Mon. Not. R. astr. Soc. (1965) **131**, 1–12.

OLBERS' PARADOX AND THE BACKGROUND RADIATION DENSITY IN AN ISOTROPIC HOMOGENEOUS UNIVERSE

E. R. Harrison

(Received 1964 October 28)*

Summary

The background radiation density in an isotropic homogeneous universe is determined using a differential equation approach rather than the usual integral equation method. For the class of models obeying $R \propto t^n$ it is found for $n > -1$ that the radiation density is $E = \epsilon \rho t/(n+1)$ after sufficient time has elapsed following the initial conditions. In this equation ρ is the density of luminous matter (the stars) and ϵ is the rate radiation energy produced per unit time per unit mass of luminous matter. (In the static model considered by Olbers, $n=0$ and t is the time for which the stars have emitted radiation; in the steady-state model $n=\frac{1}{3}$ and $t=\frac{1}{3}T_0$ where T_0^{-1} is Hubble's constant.) Olbers' paradox is that the sky is dark at night, whereas in an infinite universe the sky should be as bright as the surface of a star. But in a static universe the stars must radiate for $t \sim 10^{23}$ years in order that the radiation level is raised to that at the surface of the stars. However, radiation from the stars at the present rate for such a vast period of time violates the conservation of energy principle. It is shown that Olbers' original assumptions account satisfactorily for the present radiation level provided the stars have only a finite radiation lifetime of $t \sim 10^{10}$ years. In any model of an isotropic homogeneous universe of $n > -1$ the background radiation level is less than the average radiation level at the surface of the stars when $t < (n+1)\tau$ where τ is the " mean collision time " of a photon between emission and absorption. The present value of τ is of the order 10^{23} years. The expansion ($n > 0$) or contraction ($n < 0$) of the universe has surprisingly little effect on the present radiation level. Thermodynamic arguments allow us to estimate the present mean density of luminous matter and this is compared with the total density deduced from general relativity theory for simple models.

1. *Introduction.* In an infinite static universe, uniformly populated with stars, the background radiation density everywhere should equal the radiation density at the surface of the stars. This contradiction between theory and fact is known as Olbers' paradox (1826). In many ways Olbers' paradox is a relic of the physics of earlier days; nevertheless it remains a subject of absorbing interest and of considerable importance in cosmology.

Bondi (1960) has examined and discussed the main assumptions underlying Olbers' argument. These assumptions are: (i) the average density of stars and their average luminosity do not vary throughout space; (ii) the same quantities do not vary with time; (iii) there are no large systematic movement of the stars; (iv) space is Euclidean; (v) the known laws of physics apply. Bondi, and more recently Whitrow and Yallop (1964), have shown that in an isotropic

* Received in original form 1964 September 24.

homogeneous universe (iv) is unnecessary and the radiation density is independent of the curvature index. Assumption (iii) is no longer in accord with modern belief, and it is generally supposed that the existence of the extragalactic red-shift is a sufficient condition for the resolution of Olbers' paradox and for ensuring a low density of the background radiation. Surprisingly enough, it turns out that this is not so. As we shall show, neither the existence of the extragalactic red-shift nor the youth of the universe are in general the necessary or sufficient conditions for ensuring a low radiation density. The presence of a red-shift, or even a blue-shift, plays a relatively minor role in determining the radiation level in a variety of non-static models. The principal exception to this rule is the steady-state model of Bondi and Gold (1948) in which the expansion of the universe ensures that the radiation level remains constant; whether the actual value of the radiation level is high or low however depends on the nature of the sources and sinks of the radiation field. In general, the effect of the red-shift on the radiation level is of such minor importance that within the framework of Olbers' assumptions it is possible to calculate for a static model a value for the radiation level which is in agreement with present-day estimates. The only reservation necessary is that assumption (ii) is acceptable so long as it does not violate assumption (v). Within this framework it follows that a static universe may be of infinite age, but by the conservation of energy principle the radiating stars have existed only for a finite time.

A moment's reflection shows that it is quite impossible under the present circumstances for the universe to be filled with radiation at a density equal to that at the surface of the stars. Shapley's (1933) value for the mean density of luminous matter (the stars) in the universe is $\rho \sim 10^{-30}\,\mathrm{g\,cm^{-3}}$. If we imagine that this luminous matter is converted entirely into radiation, the radiation density is $\rho c^2 \sim 10^{-9}\,\mathrm{erg\,cm^{-3}}$. This is comparable with the radiation density of moonlight at the Earth's surface. The main source of the radiation field is the conversion of hydrogen into helium and therefore if all luminous matter were converted into helium the radiation density would be less than one per cent of ρc^2. In this case the background radiation would not affect profoundly the present brightness of the night-sky. One might therefore claim that the night-sky is dark, at least in a static universe, because the star density is low. It turns out that this is also true for a variety of non-static models. In the following discussion we use elementary thermodynamics to determine the background radiation density in static and non-static models of an isotropic homogeneous universe.

