TABLES OF HYDROGENIC PHOTOIONIZATION CROSS-SECTIONS AND RECOMBINATION COEFFICIENTS

A. BURGESS

Summary

The cross-sections for photoionization of a hydrogenic atom or ion initially in the state nl are tabulated to 5 significant figures for $n=1(1)20$, $l=0(1)n-1$ and a wide range of energies of the ejected electron. The hydrogenic radiative recombination coefficients for a Maxwellian distribution of electron velocities are tabulated to 4 significant figures for $n=1(1)20$, $l=0(1)n-1$ and electron temperatures $T=0$ to ∞.

1. *Introduction*

A knowledge of the photoionization cross-sections and recombination coefficients of hydrogenic atoms and ions is of importance in many problems of astrophysics and plasma physics, see e.g. Aller (**1**), Seaton (**2**), Bates and Dalgarno (**3**) and Kolb and Griem (**4**). Closely related quantities also occur in the problem of the atomic capture of mesons, see Martin (**5**).

One astrophysical problem of particular interest is the calculation of the intensities to be expected in recombination spectra. For such calculations on the hydrogen recombination spectrum see Searle (**6**), Burgess (**7**) and Pengelly (**8**). The hydrogenic recombination coefficients are also of use in the calculation of recombination spectra of more complex atoms and positive ions (see Burgess and Seaton (**9**) and Pengelly (**10**)), since many of the excited states of the atom or ion are very close to being hydrogenic.

The major difficulty in the determination of photoionization cross-sections and recombination coefficients lies in the calculation of the electric dipole moment matrix elements for the bound-free transitions $nl \rightarrow kl'$. These matrix elements may be expressed exactly in closed analytic form (Gordon (**11**)), but the expressions are complicated and difficult to evaluate directly for some ranges of the parameters due to cancellation, hence previous computations have usually either covered a fairly small range of parameters (see e.g. Karzas and Latter (**12**)) or introduced some simplifying approximation (see e.g. (**7**)). Here we use a method involving simple recurrence relations satisfied by the exact matrix elements which enable them to be evaluated rapidly and to high accuracy.

The photoionization cross-sections have been calculated for all levels nl with $n \leqslant 20$ and for a wide range of energies of the ejected electron. The corresponding recombination coefficients, for a Maxwellian distribution of electron velocities, are tabulated for effectively the complete electron temperature range $0 \leqslant T \leqslant \infty$. It should of course be understood that the results are not strictly valid for $T \rightarrow \infty$ since relativistic effects are not taken into account. The range of levels, $n \leqslant 20$, which the tabulations cover, was chosen to be the same as in the tables of bound–bound transition probabilities given by Green, Rush and Chandler (**13**) which is the most complete tabulation to date. A tabulation in terms of the separate orbital angular momentum states for $n > 20$ would be largely superfluous since, in cases of physical interest, collisional transitions between the separate l states of a given n appear to dominate in this region (Pengelly and Seaton (**14**)). For tables of the hydrogenic recombination coefficients summed over l, see Seaton (**15**).

2. *Photoionization cross-sections and recombination coefficients*

We consider first the ionization, by photons of energy $h\nu$, of a hydrogenic atom or ion initially in the state specified by the principal quantum number n and orbital angular momentum quantum number l. The ejected electron energy k^2 in units of the ionization energy of hydrogen I_{H} is given by the energy conservation condition $h\nu = (z^2/n^2 + k^2)I_{\mathrm{H}}$, where z is the nuclear charge number. For an ensemble of such systems, the number of ionizations per unit volume per unit time is $F_\nu d(h\nu)N_{nl}a_{nl}(k^2)$ where N_{nl} is the number density of absorbing atoms or ions in state nl, $F_\nu d(h\nu)$ is the flux of photons in the incident beam having energy lying between $h\nu$ and $h\nu + d(h\nu)$, and where $a_{nl}(k^2)$ is the photoionization cross-section which for a hydrogenic atom or ion may be written in the form (see Burgess and Seaton (**16**))

$$a_{nl}(k^2) = \left(\frac{4\pi\alpha a_0^2}{3}\right)\frac{n^2}{z^2}\sum_{l'=l\pm1}\frac{l_>}{2l+1}\,\Theta(n, l; \kappa, l'), \tag{1}$$

where

$$\Theta(n, l; \kappa, l') = (1 + n^2\kappa^2)|g(n, l; \kappa, l')|^2 \tag{2}$$

with

$$g(n, l; \kappa, l') = \frac{z^2}{n^2}\int_0^\infty P_{nl}(r)rF_{kl'}(r)\,dr, \tag{3}$$

and where l' is the orbital angular momentum quantum number of the ejected electron, $l_>$ is the greater of l and l', κ is k/z, α is the fine structure constant and a_0 is the Bohr radius. In (3), $P_{nl}(r)$ and $F_{kl}(r)$ are respectively the initial and final radial wave functions of the electron, which satisfy

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2z}{r} + \begin{Bmatrix} -z^2/n^2 \\ k^2 \end{Bmatrix}\right]\begin{Bmatrix} P_{nl}(r) \\ F_{kl}(r) \end{Bmatrix} = 0 \tag{4}$$

and are normalized such that

$$\int_0^\infty P_{nl}(r)P_{n'l}(r)\,dr = \delta_{nn'} \tag{5}$$

and

$$\int_0^\infty F_{kl}(r)F_{k'l}(r)\,dr = \pi\delta(k^2 - k'^2). \tag{6}$$

In (1) we have $(4\pi\alpha a_0^2/3) \simeq 8{\cdot}5594 \times 10^{-19}\,\mathrm{cm}^2$. The photoionization cross-section is sometimes expressed in terms of the bound-free oscillator strength $df(k, n)/d(k^2)$ or the bound-free Kramers–Gaunt factor $g_{\mathrm{II}}(n, \kappa)$. We have the relationships

$$a_{nl}(k^2) = 4\pi^2\alpha a_0^2\,\frac{df(k, n)}{d(k^2)}, \tag{7}$$

$$\sum_l\left(\frac{2l+1}{n^2}\right)a_{nl}(k^2) = \frac{64\pi\alpha a_0^2}{3\sqrt{3}}\,\frac{n}{z^2}\,\frac{g_{\mathrm{II}}(n, \kappa)}{(1 + n^2\kappa^2)^3}. \tag{8}$$

Next, we consider radiative recombination to level nl of a hydrogenic atom or ion. The number of such recombinations per unit volume per unit time is given by $N_eN_+\alpha_{nl}$, where N_e and N_+ are the number densities of the electrons and recombining ions respectively, and where α_{nl} is the recombination coefficient. With a Maxwellian distribution of velocities corresponding to a temperature T°K, we have (see (**9**))

$$\alpha_{nl} = \left(\frac{2\pi^{1/2}\alpha^4a_0^2c}{3}\right)\frac{2y^{1/2}}{n^2}\,z\sum_{l'=l\pm1}I(n, l, l', t), \tag{9}$$

where

$$I(n, l, l', t) = l_>\,y\int_0^\infty (1 + n^2\kappa^2)^2\Theta(n, l; \kappa, l')e^{-\kappa^2 y}d(\kappa^2) \tag{10}$$

and

$$y = \frac{z^2Rhc}{kT} \simeq \frac{15{\cdot}789}{t}, \tag{11}$$

with

$$t = T/10^4z^2, \tag{12}$$

and where R is the Rydberg wave number and k is Boltzmann's constant. In (9) we have

$$(2\pi^{1/2}\alpha^4 a_0^2 c/3) \simeq 2{\cdot}8128 \times 10^{-15}\ \mathrm{cm}^3\,\mathrm{sec}^{-1}.$$

For convenient tabulation we define the function

$$\begin{aligned}\Phi(n,l,l',t) &= I(n,l,l',t), \quad t \leqslant 1 \\ &= tI(n,l,l',t), \quad t \geqslant 1\end{aligned} \tag{13}$$

which has the property that it remains finite for all t. In particular

$$\Phi(n,l,l',0) = l_> \quad |g(n,l;\ 0,l')|^2 \tag{14}$$

and

$$\Phi(n,\ l,l',\ \infty) = 15{\cdot}789\ l_> \int_0^\infty (1+n^2\kappa^2)^3 |g(n,l;\ \kappa,l')|^2\, d(\kappa^2). \tag{15}$$

