{h}_{\mu 4}$ and $\mathfrak{h}_{\mu\beta}$ as follows:

$$\mathfrak{h}_{\mu 4} = n_\mu + imk_\mu \, n_\mu k^\mu = 0 \,,$$
$$\mathfrak{h}_{\mu\beta} = p_{\mu\beta} + i(q_\mu k_\beta + q_\beta k_\mu) - sk_\mu k_\beta + r\eta_{\mu\beta} \,, \qquad \text{(I.1)}$$
$$p_{\mu\beta} = p_{\beta\mu}, \qquad p^\mu{}_\mu = 0, \qquad p_{\mu\beta}k^\beta = 0, \qquad q_\mu k^\mu = 0 \,,$$

where n_μ, m, $p_{\mu\beta}$, q_μ, s, and r are functions of $\boldsymbol{k}$ and η. The moment conditions (17) imply that (i) such a splitting is possible; (ii) we can require m, q_μ, s, and r to be finite at $\boldsymbol{k} = 0$; and (iii) the splitting is then unique. We then split $\delta\mathfrak{G}^\mu{}_4$ and $\delta\mathfrak{G}^\mu{}_\beta$ in the same way. The result is

$$
\begin{aligned}
&\text{a.1)} \quad (r'a^2)' + a^4\delta\mathfrak{p} = 0 \,, \\
&\text{a.2)} \quad -a^2k^2r - [(s'k^2 - 2mk^2)a^2]' = 0 \,, \\
&\text{a.3)} \quad k_\mu[r' + a^2F(\eta)m] = 0 \,, \\
&\text{a.4)} \quad k^2r + \frac{a'}{a}(3r' + s'k^2 - 2mk^2) - a^2\delta\mathfrak{r} = 0 \,, \\
&\text{b.1)} \quad ik_\beta[(q'_\mu - n_\mu)a^2]' = 0 \,, \\
&\text{b.2)} \quad -k^2(q'_\mu - n_\mu) + 2a^2F(\eta)n_\mu = 0 \,, \\
&\text{c)} \quad a^2k^2p_{\mu\beta} + (p'_{\mu\beta}a^2)' = 0 \,,
\end{aligned}
\qquad \text{(I.2)}
$$

where primes denote η-derivatives, $k^2 = \boldsymbol{k}\cdot\boldsymbol{k} = -k_\mu k^\mu$, and all indices are raised or lowered with $\eta_{\mu\beta} = -\delta_{\mu\beta}$ as before. Because of the moment conditions equation (I.2) can be simplified; for example, (I.2.a.2) can be written

$$-a^2r - [(s' - 2m)a^2]' = 0 \,, \qquad \text{(I.3)}$$

since r, s', and m are finite at $k = 0$.

After simplifying, expressions (I.2.a), (I.2.b), and (I.2.c) can be solved directly. For example, with $a^2 =$ const. η^2 (e.g., $p = \rho/3$), equation (I.2.c) reads

$$k^2\eta^2p_{\mu\beta} + (p'_{\mu\beta}\eta^2)' = 0 \qquad \text{(I.4)}$$

with solution $p_{\mu\beta} = p_{\mu\beta}(\boldsymbol{k})e^{ik\eta}/\eta$. The Fourier transform of this last expression is $[D_{\mu\beta}(x,\eta)]/\eta$, where $D_{\mu\beta}$ is any solution of equation (20). Apart from terms of the form (11) we then get equations (22).

APPENDIX II

NEWTONIAN ANALOGUES

We will perturb the Newtonian cosmological equations and get solutions for the first-order corrections in density and velocity. As in the previous general-relativistic calculations we will use a background model which is both homogeneous and isotropic. We set the cosmological constant $\Lambda = 0$, and we consider the case of "free fall," which is analogous to the case of zero curvature ($k = 0$) in general relativity. If the Newtonian and relativistic solutions agree, one can have greater confidence in the validity of these results. For $p = 0$ the correspondence is very close.

a) Background Models

For the case of dust ($p = 0$) we calculate the equations for the background and then for the first-order perturbations Let $\boldsymbol{v}(\boldsymbol{r},t)$, $\rho(\boldsymbol{r},t)$, and $\phi(\boldsymbol{r},t)$ be the velocity, density, and potential of the dust cloud at time t and position $\boldsymbol{r}$. The Newtonian equations for dust (in Cartesian coordinates) are the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 , \tag{II.1}$$

the Navier-Stokes equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = -\nabla \phi , \tag{II.2}$$

and Poisson's equation (with $\Lambda = 0$)