2. *Equation for the radiation density.* We consider an isotropic radiation field of uniform energy density in a cavity of volume V which has perfectly reflecting walls. Hence

$$d(EV)+pdV=\delta Q \tag{1}$$

where $p=\frac{1}{3}E$ is the pressure and δQ is an incremental change in the energy produced by a uniform distribution of sources and sinks. At this stage the metric is assumed to be Euclidean. We suppose that V is a macroscopic element of volume sufficiently large to contain and represent the average conditions of an isotropic homogeneous universe. The universe can be imagined as consisting of a large number of such elements of volume, each expanding or contracting in an identical fashion, and each containing identical conditions at every instant. The perfectly reflecting walls of each element of volume can therefore be removed and the conditions will remain unchanged owing to their uniformity. We therefore use equation (1) for an arbitrary element of volume of the universe which has its boundaries defined by the positions of specified galaxies.

If e_s is the radiation density at the surface of a given star, the total rate at which energy is radiated into V is

$$\sum_{\text{stars in } V} \tfrac{1}{4}c \int_{\substack{\text{surface}\\ \text{of star}}} e_s \, dS = \tfrac{1}{4}cE_s S \tag{2a}$$

where S is the total surface area of the stars in V and E_s is their average surface radiation density. One might also write

$$\tfrac{1}{4}cE_s S = nVL = \rho\epsilon V \tag{2b}$$

where n is the number of stars per unit volume and L is their average luminosity, or ρ is the density of luminous matter and ϵ is the average rate at which radiation is produced per unit mass of luminous matter (that is, the average luminosity/average mass ratio). If there is no non-luminous matter then radiation energy is lost from V at a rate

$$-\tfrac{1}{4}cES. \tag{2c}$$

From equation (1) and the expressions (2 a, c) it follows

$$\frac{1}{V^{1/3}}\frac{d}{dt}(V^{4/3}E) = \tfrac{1}{4}cS(E_s - E). \tag{3}$$

A more general form of equation (3) is

$$\frac{1}{V^{1/3}}\frac{d}{dt}(V^{4/3}E) = \tfrac{1}{4}cS(\alpha E_s - \beta E) \tag{4}$$

which takes into account absorbtion by non-luminous matter. Thus for diffuse intergalactic matter $\beta > 1$, and for absorbing matter distributed within the galaxies $\alpha < 1$. In general, we have $\alpha \leqslant 1$, $\beta \geqslant 1$. To avoid complications of a minor nature we suppose that there is no non-luminous matter and $\alpha = 1$, $\beta = 1$*.

It can be shown that for an isotropic homogeneous universe equation (3), or (4), is quite general provided V is a proper element of volume and all densities are proper densities. Let the components of the energy-momentum tensor T_μ^ν in a co-moving coordinate system be

$$T_1^1 = T_2^2 = T_3^3 = -p_m - \tfrac{1}{3}E, \quad T_4^4 = \rho_0 + E$$

where p_m is the pressure due to the random motions of matter of proper density $\rho_0 = \rho_m + 3p_m$, and ρ_m is the density found from the rest mass of matter. Using the line element

$$ds^2 = dt^2 - \frac{R^2(t)}{(1+\frac{1}{4}kr^2)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2)$$

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* Bonner (1964) has recently considered in more detail the problem of intragalactic absorbtion. He uses galaxies instead of stars as radiating units.

and $T^{\nu}_{\mu;\nu}=0$, with $\mu=4$, we obtain the relativistic adiabatic energy equation

$$\frac{d}{dt}\{(\rho_0+E)V\}+(p_m+\tfrac{1}{3}E)\frac{dV}{dt}$$

$$=\frac{d}{dt}(\rho_m V)+\frac{3}{V^{1/3}}\frac{d}{dt}(p_m V^{4/3})+\frac{1}{V^{1/3}}\frac{d}{dt}(EV^{4/3})=0, \tag{5 a}$$

where V is a proper element of volume

$$V=\frac{R^3(t)r^2\sin\theta}{(1+\frac{1}{4}kr^2)^3}\,\delta r\,\delta\theta\,\delta r$$

and $k=1, 0, -1$, is the curvature index. Any decrease in $\rho_m V$, because of nuclear transformations for example, must equal the energy radiated and also the increase in the energy of random motions, that is

$$\frac{d}{dt}(\rho_m V)=-\tfrac{1}{4}cS(\alpha E_s-\beta E)-\frac{3}{V^{1/3}}\frac{d}{dt}(p_m V^{4/3}). \tag{5 b}$$

Combining equations (5 *a*, *b*) yields the previous equation (4).