3. *Bound-free dipole matrix elements*

We require to evaluate the dipole radial integrals (3). Writing

$$\rho = zr \tag{16}$$

$$\mathscr{P}_n(\rho) = z^{-1/2} P_{nl}(r) \tag{17}$$

$$\mathscr{F}_{\kappa l}(\rho) = z^{1/2} F_{kl}(r), \tag{18}$$

we see that

$$\left[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{2}{\rho} + \begin{Bmatrix} -1/n^2 \\ \kappa^2 \end{Bmatrix}\right] \begin{Bmatrix} \mathscr{P}_{nl}(\rho) \\ \mathscr{F}_{\kappa l}(\rho) \end{Bmatrix} = 0 \tag{19}$$

$$\int_0^\infty \mathscr{P}_{nl}(\rho)\mathscr{P}_{n'l}(\rho)\, d\rho = \delta_{nn'} \tag{20}$$

$$\int_0^\infty \mathscr{F}_{\kappa l}(\rho)\mathscr{F}_{\kappa' l}(\rho)\, d\rho = \pi\delta(\kappa^2 - \kappa'^2), \tag{21}$$

and thus $\mathscr{P}_{nl}(\rho)$, $\mathscr{F}_{\kappa l}(\rho)$ and

$$g(n,l;\ \kappa,l') = \frac{1}{n^2}\int_0^\infty \mathscr{P}_{nl}(\rho)\rho\mathscr{F}_{\kappa l}(\rho)\, d\rho \tag{22}$$

do not explicitly depend on z. We may in fact write explicitly (**11**)

$$\mathscr{P}_{nl}(\rho) = \sqrt{\frac{(n+l)!}{(n-l-1)!}}\left(\frac{2\rho}{n}\right)^{l+1} \frac{e^{-\rho/n}}{n(2l+1)!}\ {}_1F_1\left(l+1-n;\ 2l+2;\ \frac{2\rho}{n}\right) \tag{23}$$

and

$$\mathscr{F}_{\kappa l}(\rho) = \sqrt{\frac{\pi}{2(1-e^{-2\pi/\kappa})}\prod_{s=0}^{l}(1+s^2\kappa^2)}\ \frac{(2\rho)^{l+1}}{(2l+1)!}\ e^{i\kappa\rho}{}_1F_1(l+1-i/\kappa;\ 2l+2;\ -2i\kappa\rho). \tag{24}$$

N.B. The bound state radial functions, and hence $g(n,l;\ \kappa,l')$, differ by a phase factor $(-1)^{n-l-1}$ from those used in (**16**). This phase is convenient for present purposes (i.e. integer values of n) since $g(n,l;\ \kappa,l')$ then turns out to be always positive.

The integral (22) may be evaluated by choosing suitable integral representations for the confluent hypergeometric functions in (23) and (24), changing the order of integration to perform the integration over ρ first, and then expressing the resulting integral in terms of hypergeometric functions. The result has been given by several authors, notably Gordon (**11**), Biedenharn, McHale and Thaler (**17**) and Alder and Winther (**18**).

After simplifying, we have

$$g(n,l;\,\kappa,l'=l\pm 1)=\sqrt{\frac{\pi}{2}\frac{(n+l)!}{(n-l-1)!\,(1-e^{-2\pi/\kappa})}\prod_{s=0}^{l'}(1+s^2\kappa^2)}$$
$$\times\left(\frac{4n}{1+n^2\kappa^2}\right)^{l_<+1}\frac{\exp\left[-\frac{2}{\kappa}\arctan(n\kappa)\right]}{4n^2(2l\pm 1)!}Y_\pm \tag{25}$$

where

$$Y_+=i\eta\left(\frac{n-i\eta}{n+i\eta}\right)^{n-l}\left[{}_2F_1\left(l+1-n,l-i\eta;\,2l+2;\,\frac{-4ni\eta}{(n-i\eta)^2}\right)\right.$$
$$\left.-\left(\frac{n+i\eta}{n-i\eta}\right)^2{}_2F_1\left(l+1-n,l+1-i\eta;\,2l+2;\,\frac{-4ni\eta}{(n-i\eta)^2}\right)\right] \tag{26}$$

and

$$Y_-=\left(\frac{n-i\eta}{n+i\eta}\right)^{n-l-1}\left[{}_2F_1\left(l-1-n,l-i\eta;\,2l;\,\frac{-4ni\eta}{(n-i\eta)^2}\right)\right.$$
$$\left.-\left(\frac{n+i\eta}{n-i\eta}\right)^2{}_2F_1\left(l+1-n,l-i\eta;\,2l;\,\frac{-4ni\eta}{(n-i\eta)^2}\right)\right] \tag{27}$$

with $\eta=1/\kappa$.

Since the hypergeometric series involved in equations (26) and (27) terminate after a finite number of terms, all the required $g(n,l;\,\kappa,l')$ could be calculated from (25) directly. However, this procedure is not suitable in general, because of the large number of terms occurring in the series when $n-l$ is large; also, for some ranges of the parameters, there is gross cancellation between the terms of the series (see below). The following alternative procedure was adopted.

We note that for given n and κ, the following recurrence relations connect matrix elements having different values of l (see (17), (18)),

$$2n\sqrt{[n^2-(l-1)^2][1+l^2\kappa^2]}\,g(n,l-2;\,\kappa,l-1)=[4n^2-4l^2+l(2l-1)(1+n^2\kappa^2)]$$
$$\times g(n,l-1;\,\kappa,l)-2n\sqrt{[n^2-l^2][1+(l+1)^2\kappa^2]}\,g(n,l;\,\kappa,l+1), \tag{28}$$

$$2n\sqrt{[n^2-l^2][1+(l-1)^2\kappa^2]}\,g(n,l-1;\,\kappa,l-2)=[4n^2-4l^2+l(2l+1)(1+n^2\kappa^2)]$$
$$\times g(n,l;\,\kappa,l-1)-2n\sqrt{[n^2-(l+1)^2][1+l^2\kappa^2]}\,g(n,l+1;\,\kappa,l). \tag{29}$$

Also, we note that when $l\approx n$, only a small number of terms occur in the hypergeometric series in (26) (27). In particular, when $l=n-1$, $n-2$ the expressions (25) to (27) simplify to give

$$g(n,n-1;\,0,n)=\sqrt{\frac{\pi}{2(2n-1)!}}\,4\,(4n)^n e^{-2n}, \tag{30}$$

$$g(n,n-1;\,\kappa,n)=\sqrt{\frac{\prod_{s=1}^{n}(1+s^2\kappa^2)}{1-e^{-2\pi/\kappa}}}\,\frac{\exp\left[2n-2\kappa^{-1}\arctan(n\kappa)\right]}{(1+n^2\kappa^2)^{n+2}}\,g(n,n-1;\,0,n), \tag{31}$$

$$g(n,n-2;\,\kappa,n-1)=\tfrac{1}{2}\sqrt{(2n-1)(1+n^2\kappa^2)}\,g(n,n-1;\,\kappa,n), \tag{32}$$

$$g(n,n-1;\,\kappa,n-2)=\frac{1}{2n}\sqrt{\frac{1+n^2\kappa^2}{1+(n-1)^2\kappa^2}}\,g(n,n-1;\,\kappa,n), \tag{33}$$

$$g(n,n-2;\,\kappa,n-3)=\frac{4+(n-1)(1+n^2\kappa^2)}{2n}\sqrt{\frac{2n-1}{1+(n-2)^2\kappa^2}}\,g(n,n-1;\,\kappa,n-2). \tag{34}$$

Equations (28) to (34) completely define the set of matrix elements required.

A set of equations alternative to (28) to (34) which in many respects is more convenient may be obtained by putting

$$g(n,l;\,\kappa,l') = \sqrt{\frac{(n+l)!}{(n-l-1)!}}\prod_{s=0}^{l'}(1+s^2\kappa^2)\,(2n)^{l-n}G(n,l;\,\kappa,l') \tag{35}$$

and obtaining equations analogous to (30) to (34) defining $G(n,l;\,\kappa,l')$ for $l=n-1, n-2$. The quantities $G(n,l;\kappa,l')$ then satisfy the recurrence relations

$$\begin{aligned} G(n,l-2;\,\kappa,l-1) = {} & [4n^2-4l^2+l(2l-1)(1+n^2\kappa^2)]G(n,l-1;\,\kappa,l) \\ & -4n^2(n^2-l^2)[1+(l+1)^2\kappa^2]G(n,l;\,\kappa,l+1), \end{aligned} \tag{36}$$

$$\begin{aligned} G(n,l-1;\,\kappa,l-2) = {} & [4n^2-4l^2+l(2l+1)(1+n^2\kappa^2)]G(n,l;\,\kappa,l-1) \\ & -4n^2[n^2-(l+1)^2][1+l^2\kappa^2]G(n,l+1;\,\kappa,l). \end{aligned} \tag{37}$$

Since repeated use of (36) and (37) obviously does not involve division or the evaluation of square roots, this scheme is very suitable for fast computing. The only drawback is that, for large n, severe scaling problems may arise due to the very rapid variation of $G(n,l;\,\kappa,l')$ with l.

