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THERMAL INSTABILITY IN NON-DEGENERATE STARS

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ABSTRACT

In the numerical investigation of the evolution of a star of 1 $M_\odot$ through the phases in which it contains a helium-burning shell as well as a hydrogen-burning shell, an unexpected type of thermal instability has been encountered. This instability is somewhat reminiscent of the helium flash even though degeneracy plays no role in the models considered. The existence of the instability has been made reasonably certain both by a physical analysis and by a direct mathematical derivation.

The new instability appears to have the character of a thermal runaway in a non-degenerate shell containing a highly temperature-sensitive nuclear-energy source. Such a shell will be unstable only if it is sufficiently thin not to affect the over-all hydrostatic structure of the star noticeably. Under this condition the pressure within the shell cannot greatly increase during the thermal runaway, and hence cooling by an adiabatic expansion cannot stabilize the shell. Some speculations are presented regarding whether this instability could have substantial consequences for a star's evolution.

I. INTRODUCTION

During the past year the evolutionary model sequence computations for a star of 1 $M_\odot$ reported by us earlier (Härm and Schwarzschild 1964) have been extended. The new computations cover the phases after the helium flash in which the star consumes the entire helium in the core and forms a helium-burning shell that, together with the hydrogen-burning shell, provides the total luminosity of the star.

In these new computations, as soon as the phase was reached in which the star has formed a well-concentrated helium-burning shell, a new type of difficulty was encountered. The respective contributions to the total luminosity of the two burning shells fluctuated more and more violently from model to model. At first it was suspected that this fluctuation was caused by some numerical instability in spite of the use of the implicit method for the one time derivative in the heat-flow problem, a method that is usually numerically stable. However, when increasingly smaller time steps were used, evolutionary sequences were obtained that were now smooth but showed the variations in an even more violent form than the original sequences. It thus became clear that an unexpected physical phenomenon had been stumbled upon with the help of extensive use of a large computer.

Initially the numerical results seemed to suggest that the cause of the new phenomenon lay in an interaction between the two burning shells leading to an oscillatory overstability. However, when the calculations were improved and extended, the results strongly suggested that the cause of the new phenomenon was a non-oscillatory thermal instability of the highly concentrated and temperature-sensitive helium-burning shell.

It seems that the only thermal instability encountered previously in stellar-evolution computations is the one caused by degeneracy in the core of the star (Mestel 1952), an instability that can lead, for example, to the helium flash. On the other hand, as Ledoux has recently emphasized (1958), the thermal stability of non-degenerate stars appears not to have been thoroughly investigated as yet. The only well-established item regarding this problem seems to be the fact that non-degenerate stars are in general stable against homologous thermal perturbations of the star as a whole. Our recent experience now suggests that, although a star may be stable against homologous perturbations, it may under certain circumstances be unstable against more local perturbations. That this might in fact be the case may be made plausible by the following argument.

When energy is dumped into a shell within a star, this shell expands and thus lifts

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the layers above it upward. These layers then will be farther away from the center of the star and accordingly their weight will be less. Since the pressure in the shell considered is directly caused—in hydrostatic equilibrium—by the weight of the layers above it, the pressure will drop. When the shell considered is amply big, the pressure in it will drop approximately proportionally to the $\frac{4}{3}$ power of the density (simulating a homologous expansion) and hence, according to the ideal gas law, the temperature will also drop. However, if the shell in question is thin, it can undergo a substantial expansion with a large fractional drop in density within it, but nevertheless lift the higher layers only by a small fraction of their distance from the center. In consequence, the pressure drop within the thin shell will be relatively small compared to the density drop. If the shell is sufficiently thin so that the pressure drop is percentagewise smaller than the density drop, then the temperature, according to the ideal gas law, will actually rise. If the shell considered contains a substantial nuclear-energy source, the rise in temperature will cause a further energy gain because of the increased rate of the nuclear processes, but also an energy loss because of the increased flux divergence. The latter loss will be smaller than the gain from the nuclear processes if the layer is thermally thick enough. Hence, if this condition is fulfilled, the temperature rise will result in a net energy gain and thus lead to a thermal runaway. It appears plausible then that a non-degenerate layer containing a substantial nuclear-energy source could be thermally unstable if it fulfils simultaneously two conditions: (1) it must be sufficiently thin so that perturbations in it do not seriously affect the hydrostatic structure of the star, and (2) it must be thick enough so that it does not lose excess thermal energy too easily by a flux divergence.