Since $V\propto R^3(t)$, equation (3) can be written as

$$V_0\frac{d}{dt}(R^4E)=\tfrac{1}{4}cRR_0^3S(E_s-E) \tag{6}$$

where R_0 is $R(t_0)$ and t_0 denotes the present epoch.

Whitrow and Yallop (1964) (who give references to previous work on this subject) have recently derived an integral equation using a spectral distribution function, which takes account of both the effect of recession and the change in density in the time elapsed since emission; they also show that the radiation flux is independent of the curvature index. With $\frac{1}{4}cE_sS=nLV$, they find

$$E=n_0\int_{t'}^{t_0}L\,\frac{R}{R_0}\,dt. \tag{7}$$

This equation gives the same results as (6) provided $E=0$ at time $t=t'$ and there is no absorption of any kind. If the intragalactic and intergalactic absorptions are zero and the stars are point sources of radiation then $\alpha=1$, $\beta=0$. This means that in a static universe radiation equilibrium of $E=E_s$ as conceived by Olbers is ruled out. Also in a variety of non-static models it is possible to deduce from equation (7) an impossibly large value of E such that real stars would receive more radiation than they emit. Whilst it is reasonable to suppose that the stars are capable of radiating for long periods of time when $E\ll E_s$ it is quite another matter to suppose that they can continue to exist in an unchanged state when $E\gtrsim E_s$. The advantage of a differential equation such as (6) is that it contains E_s explicitly and can be integrated over the period of time during which the physically reasonable condition of $E<E_s$ exists.

It is useful to express equation (6) in terms of a "mean collision time" of a photon: $\tau=(nc\sigma)^{-1}$, where σ is the average cross-section of the stars. In traversing a proper distance of l units, where $l \ll \tau$, l/τ is the probability that a photon is intercepted. Clearly,

$$\tau = 4V/Sc \tag{8}$$

or alternatively, $\tau = E_s/nL = E_s/\rho\epsilon$, and therefore equation (6) becomes

$$\frac{d}{dt}(R^4E) = \frac{R^4}{\tau}(E_s - E) \tag{9}$$

since $\tau R^{-3} = \tau_0 R_0^{-3}$. If ρ_s and r_s are the density and radius of a representative star, $\tau_0 = 4\rho_s r_s/3\epsilon\rho_0$, and for stars such as the Sun it follows $\tau_0 \sim 10^{23}$ light years. We observe from equation (8) that if

$$\frac{1}{R}\frac{dR}{dt} = \frac{1}{T}$$

(T_0^{-1} is Hubble's constant) the radiation density increases, is constant, or decreases, according to

$$\frac{1}{E}\frac{dE}{dt} \gtreqless 0 \quad \text{when} \quad \frac{E_s - E}{E} \gtreqless 4\frac{\tau}{T}. \tag{10}$$

We commenced by supposing that a macroscopic element of volume had perfectly reflecting walls, that is, the walls were non-absorbing and did not alter the isotropy and uniformity of the radiation. By supposing that the universe is isotropic and homogeneous we are postulating that the radiation conditions within each such macroscopic element of volume are identical at each instant of cosmic time t. Hence it is possible to dispense with the reflecting walls without disturbing in any way the radiation conditions. Let us now imagine that a single given element of volume V is actually enclosed within walls, which are perfectly reflecting from both inside and outside during the entire history of the universe. It follows that an observer inside V will find exactly the same background radiation density as an observer outside V, and furthermore both observers will discover identical spectral distribution functions. The observer outside V finds that the background radiation contains contributions from distant stars which, in an expanding universe, are red-shifted and were emitted at a time when the star density was larger; whereas an observer inside V finds the background radiation contains contributions which were emitted in the past by neighbouring stars and are red-shifted owing to successive reflections from the expanding and co-moving walls. The observer outside V will tend to evaluate the radiation density or flux by integrating the various contributions from distant regions as Olbers and others have since done, and might be surprised to find that his results offer no clue as to the nature of the curvature index. The observer inside V will tend however to evaluate the radiation density or flux by using the differential equations of classical thermodynamics, and because his confined region is relatively small he will not be in the least surprised that his results are independent of the curvature index*.