4. *Computational details*

(i) *Photoionization cross-sections.*—The matrix elements $g(n,l;\,\kappa,l')$ were generated, for $n=1(1)20$, $l=0(1)n-1$, $l'=l\pm1$ and $n^2\kappa^2=0$ to 40·96 at convenient intervals, by applying the above equations in the following manner. (For convenience we describe the procedure in terms of the set of equations (28) to (34) throughout; it is obviously analogous if the set defining the quantities $G(n,l;\,\kappa,l')$ is used instead.) First n was fixed, κ set equal to 0, and (30), (32), (33) and (34) used to calculate g for the two largest possible values of l, i.e. $(n-1)$ and $(n-2)$; then (28) and (29) were applied repeatedly to obtain g for smaller values of l. Keeping n fixed, κ was then varied and the calculation repeated, this time using equations (31) to (34) then (28) and (29) repeatedly as before. Finally the whole procedure was repeated for different values of n.

(ii) *Recombination coefficients.*—The form of the integrals (10) would immediately suggest the use of the Gauss–Laguerre method for their evaluation. However, this method would require a fresh calculation of all the matrix elements $g(n,l;\,\kappa,l')$ for each value of t; so that, since a large range of values of temperature was to be covered, the less powerful Newton–Coates method was used.

The values of t used in the tabulation were chosen to be powers of 2 ($t=2^p$, with $p=-6(1)6$) since this enables a large temperature range to be covered and greatly reduces the number of exponentials which need be calculated in evaluating (10).

Examination of Table I shows that the integrand in (10) is always monotonically decreasing (approximately exponentially, except for $t=\infty$) so that it is desirable successively to increase the interval in κ^2 as κ^2 increases. Using the scheme described above for the evaluation of the quantities $g(n,l;\,\kappa,l')$, the integrals were evaluated for the following values of κ^2: $\kappa^2=0(h)4h(2h)12h(4h)28h\ldots(2^m h)\kappa^2_{\max}$, where $\kappa^2_{\max}=4h(2^{m+1}-1)$. A five point integration formula (Boole's Rule; see Buckingham (**19**)) was used over each sub-range $\kappa^2=0(h)4h$, $\kappa^2=4h(2h)12h$ etc. The starting interval h and the range $\kappa^2_{\max}$, which may vary with n but not with l, were chosen to ensure adequate convergence and accuracy of the integrals. The integrands decrease most rapidly for large l and small $t(\neq 0)$, so h was chosen to be sufficiently small to give the desired accuracy in this region. Similarly, $\kappa^2_{\max}$ was chosen sufficiently large to ensure sufficient convergence in the most slowly decreasing case, i.e. $l=0$, t large ($\neq\infty$). Aiming for at least four-figure accuracy, the values finally adopted were: $h=0{\cdot}00025/n$; $m=26$ for $n=1$, $m=25$ for $n=2$, $m=24$ for $n=3,4$, $m=23$ for $n=5,6\ldots9$, $m=22$ for $n=10,11\ldots16$, $m=21$ for $n=17,18\ldots20$.

Due to the very slow convergence, the case of l small, $t=\infty$ required special consideration. Examination of equations (28) to (34) shows that in equation (15) the integrand $\sim(\kappa^2)^{-l-3/2}$ when $\kappa\to\infty$. The range of integration for $t=\infty$ was therefore extrapolated by fitting the integrand to a series of the form $A_1/\kappa^{2l+3}+A_2/\kappa^{2l+4}+\ldots$. This form appears to give a very good fit; in all cases two terms proved sufficient.

5. *Description of tables*

In Table I we give the quantity $\Theta(N, L;\kappa, L')$ (see (2)) as a function of $N, L, L'=L\pm 1$ and X, where $X=N^2\kappa^2$.

In Table II we give the quantity $\Phi(N, L, L', t)$ (see (13)).

The values of Θ and Φ are given in the usual floating point decimal form, e.g. 5·4395, −4 denotes 0·00054395.

6. *Checks and error estimates*

(i) *Table I.*—Each individual step of the calculation wascarried out to an accuracy of about 8 significant figures so that the values of g for $l=n-1$, $n-2$ (equations (30) to (34)) should be accurate to about 7 significant figures. Extensive checks by hand computation showed this to be the case.

The main source of error lies in the repeated use of the recurrence relations (28) and (29) (or (36) and (37)). All the values of g are positive, so that the danger lies in loss of significant figures due to cancellation between the terms on the right-hand side of (28) and (29). The rate of build up of error with this type of recurrence relation depends essentially on the rate of increase (or decrease) of the terms as we go along the sequence. If the terms increase rapidly in the direction in which the sequence of terms is obtained then the error involved will be small, and vice versa. In our case the sequence operates in the direction $l=n-1$, $n-2,\ldots 2, 1, 0$. By referring to Table I it is seen that for small n the terms increase along the sequence, while for larger n they increase very rapidly at first, continue to increase over the major part of the sequence and then vary slowly over the last part. Also, the rate of increase is greater when κ is larger. We would thus expect that for the region of large n, small l and small κ, there would be small but appreciable build up of error, while outside this region the error should be negligible. Thus in addition to carrying out the usual random spot checks it is necessary to make careful independent checks for the region $n\approx 20$, $l\approx 0$, $\kappa\approx 0$.

Random spot checks were carried out by evaluating g directly from equations (25) to (27)*. Equations (26) and (27) were first cast into the following form which is more convenient for computation:

$$Y_+=\left(\frac{1+in\kappa}{-1+in\kappa}\right)^{n-l-1}\frac{4n}{1+n^2\kappa^2}\left\{2n+\frac{x(n-l-1)}{2l+3}[2n-1+n(2l+3)i\kappa]\right.$$
$$\left.+\frac{n-l-1}{2l+3}\sum_{p=2}^{n-l-1}\frac{x^p}{p!}[2n-p+n(2l+2+p)i\kappa]\prod_{q=2}^{p}\left[\frac{(n-l-q)[1+(l+q)i\kappa]}{2l+2+q}\right]\right\}, \tag{38}$$

$$Y_-=\left(\frac{1+in\kappa}{-1+in\kappa}\right)^{n-l-2}\frac{4n}{1+n^2\kappa^2}\left\{\frac{1}{l}+\frac{x(1+li\kappa)}{2l(2l+1)}[2n-2l-1+ni\kappa]\right.$$
$$\left.+\frac{1+li\kappa}{2l+1}\sum_{p=2}^{n-l}\frac{x^p}{p!}(2n-2l-p+pni\kappa)\prod_{q=2}^{p}\left[\frac{(n-l+1-q)[1+(l+q-1)i\kappa]}{2l+q}\right]\right\}, \tag{39}$$

where $x=-4n/(1+ni\kappa)^2$.

These spot calculations checked with the tables to within less than one in the fifth significant figure except for the region of n large, l small and κ small. In this region severe cancellation occurs in evaluating the series in (38) and (39) so that the spot calculations are no longer sufficiently accurate. We are thus left solely with the problem of checking the table in this special region.

* Since the preparation of these tables, calculations by Karzas and Latter (12) have been published. Their tables cover the range $n\leqslant 6$ of the present Table I. Their method of computation is essentially equivalent to the use of equations (25) to (27) cast into real form.

For this region of $n \approx 20$, $l \approx 0$, $\kappa \approx 0$ two independent checks were carried out. (*a*) If we put $\kappa = 0$ in equations (38) and (39) all the terms become real and we have

$$Y_+(\kappa=0) = (-1)^{n-l-1} 4n \left\{ 2n + \frac{(n-l-1)(-4n)(2n-1)}{(2l+3)} + \frac{n-l-1}{2l+3} \sum_{p=2}^{n-l-1} \frac{(-4n)^p}{p!} (2n-p) \prod_{q=2}^{p} \left[\frac{n-l-q}{2l+2+q} \right] \right\}, \tag{40}$$

$$Y_-(\kappa=0) = (-1)^{n-l-2} \frac{2n}{l} \left\{ 2 + \frac{(2n-2l-1)(-4n)}{(2l+1)} + \frac{1}{(2l+1)} \sum_{p=2}^{n-l} \frac{(-4n)^p}{p!} (2n-2l-p) \prod_{q=2}^{p} \left[\frac{n-l+1-q}{2l+q} \right] \right\}. \tag{41}$$

To overcome the difficulty of cancellation, (40) and (41) were calculated using double-length arithmetic (i.e. ~ 17 significant figure accuracy for each separate operation). Comparison of these double-length calculations with the single length check calculations shows that the double-length calculations should finally be accurate to about 8 significant figures. These checks show that for $n \approx 20$, $l \approx 0$, $\kappa \approx 0$ the table is in error by less than 4 in the sixth significant figure. (*b*) It is precisely in this region that the asymptotic expansions (7)

$$g(n, l;\, \kappa, l+1) \sim 1{\cdot}63479 \left[\frac{\prod_{s=1}^{l+1} (1+s^2\kappa^2)}{\prod_{s=0}^{l} (1-s^2 n^{-2})} \right]^{1/2} n^{-1/6} (1+n^2\kappa^2)^{-5/3}$$
$$\times \{1 + 0{\cdot}17282[4+5l+n^2\kappa^2(5l+6)](1+n^2\kappa^2)^{-2/3} n^{-2/3} + O(n^{-2})\}, \tag{42}$$

$$g(n, l;\, \kappa, l-1) \sim 1{\cdot}63479 \left[\frac{\prod_{s=1}^{l} (1-s^2 n^{-2})}{\prod_{s=0}^{l-1} (1+s^2\kappa^2)} \right]^{1/2} n^{-1/6} (1+n^2\kappa^2)^{-5/3}$$
$$\times \{1 - 0{\cdot}17282[5l+1+n^2\kappa^2(5l-1)](1+n^2\kappa^2)^{-2/3} n^{-2/3} + O(n^{-2})\} \tag{43}$$

are valid. Insufficient terms of the asymptotic series are available to enable the table to be checked to full accuracy. However, it is satisfying, since this is an entirely independent check, that the table agrees with the asymptotic series to within the error to be expected in the latter. For example, for $n = 20$, $l = 0$, $\kappa \approx 0$ the two agree to within ~ 2 in the fifth significant figure.