This qualitative argument will be carried through in a more quantitative form in the next two sections.

II. PHYSICAL PICTURE OF THERMAL PERTURBATION

Let us consider a shell in the star with a thickness Δr, containing the mass ΔM and with a drop in temperature ΔT from the bottom to top. The thermal conditions in the shell will be governed by the basic equations

$$L_r = -(4\pi r^2)^2 \frac{4ac}{3} \frac{T^3}{\kappa} \frac{dT}{dM_r}, \tag{1}$$

$$\epsilon - \frac{dL_r}{dM_r} = +\frac{3}{2} \frac{P}{\rho} \frac{dE}{dt} \tag{2}$$

with

$$E = \ln \frac{P}{\rho^{5/3}}. \tag{3}$$

Let us assume that the shell contains a strong nuclear source, and further that all energy sources in the layer below the shell are negligible. We may then idealize the temperature profile through the shell as shown in Figure 1. From this we derive for the flux, from equation (1),

$$\text{through bottom: } L_r = 0 , \tag{4}$$

$$\text{through top: } \quad L_r = L = (4\pi r^2)^2 \frac{4ac}{3} \frac{T^3}{\kappa} \frac{\Delta T}{\frac{1}{2}\Delta M}. \tag{5}$$

Finally, if we assume that the rate of change of the entropy in the shell is unimportant, we obtain from equation (2),

$$\epsilon = \frac{L}{\Delta M} \tag{6}$$

for the mean nuclear-energy generation in the shell.

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Let us now perturb the thermal state within the shell, assuming a temperature perturbation of the form indicated in Figure 2. Let us introduce this perturbation into the basic equations, but carry only those perturbation terms which appear likely to be the dominant ones. On the right-hand side of equation (1) we consider only the perturbation in the temperature gradient itself. Thus we obtain

$$\text{through bottom: } \delta L_r = -(4\pi r^2)^2 \frac{4ac}{3}\frac{T^3}{\kappa}\frac{\delta T}{\frac{1}{4}\Delta M}, \tag{7}$$

$$\text{through top: } \delta L_r = +(4\pi r^2)^2 \frac{4ac}{3}\frac{T^3}{\kappa}\frac{\delta T}{\frac{1}{4}\Delta M}. \tag{8}$$

This gives us for the perturbation of the divergence of the flux

$$\delta\left(\frac{dL_r}{dM_r}\right) = 4\frac{L}{\Delta M}\frac{T}{\Delta T}\frac{\delta T}{T}. \tag{9}$$

If we further represent the temperature dependence of the nuclear-energy source by

$$\epsilon \propto T^\nu , \tag{10}$$

if we ignore the density dependence of the nuclear-energy source, and if finally we consider on the right-hand side of equation (2) only the perturbation of the time derivative of the entropy, we obtain from equation (2)

$$\left(\nu - 4\frac{T}{\Delta T}\right)\cdot\frac{\delta T}{T} = \left(\frac{3}{2}\frac{P}{\rho}\frac{\Delta M}{L}\right)\cdot\frac{d}{dt}\,\delta E. \tag{11}$$