3. *Static universe.* In a static universe R is constant and therefore equation (9), on integration, gives

$$E = E_s(1 - \exp(-t/\tau_0)) \tag{11}$$

* After this paper was written the author's attention was drawn to Davidson's (1962) paper in which a differential equation treatment is used for calculating the rate of emission of radiation per unit volume. Various features of Davidson's work are similar to and anticipate the work of both Whitrow and Yallop (1964) and the present author.

for $E=0$ at $t=0$. The mean collision time τ_0 is also the characteristic time required for the background radiation to attain Olbers' equilibrium condition of $E \simeq E_s$. In a static universe t in equation (11) is the time for which the stars have emitted radiation, assuming that this has been at a constant rate, and it would be wrong to suppose that it is the age of a static universe unless there were unlimited resources of energy. If ϵ is the mean rate at which radiant energy is produced per unit of mass then necessarily $\epsilon t < c^2$, and for $\epsilon \sim 1 \text{ erg g}^{-1}\text{s}^{-1}$, then $t/\tau_0 < \sim 3\ 10^{-10}$. Therefore equation (11) becomes

$$E = E_s \frac{t}{\tau_0} \tag{12}$$

or $E = \epsilon\rho t$. Since $t \ll \tau_0$ it follows $E \ll E_s$ and the night-sky is always dark in a static universe having a star density similar to our own. With $\epsilon \sim 1 \text{ erg g}^{-1}\text{s}^{-1}$, $\rho \sim 10^{-30} \text{ g cm}^{-3}$, and using Bondi's (1960) estimate of $E = 3\ 10^{-13} \text{ erg cm}^{-3}$, it follows that the stars have emitted radiation for a time $t \sim 10^{10}$ years, which is not an unreasonable result. Thus Olbers' original assumptions (i)–(v), with the modification to (ii) that t is not infinitely large but is of the order 10^{10} years, account for the present background radiation density in a surprisingly satisfactory way.

4. *Steady-state universe.* The concept of a steady-state expanding universe, as proposed by Bondi and Gold (1948), requires that proper densities such as ρ, n, and E are constant. If we are given the background radiation density and the rate at which radiation is emitted then, clearly we also know the rate at which proper space is expanding for a steady-state condition. From either equation (9) or (10) it follows

$$\frac{1}{R}\frac{dR}{dt} = \frac{E_s - E}{4\tau_0 E} \tag{13}$$

and since $E \ll E_s$, we have

$$E = \tfrac{1}{4} E_s \frac{T}{\tau_0} \tag{14}$$

or $E = \frac{1}{4}\epsilon\rho T$. For $T = 1{\cdot}3\ 10^{10}$ years (Sandage 1958) this result is approximately the same as equation (12) for the static model. It is remarkable that two such entirely different models, the static model in which the radiation density is proportional to the emission life-time of the stars, and the steady-state model in which the radiation is proportional to T, should give reasonably similar results. It is important to notice that the extragalactic red-shift in the steady-state model merely ensures that the radiation level is constant, it does not by itself ensure that the radiation level is low. The condition for a low radiation level is that $T \ll 4\tau_0$, and this depends on the density of the stars.

5. *Expanding and contracting models.* We consider briefly the radiation density in a range of isotropic homogeneous non-static models which have a variation index n:

$$\frac{t}{R}\frac{dR}{dt} = n, \tag{15}$$

that is, $R \propto t^n$ and $t = nT$. The models which have $n > 0$ expand from $R = 0$ at $t = 0$, and those characterized by $n < 0$ contract from $R = \infty$ at $t = 0$. For $n \neq \frac{1}{3}$ equation (9) is

$$\left.\begin{aligned} \frac{d}{dt}\,(t^{4n} \exp(-\chi) E) &= t^{4n} \exp(-\chi)\,\frac{E_s}{\tau} \\ \chi &= \frac{1}{3n-1}\,\frac{t}{\tau} \end{aligned}\right\}\ (n \neq \tfrac{1}{3}) \qquad (16)$$

and for $n = \frac{1}{3}$

$$\frac{d}{dt}\,(t^{4/3+t_0/\tau_0} E) = t^{4/3+t_0/\tau_0}\,\frac{E_s}{\tau}, \quad (n = \tfrac{1}{3}) \qquad (17)$$

where $\tau t_0^{3n} = \tau_0 t^{3n}$ and as before the zero subscript denotes the present epoch.