(ii) *Table II*.—Assuming from section (i) that the integrands in (10) are accurate to at least 5 significant figures, it is a relatively easy matter to check the accuracy of $\Phi(n, l, l', t)$. Extensive checks, especially in the region of large l and small t and of small l and large t (see section 4 (ii)), using varying intervals and ranges of integration, showed the integration errors to be less than 1 in the fifth significant figure.

Both Tables I and II were checked against previous calculations (7), (12) and no systematic errors were discovered.

On the basis of the above set of checks it appears that Tables I and II are accurate to within one in the last figure given. In order to avoid typesetting errors the tables have been reproduced directly from the computer output.

Acknowledgments.—The calculations were carried out on the Ferranti Mercury computer of the University of London Computer Unit. I would like to thank the staff of the Computer Unit for their cooperation during the preparation of the tables. The work was principally carried out during the tenure of a Post-Doctoral Fellowship granted by the Atomic Energy Research Establishment (Harwell), to whom thanks are due for permission to publish, and was completed during a visit* to the Physics

* Supported by the U.S. Office of Ordnance Research under Contract No. DA-30-69-ORD-2581, and the U.S. Office of Naval Research and the Advanced Research Projects Agency under Contract Nonr-285 (49).

Department of New York University, to whom I am indebted for their hospitality. I would also like to thank Dr. M. J. Seaton for his constant encouragement.

Physics Department,
University College London,
Gower Street, W.C.1

1963 *April.*

References

(1) Aller, L. H., *Gaseous Nebulae*, Chapman and Hall, 1956.
(2) Seaton, M. J., *Rep. Prog. Phys.*, **23,** 313, 1960.
(3) Bates, D. R. and Dalgarno, A., *Atomic and Molecular Processes* (ed. Bates, D. R.), Ch. 7, Academic Press, 1962.
(4) Kolb, A. C. and Griem, H. R., *Atomic and Molecular Processes* (ed. Bates, D. R.), Ch. 5, Academic Press, 1962.
(5) Martin, A. D., *Nuovo Cimento*, **27,** 1359, 1963.
(6) Searle, L., *Ap. J.*, **128,** 489, 1958.
(7) Burgess, A., *M.N.*, **118,** 477, 1958.
(8) Pengelly, R. M., *M.N.*, **127,** 145, 1964.
(9) Burgess, A. and Seaton, M. J., *M.N.*, **121,** 76, 1960.
(10) Pengelly, R. M., to be submitted to *M.N.*
(11) Gordon, W., *Ann. d. Phys.*, **2,** 1031, 1929.
(12) Karzas, W. J. and Latter, R., *Ap. J. Suppl.*, **6,** 167, 1961.
(13) Green, L. C., Rush, P. P. and Chandler, C. D., *Ap. J. Suppl.*, **3,** 37, 1957.
(14) Pengelly, R. M. and Seaton, M. J., *M.N.*, **127,** 165, 1964.
(15) Seaton, M. J., *M.N.*, **119,** 81, 1959.
(16) Burgess, A. and Seaton, M. J., *M.N.*, **120,** 121, 1960.
(17) Biedenharn, L. C., McHale, J. L. and Thaler, R. M., *Phys. Rev.*, **100,** 376, 1955.
(18) Alder, K. and Winther, A., *K. Danske Vidensk. Selsk. mat. fys. Medd.*, **29,** No. 18, 1955.
(19) Buckingham, R. A., *Numerical Methods*, Pitman, 1957.

N	L	0	1		2		3	
	X \ L′	1	0	2	1	3	2	4
1	0.000	7.3652, 0						
	0.005	7.2678, 0						
	0.010	7.1722, 0						
	0.020	6.9859, 0						
	0.040	6.6324, 0						
	0.080	5.9939, 0						
	0.160	4.9432, 0						
	0.320	3.4765, 0						
	0.640	1.9053, 0						
	1.280	7.4789, -1						
	2.560	2.0334, -1						
	5.120	3.9700, -2						
	10.240	6.0072, -3						
	20.480	7.6096, -4						
	40.960	8.5625, -5						
2	0.000	4.3167, 0	3.5973, -1	5.7556, 0				
	0.005	4.2721, 0	3.5556, -1	5.6678, 0				
	0.010	4.2281, 0	3.5146, -1	5.5816, 0				
	0.020	4.1420, 0	3.4345, -1	5.4143, 0				
	0.040	3.9770, 0	3.2813, -1	5.0987, 0				
	0.080	3.6735, 0	3.0012, -1	4.5352, 0				
	0.160	3.1566, 0	2.5293, -1	3.6283, 0				
	0.320	2.3889, 0	1.8433, -1	2.4131, 0				
	0.640	1.4770, 0	1.0611, -1	1.2008, 0				
	1.280	6.9099, -1	4.3623, -2	4.0409, -1				
	2.560	2.3405, -1	1.1893, -2	8.7660, -2				
	5.120	5.7867, -2	2.1150, -3	1.2607, -2				
	10.240	1.0943, -2	2.5617, -4	1.2982, -3				
	20.480	1.6788, -3	2.2860, -5	1.0421, -4				
	40.960	2.2039, -4	1.6340, -6	7.0032, -6				
3	0.000	3.2688, 0	5.3368, -1	4.8031, 0	1.0674, -1	3.8425, 0		
	0.005	3.2358, 0	5.2750, -1	4.7449, 0	1.0521, -1	3.7770, 0		
	0.010	3.2033, 0	5.2142, -1	4.6876, 0	1.0371, -1	3.7130, 0		
	0.020	3.1397, 0	5.0954, -1	4.5757, 0	1.0079, -1	3.5888, 0		
	0.040	3.0179, 0	4.8687, -1	4.3625, 0	9.5251, -2	3.3558, 0		
	0.080	2.7939, 0	4.4549, -1	3.9741, 0	8.5281, -2	2.9438, 0		
	0.160	2.4122, 0	3.7600, -1	3.3249, 0	6.8982, -2	2.2931, 0		
	0.320	1.8446, 0	2.7545, -1	2.3943, 0	4.6581, -2	1.4511, 0		
	0.640	1.1664, 0	1.6118, -1	1.3557, 0	2.3455, -2	6.6130, -1		
	1.280	5.7092, -1	6.9263, -2	5.4883, -1	7.7739, -3	1.9257, -1		
	2.560	2.0932, -1	2.0780, -2	1.4950, -1	1.5541, -3	3.3596, -2		
	5.120	5.8035, -2	4.3362, -3	2.7376, -2	1.8572, -4	3.5785, -3		
	10.240	1.2582, -2	6.4455, -4	3.5344, -3	1.4149, -5	2.5156, -4		
	20.480	2.2167, -3	7.0802, -5	3.4422, -4	7.5719, -7	1.2820, -5		
	40.960	3.2954, -4	6.0452, -6	2.7012, -5	3.1257, -8	5.1501, -7		
4	0.000	2.7237, 0	6.2732, -1	4.0037, 0	2.0183, -1	4.2179, 0	3.7660, -2	2.4102, 0
	0.005	2.6962, 0	6.2006, -1	3.9592, 0	1.9909, -1	4.1596, 0	3.7035, -2	2.3651, 0
	0.010	2.6690, 0	6.1291, -1	3.9153, 0	1.9639, -1	4.1022, 0	3.6422, -2	2.3209, 0
	0.020	2.6158, 0	5.9896, -1	3.8295, 0	1.9113, -1	3.9906, 0	3.5234, -2	2.2356, 0
	0.040	2.5141, 0	5.7234, -1	3.6656, 0	1.8115, -1	3.7788, 0	3.2997, -2	2.0763, 0
	0.080	2.3272, 0	5.2378, -1	3.3656, 0	1.6312, -1	3.3966, 0	2.9021, -2	1.7972, 0
	0.160	2.0093, 0	4.4238, -1	2.8596, 0	1.3349, -1	2.7694, 0	2.2686, -2	1.3643, 0
	0.320	1.5379, 0	3.2485, -1	2.1205, 0	9.2267, -2	1.9003, 0	1.4379, -2	8.2265, -1
	0.640	9.7646, -1	1.9148, -1	1.2664, 0	4.8635, -2	9.8745, -1	6.4828, -3	3.4406, -1
	1.280	4.8408, -1	8.3884, -2	5.6031, -1	1.7526, -2	3.4677, -1	1.8001, -3	8.6909, -2
	2.560	1.8288, -1	2.6266, -2	1.7376, -1	4.0036, -3	7.6126, -2	2.7856, -4	1.2219, -2
	5.120	5.3588, -2	5.9343, -3	3.7386, -2	5.7067, -4	1.0340, -2	2.3793, -5	9.6542, -4
	10.240	1.2603, -2	9.9631, -4	5.7308, -3	5.2666, -5	9.1219, -4	1.2048, -6	4.6375, -5
	20.480	2.4473, -3	1.2714, -4	6.5546, -4	3.3687, -6	5.6429, -5	4.0242, -8	1.5012, -6
	40.960	4.0149, -4	1.2606, -5	5.9001, -5	1.6143, -7	2.6479, -6	9.8343, -10	3.6060, -8