$$\nabla^2 \phi = 4\pi G \rho . \tag{II.3}$$

In the zero-order approximation these equations will yield a class of evolving background models. From the postulate that the background flow be homogeneous and isotropic we infer that $\boldsymbol{v} = H(t)\boldsymbol{r}$ and $\rho = \rho(t)$ (Heckmann and Schücking 1959), where $\rho(t)$ represents the smoothed-out background matter density. We let $H(t) = [da(t)/dt]/a(t)$, where $a(t)$ is the usual expansion parameter which is called $R(t)$ in Heckmann and Schücking. These expressions are then substituted into equations (II.1), (II.2), and (II.3). The subsequent calculations have been done by Heckmann and Schücking. We consider the case where h, the energy of a unit mass particle, vanishes. This is known as "free fall" since the particle can just escape to infinity. The energy equation being identical to the relativistic Friemann equation, we can identify h with the curvature of the spacelike hypersurfaces $t =$ const. Hence we have the same background model that we used in our relativistic calculations. We now give the results of Heckmann and Schücking for "free fall." We use the result that $M = 4/GH_R$ where M is the "mass" of the universe and H_R is the present Hubble constant. We also define the variable η by $t = (GM/6)\eta^3$. In this notation $\eta(t)$ is the same time coordinate we used in the relativistic calculations with the present value of η, $\eta_R = 1$. Therefore:

$$a(\eta) = \frac{2\eta^2}{H_R}, \qquad 8\pi G \rho(\eta) = \frac{3H^2{}_R}{\eta^6}, \qquad \boldsymbol{v}(\boldsymbol{r}, \eta) = \frac{H_R \boldsymbol{r}}{\eta^3},$$
$$\phi(r, \eta) = \frac{H^2{}_R}{4\eta^6} r^2, \qquad H(\eta) = \frac{H_R}{\eta^3}. \tag{II.4}$$

We now have complete knowledge of the time evolution and spatial dependence of the background expansion parameter, density, velocity, potential, and Hubble constant. These zero-order quantities will appear in the first-order calculations.

b) First-Order Solutions ($p = 0$)

If a small perturbation is placed on the background, first-order corrections appear in the velocity, density, and potential. We will call these $\delta \boldsymbol{v}(\boldsymbol{r},t)$, $\delta\rho(\boldsymbol{r},t)$. and $\delta\phi(\boldsymbol{r},t)$. The Newtonian equations are now solved to first order. The perturbed Navier-Stokes equation is

$$\frac{\partial(\delta \boldsymbol{v})}{\partial t} + H(\boldsymbol{r} \cdot \nabla)\partial(\delta \boldsymbol{v})/\partial \boldsymbol{r} + H\delta \boldsymbol{v} = -\nabla(\delta\phi) . \tag{II.5}$$

The perturbed continuity equation is [for convenience we define the density fluctuation, $D(\boldsymbol{r},t) \equiv \delta\rho(\boldsymbol{r},t)/\rho(t)$]

$$\frac{\partial D}{\partial t} + H\boldsymbol{r} \cdot \nabla D + \nabla \cdot (\delta \boldsymbol{v}) = 0 . \tag{II.6}$$

Finally the perturbed Poisson equation is

$$\nabla^2(\delta\phi) = 4\pi G\rho(t)D\,. \tag{II.7}$$

Equations (II.5), (II.6), and (II.7) can be combined to yield the following differential equation for $D(r,t)$:

$$\left(\frac{\partial}{\partial t}+2H+H\boldsymbol{r}\cdot\boldsymbol{\nabla}\right)\left(\frac{\partial D}{\partial t}+H\boldsymbol{r}\cdot\boldsymbol{\nabla}D\right)=4\pi G\rho(t)D\,. \tag{II.8}$$

The method used in getting this differential equation is given in Peebles (1965).

In order to compare the Newtonian and relativistic perturbations we transform equation (II.8) to a coordinate system comoving in zero order by going from the components r^β of the Cartesian $\boldsymbol{r}$ to coordinates x^β:

$$x^\beta = r^\beta/a(t)\,. \tag{II.9}$$

We shall use the symbol $\boldsymbol{\nabla}_x$ to indicate $\partial/\partial x^\beta$. Under the coordinate transformation (II.9) equation (II.8) goes into the form

$$\frac{\partial^2 D}{\partial t^2}+2H(t)\frac{\partial D}{\partial t}-4\pi G\rho D=0\,. \tag{II.10}$$

Transforming from t to η and substituting the background values for H and ρ gives

$$\frac{\partial^2 D}{\partial \eta^2}+\frac{2}{\eta}\frac{\partial D}{\partial \eta}-\frac{6}{\eta^2}D=0\,. \tag{II.11}$$

The general solution of this equation can be written

$$D(x^\beta,\eta)=\frac{\nabla^2{}_x}{12}\left[\frac{6A(x^\beta)}{\eta^3}-\tfrac{3}{5}B(x^\beta)\eta^2\right], \tag{II.12}$$

where A and B are any functions of the x^β alone and the numerics have been chosen to facilitate comparison with the relativistic solutions. From equation (II.12) and the definition of D we get

$$\delta\rho=\frac{H^2{}_R}{32\pi G}\nabla^2{}_x\left(\frac{6A}{\eta^9}-\tfrac{3}{5}\frac{B}{\eta^4}\right). \tag{II.13}$$