We assume that it is physically reasonable to integrate equations (16) and (17), with E_s constant, only in the range of t for which $E \leqslant E_s$. Also, at time t_1 when the stars begin to radiate, the initial radiation density is E_1. Let us suppose provisionally that t/τ is small if $E < E_s$. In that case equations (16) and (17) are of the same form and can be integrated to give

$$E = E_1 \left(\frac{t_1}{t}\right)^{4n} + \frac{E_0}{n+1}\,\frac{t}{\tau}\left\{1 - \left(\frac{t_1}{t}\right)^{n+1}\right\}, (n \neq -1) \qquad (18)$$

for all values of n except $n = -1$, when

$$E = E_1 \left(\frac{t_1}{t}\right)^{4n} + E_s\,\frac{t}{\tau}\,\ln\frac{t}{t_1}, \quad (n = -1). \qquad (19)$$

By definition E_1 depends on the radiation conditions prior to t_1 and it may have any value between zero and E_s. Of the many possibilities we shall suppose, mainly for the sake of illustration, that the initial condition is either $E_1 = 0$ or $E_1 = E_s$.

A. *Expanding models,* $n > 0$

(i) $n > \frac{1}{3}$. For $E_1 = 0$ the radiation level first increases, reaches a maximum, and then decreases. For $E_1 = E_s$ the radiation level diminishes monotonically. Provided $t_1 \geqslant t_c$, where

$$t_c = \left(\frac{1}{n+1}\,\frac{t_c^{3n}}{\tau_0}\right)^{1/(3n-1)} \qquad (20)$$

then $E < E_s$ for $t > t_1$. Owing to the uncertainty $0 \leqslant E_1 \leqslant E_s$ in our initial condition the stars must originate at a time later than the critical epoch t_c in order that the radiation level remains less than E_s. For $t \gg t_1$ the radiation diminishes as

$$E = \frac{E_s}{n+1}\,\frac{t_0^{3n}}{\tau_0 t^{3n-1}} = \frac{E_s}{n+1}\,\frac{t}{\tau}, \qquad (n > \tfrac{1}{3}) \qquad (21)$$

as shown in Fig. 1 (*a*). Because $t \geqslant t_1 > t_c$ it follows from equations (16) and (20) that $\chi < (n+1)/(3n-1)$, thus justifying to a large extent the neglect of $\exp(\chi)$ in the integration.