N	L	0	1		2		3	
	X \ L′	1	0	2	1	3	2	4
5	0.000	2.3832, 0	6.8152, −1	3.4451, 0	2.7661, −1	3.9797, 0	8.4200, −2	3.2890, 0
	0.005	2.3589, 0	6.7364, −1	3.4081, 0	2.7296, −1	3.9304, 0	8.2894, −2	3.2380, 0
	0.010	2.3349, 0	6.6588, −1	3.3717, 0	2.6936, −1	3.8819, 0	8.1613, −2	3.1880, 0
	0.020	2.2881, 0	6.5072, −1	3.3004, 0	2.6236, −1	3.7871, 0	7.9123, −2	3.0907, 0
	0.040	2.1984, 0	6.2181, −1	3.1642, 0	2.4907, −1	3.6067, 0	7.4425, −2	2.9071, 0
	0.080	2.0338, 0	5.6912, −1	2.9147, 0	2.2503, −1	3.2785, 0	6.6028, −2	2.5789, 0
	0.160	1.7544, 0	4.8087, −1	2.4924, 0	1.8538, −1	2.7315, 0	5.2501, −2	2.0499, 0
	0.320	1.3412, 0	3.5359, −1	1.8721, 0	1.2987, −1	1.9513, 0	3.4384, −2	1.3411, 0
	0.640	8.5109, −1	2.0929, −1	1.1464, 0	7.0297, −2	1.0881, 0	1.6462, −2	6.4007, −1
	1.280	4.2313, −1	9.2688, −2	5.3113, −1	2.6624, −2	4.2740, −1	5.0565, −3	1.9516, −1
	2.560	1.6171, −1	2.9725, −2	1.7771, −1	6.6128, −3	1.0955, −1	9.0711, −4	3.4558, −2
	5.120	4.8702, −2	7.0263, −3	4.2585, −2	1.0641, −3	1.7872, −2	9.2955, −5	3.4838, −3
	10.240	1.2022, −2	1.2709, −3	7.4266, −3	1.1374, −4	1.9015, −3	5.6910, −6	2.1011, −4
	20.480	2.4975, −3	1.8005, −4	9.6970, −4	8.4523, −6	1.3947, −4	2.2610, −7	8.2575, −6
	40.960	4.4250, −4	2.0120, −5	9.8472, −5	4.6300, −7	7.5470, −6	6.3872, −9	2.3172, −7
6	0.000	2.1471, 0	7.1418, −1	3.0439, 0	3.3444, −1	3.6540, 0	1.2951, −1	3.5020, 0
	0.005	2.1251, 0	7.0593, −1	3.0118, 0	3.3011, −1	3.6114, 0	1.2759, −1	3.4538, 0
	0.010	2.1033, 0	6.9779, −1	2.9801, 0	3.2586, −1	3.5695, 0	1.2571, −1	3.4063, 0
	0.020	2.0607, 0	6.8191, −1	2.9181, 0	3.1755, −1	3.4875, 0	1.2204, −1	3.3139, 0
	0.040	1.9793, 0	6.5163, −1	2.7997, 0	3.0179, −1	3.3313, 0	1.1511, −1	3.1385, 0
	0.080	1.8300, 0	5.9646, −1	2.5826, 0	2.7324, −1	3.0461, 0	1.0269, −1	2.8219, 0
	0.160	1.5769, 0	5.0410, −1	2.2150, 0	2.2607, −1	2.5674, 0	8.2537, −2	2.3015, 0
	0.320	1.2035, 0	3.7100, −1	1.6741, 0	1.5976, −1	1.8753, 0	5.5200, −2	1.5789, 0
	0.640	7.6227, −1	2.2019, −1	1.0384, 0	8.7990, −2	1.0894, 0	2.7482, −2	8.1765, −1
	1.280	3.7873, −1	9.8203, −2	4.9345, −1	3.4431, −2	4.5888, −1	9.0351, −3	2.8291, −1
	2.560	1.4532, −1	3.1980, −2	1.7291, −1	9.0486, −3	1.3077, −1	1.8006, −3	5.9388, −2
	5.120	4.4373, −2	7.7808, −3	4.4565, −2	1.5869, −3	2.4493, −2	2.1213, −4	7.2836, −3
	10.240	1.1277, −2	1.4775, −3	8.5608, −3	1.8987, −4	3.0385, −3	1.5194, −5	5.3444, −4
	20.480	2.4547, −3	2.2521, −4	1.2458, −3	1.5986, −5	2.5866, −4	7.0312, −7	2.5037, −5
	40.960	4.6161, −4	2.7632, −5	1.4057, −4	9.8638, −7	1.5946, −5	2.2689, −8	8.1261, −7
7	0.000	1.9721, 0	7.3415, −1	2.7438, 0	3.7940, −1	3.3499, 0	1.7052, −1	3.4612, 0
	0.005	1.9517, 0	7.2566, −1	2.7151, 0	3.7456, −1	3.3122, 0	1.6807, −1	3.4171, 0
	0.010	1.9315, 0	7.1730, −1	2.6867, 0	3.6980, −1	3.2752, 0	1.6567, −1	3.3736, 0
	0.020	1.8921, 0	7.0097, −1	2.6313, 0	3.6051, −1	3.2027, 0	1.6099, −1	3.2888, 0
	0.040	1.8168, 0	6.6986, −1	2.5253, 0	3.4287, −1	3.0644, 0	1.5214, −1	3.1275, 0
	0.080	1.6788, 0	6.1317, −1	2.3310, 0	3.1091, −1	2.8115, 0	1.3623, −1	2.8347, 0
	0.160	1.4451, 0	5.1832, −1	2.0020, 0	2.5802, −1	2.3857, 0	1.1032, −1	2.3484, 0
	0.320	1.1011, 0	3.8171, −1	1.5177, 0	1.8346, −1	1.7657, 0	7.4859, −2	1.6593, 0
	0.640	6.9586, −1	2.2698, −1	9.4784, −1	1.0228, −1	1.0519, 0	3.8303, −2	9.0604, −1
	1.280	3.4507, −1	1.0173, −1	4.5713, −1	4.0956, −2	4.6365, −1	1.3218, −2	3.4208, −1
	2.560	1.3248, −1	3.3487, −2	1.6491, −1	1.1206, −2	1.4228, −1	2.8466, −3	8.1499, −2
	5.120	4.0730, −2	8.3124, −3	4.4684, −2	2.0939, −3	2.9584, −2	3.7379, −4	1.1701, −2
	10.240	1.0537, −2	1.6328, −3	9.2307, −3	2.7343, −4	4.1612, −3	3.0484, −5	1.0161, −3
	20.480	2.3690, −3	2.6231, −4	1.4675, −3	2.5536, −5	4.0332, −4	1.6125, −6	5.5779, −5
	40.960	4.6642, −4	3.4602, −5	1.8150, −4	1.7503, −6	2.8002, −5	5.8780, −8	2.0738, −6
8	0.000	1.8361, 0	7.4618, −1	2.5112, 0	4.1471, −1	3.0869, 0	2.0662, −1	3.3286, 0
	0.005	1.8169, 0	7.3756, −1	2.4850, 0	4.0948, −1	3.0530, 0	2.0373, −1	3.2882, 0
	0.010	1.7980, 0	7.2906, −1	2.4591, 0	4.0433, −1	3.0196, 0	2.0089, −1	3.2485, 0
	0.020	1.7611, 0	7.1246, −1	2.4085, 0	3.9428, −1	2.9543, 0	1.9535, −1	3.1710, 0
	0.040	1.6906, 0	6.8085, −1	2.3117, 0	3.7519, −1	2.8296, 0	1.8486, −1	3.0233, 0
	0.080	1.5613, 0	6.2326, −1	2.1344, 0	3.4060, −1	2.6015, 0	1.6599, −1	2.7543, 0
	0.160	1.3427, 0	5.2692, −1	1.8343, 0	2.8330, −1	2.2165, 0	1.3515, −1	2.3048, 0
	0.320	1.0215, 0	3.8823, −1	1.3927, 0	2.0236, −1	1.6540, 0	9.2695, −2	1.6603, 0
	0.640	6.4414, −1	2.3119, −1	8.7296, −1	1.1384, −1	1.0015, 0	4.8395, −2	9.3968, −1
	1.280	3.1868, −1	1.0400, −1	4.2473, −1	4.6372, −2	4.5512, −1	1.7315, −2	3.7739, −1
	2.560	1.2223, −1	3.4506, −2	1.5614, −1	1.3077, −2	1.4722, −1	3.9558, −3	9.8904, −2
	5.120	3.7692, −2	8.6929, −3	4.3814, −2	2.5641, −3	3.3153, −2	5.6595, −4	1.6122, −2
	10.240	9.8578, −3	1.7505, −3	9.5627, −3	3.5871, −4	5.1677, −3	5.1416, −5	1.6174, −3
	20.480	2.2669, −3	2.9242, −4	1.6339, −3	3.6514, −5	5.6066, −4	3.0577, −6	1.0241, −4
	40.960	4.6239, −4	4.0811, −5	2.1883, −4	2.7456, −6	4.3369, −5	1.2459, −7	4.3215, −6