We can now obtain $\delta\phi$ by putting equation (II.7) into comoving coordinates x^β. The solution comes out

$$\delta\phi = 3[(A/\eta^5)-(B/10)]+J(x^\beta,\eta)\,, \tag{II.14}$$

where J is any solution of $\nabla^2{}_x J = 0$. If we impose on ϕ conditions analogous to the moment conditions described in the text, J must be zero and we henceforth assume that such is the case.

Finally, we can now solve the perturbed Navier-Stokes equations (II.5) to get $\delta\boldsymbol{v}$. We shall compute the Cartesian components of $\delta\boldsymbol{v}$ as functions of the comoving coordinates x^β. If we transform variables in equation (II.5), it becomes

$$\frac{\partial\,\delta v^\beta}{\partial t}+H(t)\,\delta v^\beta=-\frac{1}{a}\frac{\partial\,\delta\phi}{\partial x^\beta}, \tag{II.15}$$

where δv^β are still the Cartesian components. When we introduce η and the relevant background values this equation reduces to

$$\frac{\partial\,\delta v^\beta}{\partial \eta}+\frac{2}{\eta}\,\delta v^\beta=-3\frac{\partial}{\partial x^\beta}\left(\frac{A}{\eta^5}-\frac{B}{10}\right). \tag{II.16}$$

Let $C^\beta(x^\alpha)$ be any function of the x^β alone. Then the general solution of equation (II.16) can be written in the form

$$\delta v^\beta = \frac{\nabla^2{}_x C^\beta}{\eta^2} + \frac{1}{2}\frac{\partial}{\partial x^\beta}\left(\frac{3A}{\eta^4} + \frac{B\eta}{5}\right). \tag{II.17}$$

We can now compare with the relativistic solutions by setting $8\pi G = 1$; moreover, the coordinates x^β are fully analogous in the two cases because they are comoving to zero order in both cases and small corrections to x^β are irrelevant when considering terms already small to first order. For $\delta\rho$ the Newtonian and relativistic expressions are simply identical:

$$\delta\rho = \frac{H^2{}_R}{4}\nabla^2{}_x\left[\frac{6A(x^\beta)}{\eta^9} - \frac{3}{5}\frac{B(x^\beta)}{\eta^4}\right] \quad (A, B \text{ arbitrary}). \tag{II.18}$$

There are no simple relativistic analogues of $\boldsymbol{v}$ and ϕ which are gauge-invariant, though quantities analogous to the Newtonian $\delta\boldsymbol{v}$ do appear in the redshift equations. However a simple relationship exists between the relativistic vorticity tensor ω_{ab} and the Newtonian analogue $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{\nabla}_x \times \boldsymbol{v}/a(\eta)$. For a comparison we may work out the scalar magnitude of both quantities, which is first order in both cases since the zero-order vorticity vanishes. A short calculation shows that the magnitudes are in fact equal:

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \omega_{ab}\omega_{cd}g^{ac}g^{bd} = \frac{H^2{}_R}{4\eta^8}[\nabla^2{}_x(\boldsymbol{\nabla}_x \times \boldsymbol{C})]^2; \qquad \boldsymbol{C} = \boldsymbol{C}(x^\beta) \text{ arbitrary}. \tag{II.19}$$

We note that $\boldsymbol{C}$ as in the relativistic case has no longitudinal part. This may be seen by substituting the solutions for δv^α and $\delta\rho/\rho$ into the perturbed continuity equation expressed in comoving coordinates. This completes the Newtonian analogue. The only term in (22) (i) which has no direct analogue here is the gravitational radiation term $D_{\alpha\beta}$, which must of course be missing in Newtonian approximation.

c) The Case of Radiation ($p = \rho/3$)

The same perturbation scheme was tried with the following results. The background Newtonian equations (II.1), (II.2), and (II.3) had to be modified (Harrison 1965) to get the correct background solutions. When these modified equations were perturbed to first order, their solutions did not agree with the relativistic results, even qualitatively.

Note added in proof: (1) The Newtonian calculations have been done by Doroshkevich and Zeldvich in 1963 (*Astr. Zh.*, **40,** 807). (2) The temperature-shift argument relating to equation (40) was given previously by Etherington in 1933 (*Phil. Mag.*, ser. 7, **15,** 761). (3) We have investigated the density perturbations of the relatively increasing type (B type) in detail, and find that the mass of these lumps increases with time at the same rate as the background expansion parameter a. We also find that the increasing mass is supplied by a perturbed matter flow from the background.

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