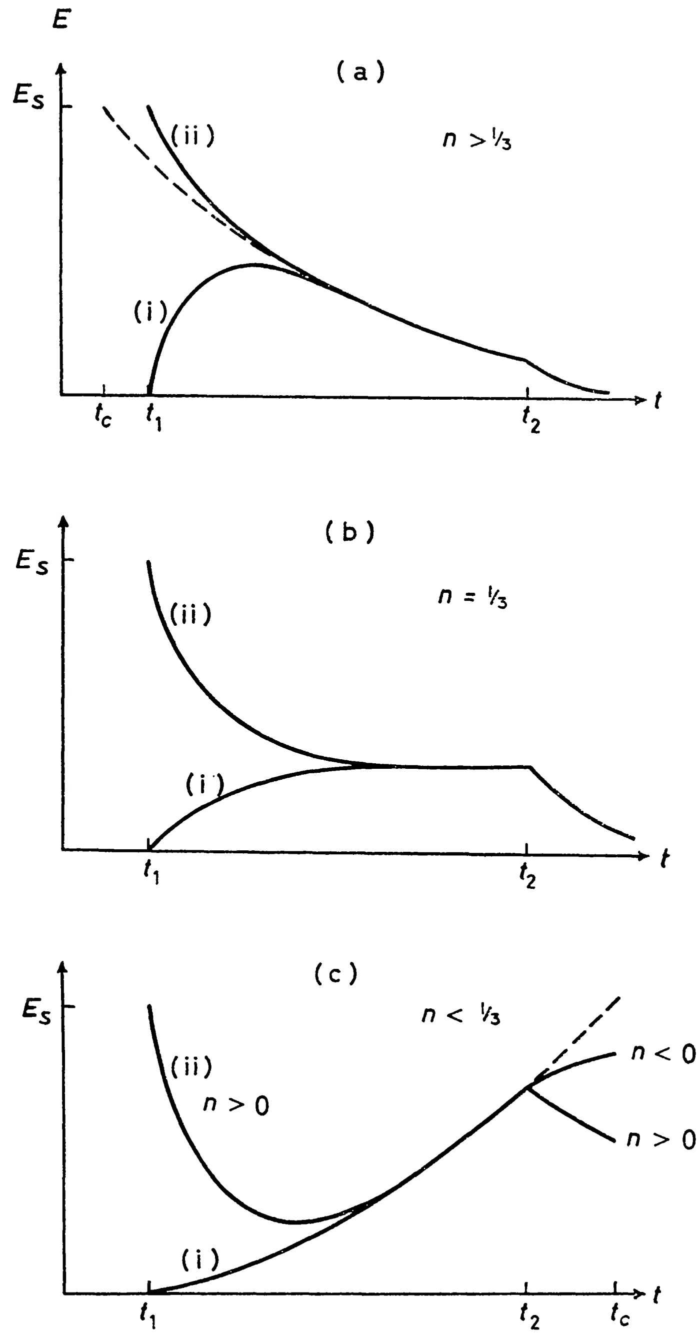

FIG. 1. (*a*) $n > \frac{1}{3}$; (*b*) $n = \frac{1}{3}$; (*c*) $n < \frac{1}{3}$. *Curve shows how the radiation density varies with time. Stars commence radiating at time* t_1 *and cease at time* t_2. *For the curves marked* (i) *the radiation level is initially zero and for the* (ii) *curves the radiation level is initially* $E_1 = E_S$.

(ii) $n = \frac{1}{3}$. The radiation level increases monotonically for $E_1 = 0$ and decreases monotonically for $E_1 = E_s$, and in both cases approaches asymptotically the constant value

$$E = \frac{3t_0 E}{4\tau_0 + 3t_0} \sim \tfrac{3}{4} E_s \frac{t_0}{\tau_0} = \tfrac{1}{4} E_s \frac{T}{\tau_0}, \qquad (n = \tfrac{1}{3}) \tag{22}$$

as shown in Fig. 1 (*b*).

(iii) $0 < n < \frac{1}{3}$. For $E_1 = 0$ the radiation level increases monotonically and for $E_1 = E_s$ it first diminishes and then increases, as in Fig. 1 (*c*). For $t \gg t_1$ the radiation increases as

$$E = \frac{E_s}{n+1} \frac{t_0^{3n}}{\tau_0} t^{1-3n} = \frac{E_s}{n+1} \frac{t}{\tau} \quad (0 < n < \tfrac{1}{3}). \tag{23}$$

In these models the expansion is sufficiently slow for the radiation to increase. The critical epoch t_c given by equation (20), at which $E = E_s$ now lies in the future: $t_c > t_0$. For $t < t_c$ we observe that $|\chi|$ is again reasonably small.

(iv) $n = 0$. This is the static model previously considered. The only reasonable initial condition is $E_1 = 0$ at $t = t_1 = 0$ and therefore $E = E_s t/\tau_0$ as given by equation (12). So long as $t < \tau_0$ it is seen that $|\chi|$ is also small.

B. *Contracting models*, $n < 0$

(v) $-1 < n < 0$. It would be absurd to suppose that the stars originated at $t = 0$ in an infinitely dispersed universe of $R = \infty$, and therefore we suppose that they were formed at time $t_1 > 0$ when the mean density of matter had a finite value. Also we suppose that prior to t_1 there was no radiation and hence $E_1 = 0$. The radiation level increases monotonically as given by equation (18) and for $t \gg t_1$

$$E = \frac{E_s}{n+1} \frac{t^{1-3n}}{\tau_0 t_0^{-3n}} = \frac{E_s}{n+1} \frac{t}{\tau} \quad (-1 < n < 0). \tag{24}$$

There is again a critical epoch of $t_c > t_0$ at which $E = E_s$, and $|\chi|$ is small for $t < t_c$.

(vi) $n = -1$. (vii) $n < -1$. The radiation level is given by equations (18) and (19), respectively, and the critical epochs of t_c are readily found. Again $|\chi|$ is small for $t < t_c$.

In general, after a sufficient period of time has elapsed, the radiation level diminishes for $n > \frac{1}{3}$ and increases for $n < \frac{1}{3}$. For all models fulfilling the condition $n > -1$ we have

$$E = \frac{E_s}{n+1} \frac{t}{\tau} = \frac{1}{n+1} \epsilon \rho t = \frac{n}{n+1} \epsilon \rho T. \tag{25}$$

The static model corresponds to $n = 0$ and the steady-state model to $n = \frac{1}{3}$. (Whitrow and Yallop's results are in agreement with equation (25). Their equation (7) however appears to require modification in order that the initial conditions can be taken fully into account.) For this range of models the present background radiation level is low compared with the radiation level at the surface of the stars provided

$$t \ll (n+1)\tau_0. \tag{26}$$

Very approximately, this is the same condition $t \ll \tau_0$ as for a static model. The existence of a red-shift ($n > 0$) or a blue-shift ($n < 0$) has on the whole a surprisingly small effect on the radiation level. For the $n \leqslant -1$ models the radiation level increases at a more rapid rate than the t/τ law and $t < \tau$ is now a necessary condition to ensure $E \leqslant E_s$.