4		5		6		7	
3	5	4	6	5	7	6	8
1.4618, −2	1.4618, 0						
1.4345, −2	1.4320, 0						
1.4079, −2	1.4029, 0						
1.3563, −2	1.3467, 0						
1.2598, −2	1.2424, 0						
1.0904, −2	1.0613, 0						
8.2645, −3	7.8541, −1						
4.9473, −3	4.5156, −1						
2.0181, −3	1.7346, −1						
4.7679, −4	3.8043, −2						
5.8214, −5	4.3144, −3						
3.6204, −6	2.5300, −4						
1.2362, −7	8.3079, −6						
2.6015, −9	1.7086, −7						
3.7842, −11	2.4544, −9						
3.7300, −2	2.3872, 0	6.0282, −3	8.6806, −1				
3.6654, −2	2.3462, 0	5.9042, −3	8.4891, −1				
3.6022, −2	2.3060, 0	5.7831, −3	8.3025, −1				
3.4796, −2	2.2282, 0	5.5496, −3	7.9436, −1				
3.2495, −2	2.0819, 0	5.1153, −3	7.2794, −1				
2.8425, −2	1.8229, 0	4.3610, −3	6.1377, −1				
2.1993, −2	1.4128, 0	3.2110, −3	4.4290, −1				
1.3687, −2	8.8155, −1	1.8214, −3	2.4285, −1				
5.9826, −3	3.8659, −1	6.7586, −4	8.5719, −2				
1.5807, −3	1.0239, −1	1.3689, −4	1.6331, −2				
2.2588, −4	1.4631, −2	1.3301, −5	1.4945, −3				
1.6948, −5	1.0951, −3	6.0704, −7	6.5069, −5				
7.0113, −7	4.5157, −5	1.4059, −8	1.4609, −6				
1.7554, −8	1.1277, −6	1.8707, −10	1.9090, −8				
2.9510, −10	1.8925, −8	1.6230, −12	1.6401, −10				
6.3108, −2	2.8433, 0	1.7193, −2	1.6512, 0	2.5922, −3	5.0807, −1		
6.2072, −2	2.8000, 0	1.6866, −2	1.6202, 0	2.5341, −3	4.9603, −1		
6.1057, −2	2.7575, 0	1.6546, −2	1.5898, 0	2.4775, −3	4.8431, −1		
5.9087, −2	2.6748, 0	1.5927, −2	1.5310, 0	2.3686, −3	4.6183, −1		
5.5380, −2	2.5186, 0	1.4771, −2	1.4210, 0	2.1671, −3	4.2042, −1		
4.8790, −2	2.2386, 0	1.2745, −2	1.2281, 0	1.8210, −3	3.4990, −1		
3.8280, −2	1.7855, 0	9.6045, −3	9.2822, −1	1.3039, −3	2.4621, −1		
2.4463, −2	1.1741, 0	5.6888, −3	5.5246, −1	7.0220, −4	1.2878, −1		
1.1222, −2	5.6304, −1	2.2771, −3	2.2267, −1	2.3776, −4	4.1777, −2		
3.2148, −3	1.7091, −1	5.1997, −4	5.1244, −2	4.1458, −5	6.9155, −3		
5.1781, −4	2.9248, −2	5.9575, −5	5.9104, −3	3.2208, −6	5.1084, −4		
4.5231, −5	2.6856, −3	3.2974, −6	3.2851, −4	1.0830, −7	1.6516, −5		
2.2068, −6	1.3551, −4	9.2785, −8	9.2631, −6	1.7061, −9	2.5357, −7		
6.4631, −8	4.0472, −6	1.4709, −9	1.4698, −7	1.4380, −11	2.1055, −9		
1.2436, −9	7.8706, −8	1.4758, −11	1.4753, −9	7.4487, −14	1.0818, −11		
8.9347, −2	3.0281, 0	3.1603, −2	2.1799, 0	8.1492, −3	1.1034, 0	1.1494, −3	2.9425, −1
8.7939, −2	2.9857, 0	3.1035, −2	2.1435, 0	7.9802, −3	1.0809, 0	1.1216, −3	2.8679, −1
8.6558, −2	2.9441, 0	3.0479, −2	2.1077, 0	7.8152, −3	1.0588, 0	1.0945, −3	2.7955, −1
8.3874, −2	2.8630, 0	2.9403, −2	2.0383, 0	7.4970, −3	1.0162, 0	1.0426, −3	2.6568, −1
7.8816, −2	2.7092, 0	2.7386, −2	1.9078, 0	6.9056, −3	9.3706, −1	9.4710, −4	2.4027, −1
6.9794, −2	2.4317, 0	2.3832, −2	1.6758, 0	5.8795, −3	7.9942, −1	7.8467, −4	1.9739, −1
5.5313, −2	1.9765, 0	1.8258, −2	1.3063, 0	4.3177, −3	5.8923, −1	5.4680, −4	1.3545, −1
3.6035, −2	1.3464, 0	1.1155, −2	8.2251, −1	2.4366, −3	3.3456, −1	2.7992, −4	6.7589, −2
1.7127, −2	6.8720, −1	4.7203, −3	3.6458, −1	8.9463, −4	1.2395, −1	8.6655, −5	2.0155, −2
5.2131, −3	2.3061, −1	1.1805, −3	9.6855, −2	1.7700, −4	2.4790, −2	1.3042, −5	2.8994, −3
9.2163, −4	4.5463, −2	1.5418, −4	1.3476, −2	1.6314, −5	2.3082, −3	8.1229, −7	1.7290, −4
9.1147, −5	4.9535, −3	1.0033, −5	9.2434, −4	6.6837, −7	9.5280, −5	2.0173, −8	4.1516, −6
5.1255, −6	2.9874, −4	3.3524, −7	3.2019, −5	1.2828, −8	1.8373, −6	2.1650, −10	4.3590, −8
1.7287, −7	1.0521, −5	6.2418, −9	6.0911, −7	1.2901, −10	1.8526, −8	1.1570, −12	2.3000, −10
3.7724, −9	2.3520, −7	7.1835, −11	7.0935, −9	7.7343, −13	1.1122, −10	3.5800, −15	7.0680, −13