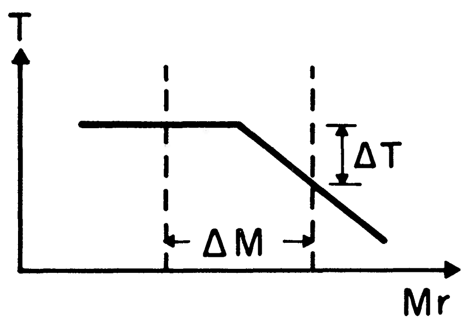

FIG. 1.—Idealized temperature profile through shell

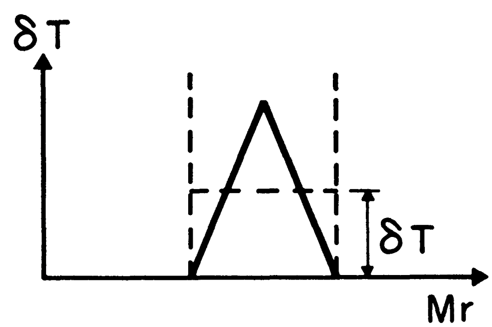

FIG. 2.—Idealized temperature perturbation through shell

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The parentheses on the right-hand side of this equation contain the Kelvin contraction time of the shell considered, which is a good measure for the rate of change of the perturbation in time. The left-hand side of the last equation contains two terms: the first one represents the gain in energy for a positive temperature perturbation caused by the increased nuclear-energy source, while the second term gives the increased energy loss caused by the increased flux divergence. Thus equation (11) says that, if the energy gain caused by the nuclear-energy source outweighs the energy loss caused by the flux divergence, a positive temperature perturbation will cause a positive rate of change of the entropy. This is a physically obvious result. The usefulness of equation (11) lies in its left-hand side, which provides a quantitative estimate for the condition that the effect of the nuclear-energy source outweighs the effect of the flux divergence. This condition turns out to be

$$\frac{\Delta T}{T} > \frac{4}{\nu} . \tag{12}$$

This condition states that the perturbation of the energy source will outweigh the perturbation of the flux divergence only if the layer is thick enough to contain a temperature drop fulfilling equation (12). This condition is not likely fulfilled by any reasonable shell containing hydrogen burning by the proton-proton reaction ($\nu \approx 4$). However, for the more temperature-sensitive nuclear sources, particularly for helium burning, condition (12) requires a quite modest temperature drop throughout the shell considered.

Equation (11) by itself does not tell us whether the shell under consideration is thermally unstable. We still have to investigate whether or not a positive entropy perturbation in the shell will cause a hydrostatic readjustment of the star of such a character that it will lead to a further positive temperature perturbation in the shell.

III. PHYSICAL PICTURE OF HYDROSTATIC READJUSTMENT

The readjustment of the structure of a star, made necessary by an increase of entropy in some layer of the star, is governed by the basic hydrostatic equilibrium conditions

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}, \qquad \frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} . \tag{13}$$

Linearization of these equations gives the hydrostatic conditions for the perturbations:

$$M_r \frac{d(\delta r/r)}{dM_r} = \frac{1}{U}\left(-3\frac{\delta r}{r} - \frac{3}{5}\frac{\delta P}{P} + \frac{3}{5}\delta E \right), \tag{14}$$

$$M_r \frac{d(\delta P/P)}{dM_r} = \frac{V}{U}\left(+4\frac{\delta r}{r} + \frac{\delta P}{P} \right). \tag{15}$$

Here U and V are the usual homology invariants, and in equation (14) the density perturbation has been expressed in terms of the perturbations of the pressure and the entropy. The solution of these perturbation equations for the position adjustment δr and the pressure adjustment δP as a function of any given entropy perturbation δE can be expressed in terms of two independent solutions of the corresponding homogeneous equations (i.e., with δE set to zero in eq. [14]). One of these solutions (subscript "1") should be regular at the center. It is given near the center by the development

$$\left(\frac{\delta r}{r}\right)_1 = -\frac{1}{5} - \frac{3}{250}V - \ldots , \qquad \left(\frac{\delta P}{P}\right)_1 = +1 + \frac{1}{10}V + \ldots . \tag{16}$$

The other solution (subscript "2") should be regular at the surface. If the stellar model under consideration has an envelope represented by a convective zero solution—as is

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the case for the red-giant models recently studied—the second solution can be represented near the surface by the development

$$\left(\frac{\delta r}{r}\right)_2 = -\tfrac{1}{4} - \tfrac{3}{20}\left(\frac{R-r}{R}\right) - \tfrac{6}{35}\left(\frac{R-r}{R}\right)^2 - \dots ,$$
$$\left(\frac{\delta P}{P}\right)_2 = +1 + \tfrac{3}{7}\left(\frac{R-r}{R}\right) + \tfrac{10}{21}\left(\frac{R-r}{R}\right)^2 + \dots . \quad (17)$$