If at some time t_2 the stars cease to radiate the radiation level will thereafter vary as t^{-4n}. We have seen that for $n > \frac{1}{3}$ the stars must commence radiating at $t_1 > t_c$ in order that $E < E_s$; but it also follows that for $n < \frac{1}{3}$ the stars must cease radiating at $t_2 < t_c$ if the condition $E < E_s$ is to apply during their radiation lifetime.

Some values of t_c (using $t_0 \sim 10^{23}$ years, $t_0 \sim 10^{10}$ years) are: $n = 1$ (Milne's model), $t_c \sim 3\ 10^3$ years; $n = \frac{2}{3}$ (Einstein–de Sitter model), $t_c \sim 1$ day; $n = \frac{1}{3}$ (steady-state and Dirac's models), $t_c = \infty$; $n = 0$ (static model), $t_c \sim 10^{23}$ years; $n = -1$, $t_c \sim 10^{13}$ years; and $n = -2$, $t_c \sim 10^{12}$ years. The corresponding densities at these critical epochs are $\rho_c = \rho_0(t_0/t_c)^{3n}$ or: $10^{20}\rho_0(n=1)$, $10^{26}\rho_0(n=\frac{2}{3})$, $\rho_0(n=0)$, $10^9\rho_0(n=-1)$, and $10^{12}\rho_0(n=-2)$.

6. *Density of luminous matter and total density of matter.* From the luminosity distribution function for stars in the solar neighbourhood it is found that ϵ is approximately 2·5 times that of the Sun, or $\epsilon = 5\ \mathrm{erg\,g^{-1}\,s^{-1}}$ (Schwarzschild 1958). Using Bondi's estimate of $E = 3\ 10^{-13}\ \mathrm{erg\,cm^{-1}}$ and $T = 1{\cdot}3\ 10^{10}$ years (Sandage 1958), it follows that $\rho = 1{\cdot}5\ 10^{-31}(n+1)/n\ \mathrm{g\,cm^{-3}}$. Even if we accept a value for T of approximately 10^{10} years there remains considerable uncertainty in ρ owing to our meagre knowledge of ϵ and E.

Kiang (1961) estimates that $\epsilon\rho$ is $7{\cdot}6\ 10^{-31} H\ \mathrm{erg\,cm^{-3}\,s^{-1}}\,A^{-1}$ near 4 300 A, where H is Hubble's constant in km s^{-1} Mpc^{-1}. Van den Bergh's (1961) estimate is in agreement within a factor of 2. Hence, $\epsilon\rho T = 1\ 10^{-14}\ \mathrm{erg\,cm^{-3}}$ $(H=75)$ and for $n > -1$, $n \neq 0$,

$$E = \frac{n}{n+1}\ 1\ 10^{-14}\,\mathrm{erg\,cm^{-3}}. \tag{27}$$

If we assume that the main source of radiation is the conversion of hydrogen into helium then $\epsilon t = 7\ 10^{-5} Xc^2$, where X is the fractional burn-up of luminous matter, and therefore

$$E = 7\ 10^{-3}\rho c^2 \frac{X}{n+1}. \tag{28}$$

From (27) and (28) it follows

$$\rho = 1\ 10^{-33}\frac{n}{X}. \tag{29}$$

On the basis of observations of the solar neighbourhood $X > 10$ per cent. If therefore we suppose that in general $X > 1$ per cent then we have $\rho < 10^{-31} n\ \mathrm{g\,cm^{-3}}$ and $\epsilon > 0{\cdot}2\,n^{-1}\ \mathrm{erg\,g^{-1}\,s^{-1}}$. Hoyle, Fowler, Burbidge and Burbidge (1964) estimate that the light-to-mass ratio of the galaxies is of the order 0·3.

Since it is possible to deduce a reasonable upper limit for the density of luminous matter by thermodynamic arguments it is of interest to compare our results with the total density deduced for simple models from general relativity theory. In an isotropic homogeneous universe of negligible pressure and zero cosmological constant

$$R(\dot{R}^2 + k) = \frac{8\pi G\rho_m R^3}{3c^2} = \mathrm{const} \tag{30}$$

(Tolman 1934) where k is the curvature index and ρ_m is the total density of matter. If $k=0$, as in the Einstein–de Sitter model, then

$$8\pi G\rho_m T^2 = 3 \tag{31}$$

and $n = \frac{2}{3}$. For $T = 1{\cdot}3\ 10^{10}$ years this gives $\rho_m = 1\ 10^{-29}\ \mathrm{g\,cm^{-3}}$ and therefore $\rho_m/\rho > 10^2$. For the steady-state theory Hoyle (1948) has derived a relation for ρ_m which is the same as equation (31).