N	L	0	1		2		3	
	L′ / t	1	0	2	1	3	2	4
1	∞	7.400, 2						
	64	1.827, 2						
	32	1.238, 2						
	16	7.804, 1						
	8	4.599, 1						
	4	2.559, 1						
	2	1.365, 1						
	1	7.077, 0						
	1/2	7.216, 0						
	1/4	7.289, 0						
	1/8	7.327, 0						
	1/16	7.346, 0						
	1/32	7.356, 0						
	1/64	7.361, 0						
	0	7.365, 0						
2	∞	3.863, 2	6.132, 0	9.076, 1				
	64	1.021, 2	4.179, 0	7.025, 1				
	32	7.037, 1	3.376, 0	6.045, 1				
	16	4.514, 1	2.513, 0	4.879, 1				
	8	2.698, 1	1.708, 0	3.647, 1				
	4	1.514, 1	1.061, 0	2.505, 1				
	2	8.097, 0	6.093, −1	1.581, 1				
	1	4.191, 0	3.301, −1	9.253, 0				
	1/2	4.259, 0	3.445, −1	1.020, 1				
	1/4	4.291, 0	3.521, −1	1.080, 1				
	1/8	4.305, 0	3.559, −1	1.114, 1				
	1/16	4.311, 0	3.578, −1	1.132, 1				
	1/32	4.314, 0	3.588, −1	1.141, 1				
	1/64	4.316, 0	3.593, −1	1.146, 1				
	0	4.317, 0	3.597, −1	1.151, 1				
3	∞	2.616, 2	6.010, 0	7.418, 1	6.415, −1	2.395, 1		
	64	7.112, 1	4.336, 0	5.775, 1	5.912, −1	2.247, 1		
	32	4.952, 1	3.610, 0	4.984, 1	5.531, −1	2.131, 1		
	16	3.217, 1	2.798, 0	4.037, 1	4.952, −1	1.949, 1		
	8	1.951, 1	2.000, 0	3.030, 1	4.163, −1	1.690, 1		
	4	1.112, 1	1.314, 0	2.090, 1	3.228, −1	1.365, 1		
	2	6.024, 0	7.971, −1	1.323, 1	2.279, −1	1.012, 1		
	1	3.147, 0	4.524, −1	7.753, 0	1.462, −1	6.837, 0		
	1/2	3.215, 0	4.883, −1	8.550, 0	1.722, −1	8.445, 0		
	1/4	3.246, 0	5.096, −1	9.039, 0	1.901, −1	9.677, 0		
	1/8	3.259, 0	5.213, −1	9.312, 0	2.009, −1	1.049, 1		
	1/16	3.265, 0	5.274, −1	9.457, 0	2.070, −1	1.098, 1		
	1/32	3.267, 0	5.306, −1	9.531, 0	2.102, −1	1.124, 1		
	1/64	3.268, 0	5.321, −1	9.569, 0	2.118, −1	1.138, 1		
	0	3.269, 0	5.337, −1	9.606, 0	2.135, −1	1.153, 1		
4	∞	1.977, 2	5.313, 0	5.986, 1	9.128, −1	2.615, 1	1.175, −1	8.026, 0
	64	5.455, 1	3.943, 0	4.678, 1	8.520, −1	2.455, 1	1.145, −1	7.851, 0
	32	3.821, 1	3.334, 0	4.046, 1	8.050, −1	2.329, 1	1.118, −1	7.690, 0
	16	2.503, 1	2.642, 0	3.288, 1	7.319, −1	2.132, 1	1.069, −1	7.400, 0
	8	1.534, 1	1.942, 0	2.477, 1	6.296, −1	1.851, 1	9.871, −2	6.909, 0
	4	8.848, 0	1.321, 0	1.716, 1	5.035, −1	1.497, 1	8.646, −2	6.153, 0
	2	4.856, 0	8.319, −1	1.092, 1	3.690, −1	1.112, 1	7.032, −2	5.124, 0
	1	2.567, 0	4.895, −1	6.428, 0	2.464, −1	7.517, 0	5.217, −2	3.915, 0
	1/2	2.647, 0	5.446, −1	7.111, 0	3.014, −1	9.290, 0	7.006, −2	5.430, 0
	1/4	2.688, 0	5.809, −1	7.530, 0	3.429, −1	1.064, 1	8.563, −2	6.844, 0
	1/8	2.707, 0	6.026, −1	7.763, 0	3.699, −1	1.153, 1	9.702, −2	7.949, 0
	1/16	2.716, 0	6.145, −1	7.884, 0	3.858, −1	1.206, 1	1.042, −1	8.690, 0
	1/32	2.720, 0	6.208, −1	7.946, 0	3.944, −1	1.235, 1	1.084, −1	9.133, 0
	1/64	2.722, 0	6.241, −1	7.977, 0	3.990, −1	1.250, 1	1.106, −1	9.378, 0
	0	2.724, 0	6.273, −1	8.007, 0	4.037, −1	1.265, 1	1.130, −1	9.641, 0

N	t \ L	0	1		2		3	
	L′	1	0	2	1	3	2	4
5	∞	1.588, 2	4.654, 0	4.968, 1	1.005, 0	2.438, 1	2.134, −1	1.094, 1
	64	4.422, 1	3.511, 0	3.893, 1	9.450, −1	2.290, 1	2.086, −1	1.070, 1
	32	3.110, 1	2.998, 0	3.372, 1	8.982, −1	2.174, 1	2.043, −1	1.048, 1
	16	2.047, 1	2.407, 0	2.746, 1	8.246, −1	1.992, 1	1.964, −1	1.009, 1
	8	1.264, 1	1.803, 0	2.076, 1	7.197, −1	1.731, 1	1.832, −1	9.420, 0
	4	7.360, 0	1.254, 0	1.444, 1	5.873, −1	1.403, 1	1.630, −1	8.394, 0
	2	4.083, 0	8.111, −1	9.233, 0	4.418, −1	1.043, 1	1.354, −1	6.993, 0
	1	2.183, 0	4.906, −1	5.463, 0	3.040, −1	7.069, 0	1.033, −1	5.347, 0
	1/2	2.273, 0	5.599, −1	6.072, 0	3.832, −1	8.751, 0	1.429, −1	7.419, 0
	1/4	2.326, 0	6.098, −1	6.452, 0	4.474, −1	1.004, 1	1.799, −1	9.352, 0
	1/8	2.355, 0	6.418, −1	6.666, 0	4.923, −1	1.088, 1	2.087, −1	1.086, 1
	1/16	2.369, 0	6.605, −1	6.778, 0	5.201, −1	1.138, 1	2.279, −1	1.187, 1
	1/32	2.376, 0	6.707, −1	6.834, 0	5.359, −1	1.165, 1	2.394, −1	1.247, 1
	1/64	2.380, 0	6.760, −1	6.862, 0	5.444, −1	1.179, 1	2.458, −1	1.280, 1
	0	2.383, 0	6.815, −1	6.890, 0	5.532, −1	1.194, 1	2.526, −1	1.316, 1
6	∞	1.327, 2	4.106, 0	4.230, 1	1.017, 0	2.204, 1	2.764, −1	1.159, 1
	64	3.717, 1	3.132, 0	3.321, 1	9.616, −1	2.072, 1	2.709, −1	1.134, 1
	32	2.621, 1	2.691, 0	2.881, 1	9.176, −1	1.967, 1	2.658, −1	1.111, 1
	16	1.732, 1	2.180, 0	2.350, 1	8.478, −1	1.803, 1	2.566, −1	1.069, 1
	8	1.075, 1	1.653, 0	1.781, 1	7.475, −1	1.569, 1	2.410, −1	9.988, 0
	4	6.303, 0	1.169, 0	1.243, 1	6.188, −1	1.273, 1	2.166, −1	8.904, 0
	2	3.527, 0	7.708, −1	7.982, 0	4.743, −1	9.489, 0	1.829, −1	7.424, 0
	1	1.904, 0	4.764, −1	4.747, 0	3.338, −1	6.443, 0	1.424, −1	5.681, 0
	1/2	2.002, 0	5.554, −1	5.302, 0	4.311, −1	7.996, 0	2.019, −1	7.890, 0
	1/4	2.066, 0	6.162, −1	5.659, 0	5.150, −1	9.190, 0	2.603, −1	9.954, 0
	1/8	2.104, 0	6.579, −1	5.865, 0	5.772, −1	9.979, 0	3.085, −1	1.156, 1
	1/16	2.125, 0	6.835, −1	5.975, 0	6.178, −1	1.044, 1	3.425, −1	1.264, 1
	1/32	2.136, 0	6.981, −1	6.032, 0	6.417, −1	1.070, 1	3.635, −1	1.328, 1
	1/64	2.142, 0	7.060, −1	6.060, 0	6.548, −1	1.083, 1	3.755, −1	1.363, 1
	0	2.147, 0	7.142, −1	6.088, 0	6.689, −1	1.096, 1	3.885, −1	1.401, 1
7	∞	1.140, 2	3.659, 0	3.678, 1	9.947, −1	1.985, 1	3.145, −1	1.137, 1
	64	3.205, 1	2.812, 0	2.892, 1	9.434, −1	1.867, 1	3.088, −1	1.112, 1
	32	2.264, 1	2.427, 0	2.510, 1	9.028, −1	1.773, 1	3.035, −1	1.090, 1
	16	1.500, 1	1.979, 0	2.050, 1	8.381, −1	1.626, 1	2.939, −1	1.049, 1
	8	9.346, 0	1.514, 0	1.557, 1	7.443, −1	1.416, 1	2.773, −1	9.806, 0
	4	5.511, 0	1.083, 0	1.090, 1	6.227, −1	1.151, 1	2.513, −1	8.747, 0
	2	3.106, 0	7.253, −1	7.024, 0	4.842, −1	8.592, 0	2.147, −1	7.299, 0
	1	1.691, 0	4.559, −1	4.196, 0	3.469, −1	5.848, 0	1.698, −1	5.591, 0
	1/2	1.794, 0	5.410, −1	4.711, 0	4.569, −1	7.276, 0	2.455, −1	7.774, 0
	1/4	1.866, 0	6.103, −1	5.052, 0	5.565, −1	8.383, 0	3.232, −1	9.819, 0
	1/8	1.912, 0	6.605, −1	5.256, 0	6.344, −1	9.121, 0	3.905, −1	1.142, 1
	1/16	1.940, 0	6.930, −1	5.368, 0	6.877, −1	9.558, 0	4.401, −1	1.249, 1
	1/32	1.956, 0	7.122, −1	5.428, 0	7.203, −1	9.797, 0	4.720, −1	1.312, 1
	1/64	1.964, 0	7.228, −1	5.458, 0	7.387, −1	9.922, 0	4.907, −1	1.347, 1
	0	1.972, 0	7.341, −1	5.488, 0	7.588, −1	1.005, 1	5.116, −1	1.385, 1
8	∞	9.987, 1	3.293, 0	3.250, 1	9.569, −1	1.795, 1	3.359, −1	1.083, 1
	64	2.817, 1	2.544, 0	2.558, 1	9.100, −1	1.689, 1	3.302, −1	1.060, 1
	32	1.992, 1	2.203, 0	2.222, 1	8.727, −1	1.605, 1	3.249, −1	1.039, 1
	16	1.323, 1	1.806, 0	1.817, 1	8.130, −1	1.472, 1	3.152, −1	1.000, 1
	8	8.267, 0	1.391, 0	1.382, 1	7.260, −1	1.283, 1	2.986, −1	9.350, 0
	4	4.895, 0	1.004, 0	9.696, 0	6.123, −1	1.044, 1	2.722, −1	8.345, 0
	2	2.775, 0	6.802, −1	6.269, 0	4.815, −1	7.806, 0	2.347, −1	6.968, 0
	1	1.521, 0	4.335, −1	3.760, 0	3.498, −1	5.325, 0	1.881, −1	5.344, 0
	1/2	1.627, 0	5.221, −1	4.241, 0	4.684, −1	6.642, 0	2.763, −1	7.441, 0
	1/4	1.705, 0	5.976, −1	4.570, 0	5.802, −1	7.674, 0	3.703, −1	9.410, 0
	1/8	1.759, 0	6.550, −1	4.774, 0	6.717, −1	8.369, 0	4.551, −1	1.096, 1
	1/16	1.793, 0	6.939, −1	4.891, 0	7.369, −1	8.786, 0	5.202, −1	1.199, 1
	1/32	1.813, 0	7.177, −1	4.955, 0	7.784, −1	9.016, 0	5.637, −1	1.261, 1
	1/64	1.824, 0	7.313, −1	4.989, 0	8.024, −1	9.137, 0	5.898, −1	1.295, 1
	0	1.836, 0	7.462, −1	5.022, 0	8.294, −1	9.261, 0	6.199, −1	1.331, 1