Now the final solution of the inhomogeneous equations (14) and (15) can be expressed by

$$\frac{\delta r}{r} = \left(\frac{\delta r}{r}\right)_1 \int_r^R \frac{Q}{(\delta P/P)_1} \cdot \tfrac{3}{5}\delta E \frac{dr}{r} + \left(\frac{\delta r}{r}\right)_2 \int_0^r \frac{Q}{(\delta P/P)_2} \cdot \tfrac{3}{5}\delta E \frac{dr}{r}, \quad (18)$$

$$\frac{\delta P}{P} = \left(\frac{\delta P}{P}\right)_1 \int_r^R \frac{Q}{(\delta P/P)_1} \cdot \tfrac{3}{5}\delta E \frac{dr}{r} + \left(\frac{\delta P}{P}\right)_2 \int_0^r \frac{Q}{(\delta P/P)_2} \cdot \tfrac{3}{5}\delta E \frac{dr}{r}, \quad (19)$$

with

$$Q = \left[\left(\frac{\delta r}{r}\right)_2 \Big/ \left(\frac{\delta P}{P}\right)_2 - \left(\frac{\delta r}{r}\right)_1 \Big/ \left(\frac{\delta P}{P}\right)_1\right]^{-1}. \quad (20)$$

Numerical integrations carried out for one example of red-giant models of the type recently studied result in a Q varying smoothly throughout the star and staying between the limits -4 and -8 everywhere except in small regions near the center and the surface. A value of -6 may therefore be accepted as typical for Q.

If we now consider again a shell of limited thickness and if we further take it that the entropy perturbation is restricted to this shell, then equation (19) may be approximated by

$$\frac{\delta P}{P} = \tfrac{3}{5} Q \frac{\Delta r}{r} \cdot \delta E. \quad (21)$$

With a negative value for Q, this equation shows that a positive entropy perturbation causes a negative pressure perturbation in the shell, as was to be expected. Furthermore, this equation shows that the size of the pressure perturbation decreases as the thickness of the shell decreases, as was made plausible by the discussion in § I.

Finally, we can compute the temperature perturbation within the shell since this perturbation can be derived from the pressure perturbation and the entropy perturbation with the help of equation (3) and of the ideal gas law. The result is

$$\frac{\delta T}{T} = \tfrac{3}{5}\left(1 + \tfrac{2}{5} Q \frac{\Delta r}{r}\right) \cdot \delta E. \quad (22)$$

Were the shell considered very wide ($\Delta r \approx r$), equation (22) would give—with the negative value of Q just quoted—a negative temperature perturbation for a positive entropy perturbation, just as for a homologous perturbation for the star as a whole. However, if the shell considered is very thin, clearly equation (22) gives a positive temperature perturbation for a positive entropy perturbation—much as in a degenerate region. For this positive relation between temperature and entropy perturbations to hold, the thickness of the layer must clearly fulfil the inequality

$$\frac{\Delta r}{r} < \tfrac{5}{2} \frac{1}{|Q|}. \quad (23)$$

Now let us combine the result of the preceding section with the one just derived. If the layer is thick enough to fulfil equation (12), a positive temperature perturbation

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causes an increase in time of the entropy in the shell. If the shell is thin enough to fulfil equation (23), a positive entropy perturbation causes a positive temperature perturbation. Hence if the shell manages to fulfil both equation (12) and equation (23), a thermal runaway threatens to result. Since for a typical shell containing a major nuclear-energy source in red-giant models the relative temperature thickness ($\Delta T/T$) is numerically roughly equal to the relative geometrical thickness ($\Delta r/r$), conditions (12) and (23) can be simultaneously fulfilled by such a shell as long as ν is larger than 10, which is the case for the carbon cycle as well as for helium burning. It appears then possible that the helium-burning shell and possibly even the hydrogen-burning shell in red-giant models may be thermally unstable.