We are thus confronted with three possibilities: (*a*) The first is that $\rho \lesssim 10^{-31}\,\mathrm{g\,cm^{-3}}$ and $\rho_m \simeq 10^{-29}\,\mathrm{g\,cm^{-3}}$ are approximately correct and there is a large preponderance of non-luminous matter which is diffuse and also possibly in the form of bodies of low luminosity. Apart from the intriguing prospects which this possibility offers (see, for example, Hoyle, Fowler Burbidge and Burbidge 1964) it allows us to consider the formation of galaxies and the condensation into stars in a universe relatively rich in diffuse matter. (*b*) The second is that $\rho \simeq \rho_m \lesssim 10^{-30}\,\mathrm{g\,cm^{-3}}$. In this case $8\pi G\rho_m T^2 \ll 3$ and it is evident that the curvature index must be negative and $n = 1$ at present. Agacy and McCrea (1962) have generalized Hoyle's (1948) treatment of the steady-state model and have shown that ρ_m need not necessarily be given by equation (31) but in actual fact it may have any desired value. If general relativity theory can tell us very little about the present mean density of matter then we must turn to thermodynamic theory for information on this subject. (*d*) The third is that $\rho \simeq \rho_m \simeq 10^{-29}\,\mathrm{g\,cm^{-3}}$. This possibility revives Olbers' paradox in a slightly different guise: why is the background radiation level not two orders of magnitude higher?

These conclusions will obviously be modified to some extent if more appropriate values than unity are used in equation (4) for the intragalactic coefficient α and the intergalactic coefficient β.

7. *Summary*. The basic assumptions (i)–(v) lead to Olbers' paradox if the stars have emitted radiation continuously at the present rate for a time of the order 10^{23} years or longer. But radiation emitted at the present rate for such a vast period of time violates the conservation of energy principle and therefore the paradox arises from an inconsistency in the assumptions which was not apparent in pre-relativistic days. If however the stars have emitted radiation at their present rate for a finite period of time of the order 10^{10} years, then Olbers' assumptions with (ii) only slightly modified lead to results in accord with present-day observations.

It is often thought that Olbers' paradox can be resolved by rejecting assumption (iii) and the existence of the extragalactic red-shift is the sufficient condition for a low background radiation level. But it is clear that the existence of a red-shift cannot by itself guarantee that the radiation level is less than some arbitrary value. The situation is analogous to the radiation in a cavity of variable volume and having perfectly reflecting walls. The total energy of the radiation depends not only on the work done by varying the volume but also on the initial conditions and the nature of the sources of the radiation field.

For the class of isotropic homgeneous models of $R \propto t^n$ the radiation density for $n > -1$ is

$$E = \frac{E_s}{n+1}\,\frac{\mathrm{t}}{\tau}$$

after a sufficient time has elapsed for the initial conditions to become relatively unimportant. The static model is given by $n = 0$ and the steady-state model by $n = \frac{1}{3}$ with $t = \frac{1}{3}T$. Thus the condition for $E < E_s$ is that $t < (n+1)\tau$, where $\tau t^3{}_0{}^n = \tau t_0{}^{3n}$. For various reasons this condition may need modification; for example, our integrations are inexact and also we have assumed $\alpha = \beta = 1$ in equation (4). However, the essential physics of the problem in which we are mainly interested are contained in the condition $t < (n+1)\tau$. From this it is seen that the background radiation level at present is low compared with that at stellar surfaces not so much because of the red-shift (when $n > 0$) but primarily because $\tau_0 \sim 10^{23}$ years is so very large. In fact, the expansion ($n > 0$) and the contraction ($n < 0$) of the universe have surprisingly little effect on the radiation level, and the results deduced for the static model are sufficiently exact for most purposes.

Finally, one might remark that the treatment outlined in this paper can be applied to the X-ray and radio regions of the spectrum provided care is taken in separating off a specific range of frequencies. For a non-degenerate background of neutrinos the same arguments apply as for photons except that $\alpha = 1$ and $\beta \simeq 0$ owing to the small neutrino absorption cross-section. A fuller treatment is given by Weinberg (1962).

6 *Clifton Drive,*
Abingdon,
Berkshire,
England:
1964 *October.*

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