4		5		6		7	
3	5	4	6	5	7	6	8
2.817, −2	3.028, 0						
2.786, −2	2.998, 0						
2.757, −2	2.970, 0						
2.700, −2	2.915, 0						
2.596, −2	2.814, 0						
2.418, −2	2.640, 0						
2.141, −2	2.364, 0						
1.762, −2	1.977, 0						
2.643, −2	3.028, 0						
3.580, −2	4.194, 0						
4.401, −2	5.266, 0						
5.003, −2	6.088, 0						
5.385, −2	6.629, 0						
5.604, −2	6.948, 0						
5.847, −2	7.309, 0						
6.126, −2	4.943, 0	7.922, −3	1.227, 0				
6.067, −2	4.895, 0	7.878, −3	1.220, 0				
6.009, −2	4.848, 0	7.834, −3	1.214, 0				
5.899, −2	4.759, 0	7.748, −3	1.202, 0				
5.696, −2	4.595, 0	7.585, −3	1.178, 0				
5.345, −2	4.311, 0	7.284, −3	1.135, 0				
4.788, −2	3.861, 0	6.767, −3	1.061, 0				
4.008, −2	3.230, 0	5.961, −3	9.426, −1				
6.144, −2	4.948, 0	9.727, −3	1.558, 0				
8.523, −2	6.855, 0	1.441, −2	2.344, 0				
1.072, −1	8.607, 0	1.923, −2	3.181, 0				
1.240, −1	9.948, 0	2.332, −2	3.915, 0				
1.352, −1	1.083, 1	2.622, −2	4.453, 0				
1.417, −1	1.135, 1	2.802, −2	4.795, 0				
1.492, −1	1.194, 1	3.014, −2	5.208, 0				
9.029, −2	5.874, 0	1.990, −2	2.333, 0	2.478, −3	5.210, −1		
8.949, −2	5.817, 0	1.980, −2	2.321, 0	2.470, −3	5.195, −1		
8.872, −2	5.762, 0	1.970, −2	2.309, 0	2.462, −3	5.179, −1		
8.725, −2	5.656, 0	1.950, −2	2.286, 0	2.446, −3	5.148, −1		
8.451, −2	5.462, 0	1.913, −2	2.242, 0	2.415, −3	5.087, −1		
7.973, −2	5.125, 0	1.844, −2	2.160, 0	2.357, −3	4.972, −1		
7.207, −2	4.592, 0	1.724, −2	2.018, 0	2.250, −3	4.760, −1		
6.113, −2	3.843, 0	1.534, −2	1.794, 0	2.068, −3	4.397, −1		
9.536, −2	5.889, 0	2.540, −2	2.964, 0	3.583, −3	7.676, −1		
1.350, −1	8.163, 0	3.831, −2	4.462, 0	5.708, −3	1.236, 0		
1.731, −1	1.025, 1	5.213, −2	6.056, 0	8.214, −3	1.801, 0		
2.038, −1	1.185, 1	6.432, −2	7.452, 0	1.064, −2	2.363, 0		
2.248, −1	1.290, 1	7.329, −2	8.475, 0	1.257, −2	2.823, 0		
2.376, −1	1.352, 1	7.902, −2	9.124, 0	1.388, −2	3.140, 0		
2.524, −1	1.422, 1	8.597, −2	9.908, 0	1.555, −2	3.557, 0		
1.133, −1	6.231, 0	3.266, −2	3.077, 0	7.024, −3	1.132, 0	8.364, −4	2.290, −1
1.123, −1	6.171, 0	3.251, −2	3.061, 0	7.003, −3	1.128, 0	8.347, −4	2.286, −1
1.114, −1	6.113, 0	3.236, −2	3.045, 0	6.982, −3	1.125, 0	8.329, −4	2.281, −1
1.097, −1	6.001, 0	3.207, −2	3.015, 0	6.940, −3	1.118, 0	8.296, −4	2.273, −1
1.065, −1	5.796, 0	3.150, −2	2.957, 0	6.860, −3	1.105, 0	8.229, −4	2.255, −1
1.009, −1	5.440, 0	3.045, −2	2.849, 0	6.706, −3	1.080, 0	8.099, −4	2.222, −1
9.188, −2	4.875, 0	2.861, −2	2.662, 0	6.425, −3	1.034, 0	7.855, −4	2.158, −1
7.876, −2	4.082, 0	2.567, −2	2.366, 0	5.942, −3	9.553, −1	7.418, −4	2.044, −1
1.246, −1	6.260, 0	4.300, −2	3.912, 0	1.039, −2	1.668, 0	1.339, −3	3.706, −1
1.794, −1	8.682, 0	6.588, −2	5.889, 0	1.677, −2	2.685, 0	2.256, −3	6.288, −1
2.343, −1	1.091, 1	9.119, −2	7.995, 0	2.451, −2	3.914, 0	3.463, −3	9.744, −1
2.801, −1	1.262, 1	1.143, −1	9.839, 0	3.226, −2	5.134, 0	4.781, −3	1.359, 0
3.127, −1	1.374, 1	1.320, −1	1.119, 1	3.863, −2	6.133, 0	5.956, −3	1.710, 0
3.331, −1	1.440, 1	1.436, −1	1.205, 1	4.306, −2	6.822, 0	6.824, −3	1.974, 0
3.574, −1	1.514, 1	1.580, −1	1.308, 1	4.889, −2	7.724, 0	8.046, −3	2.354, 0