IV. STABILITY TEST

To derive a more definitive test of thermal stability for any sample model it is necessary to consider the four basic equations (1), (2), and (13) all simultaneously, not in two separate pairs of equations as was done in the preceding semiquantitative considerations. We may introduce into the four basic equations for each of the independent variables the appropriate values from the unperturbed sample model plus an as yet undetermined perturbation. We may further isolate separate instability modes by assuming that the time variation of all perturbations in one mode follows a simple exponential, i.e., that

$$\delta L_r,\ \delta r,\ \delta P,\ \delta T \propto e^{+t/\tau}\,. \tag{24}$$

We shall here consider only real values of τ. This means that we are testing the sample models for thermal runaways, not oscillatory instabilities (this does not mean, however, that a thermal runaway when it reaches substantial amplitudes might not lead to a repetitive phenomenon such as relaxation oscillations).

When the time variation given by equation (24) is introduced into the basic equation (2), this equation takes the form

$$\delta(\epsilon)-\delta\left(\frac{dL_r}{dM_r}\right)=\delta\left(\frac{3}{2}\frac{P}{\rho}\right)\cdot\frac{dE}{dt}+\frac{3}{2}\frac{P}{\rho}\cdot\frac{\delta(E)}{\tau}\,. \tag{25}$$

Here the first term on the right-hand side represents the perturbation in the energy release by contraction in the unperturbed model, while the second term on the right-hand side arises from the exponential increase in the entropy perturbation. The other three basic equations do not contain time derivatives explicitly and therefore give simple equations for the perturbations without extra terms like the last one in equation (25). We ignore here the perturbations of the chemical composition caused by the perturbation in ϵ, on the plausible assumption that this effect is a minor one.

The various perturbations occurring in the four basic equations in their perturbed forms can all be expressed in terms of the four principle perturbations listed in equation (24). Thus four first-order, linear, homogeneous differential equations for these four perturbations are obtained. In addition, the four usual boundary conditions can be transformed to give four new boundary conditions for the perturbations. Thus a full fourth-order eigenvalue problem arises with the reciprocal of the e-folding time, $1/\tau$, as the eigenparameter. (The detailed equations are given in the Appendix.)

This problem can be solved numerically for any particular model by the following technique. The differential equations for the four principle perturbations are replaced by difference equations (these difference equations are identical with those used in the Henyey method with only one exception, caused by the last term in eq. [25]; hence these difference equations are essentially already available if the Henyey method was used to produce the unperturbed model in the first place). The difference equations, together with four boundary conditions, form a set of linear, homogeneous algebraic equations just equal in number to the number of unknown perturbation values. To permit a non-

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trivial solution the determinant of the matrix of this set of equations must be zero, a condition that will be fulfilled for particular values of the eigenparameter. The numerical work then consists of computing the determinant for various trial values of the eigenparameter until the largest value is found for which the determinant goes to zero. The necessary numerical work is tedious since it turns out that many trial values for the eigenparameter have to be considered to be reasonably certain that the largest positive eigenvalue is not overlooked. On the other hand, with a fast computer like the IBM 7094 the computation of the determinant for one value of the eigenparameter (or equivalently the determination of the last coefficient in the Gaussian elimination scheme) takes only 4 sec even in a case in which the order of 500 points are needed to cover the star properly.

This technique has been applied to ten models representing various phases in the evolution of a star of 1 $M_{\odot}$ with an initial composition appropriate for Population II stars. The numerical results are summarized in Table 1. The first three models corre-

TABLE 1

RESULTS OF STABILITY TEST FOR TEN MODELS REPRESENTING VARIOUS EVOLUTIONARY PHASES* FOR A STAR OF 1 $M_{\odot}$

Model	q_{Hy}	q_{He}	$L_{Hy}/L_{\odot}$	$L_{He}/L_{\odot}$	Phase	t (yrs)	Δt (yrs)	τ (yrs)
A	0 264	.	33	0	Before He flash	− 1 2×10^8	9×10^6	−3 7×10^7
B	393		490	0		− 1 0×10^7	7×10^5	−1 6×10^7
C ...	464	.	1450	0		− 2 3×10^6	3×10^5	−1 0×10^7
D	510		2600	0 8	Start of He flash	− 1 4×10^5	5×10^4	+2 8×10^5
E ..	.510	.	2600	0		—	—	−0 8×10^7
F ...	533		45	21	He core burning	+ 4 1×10^7	15×10^5	−1 8×10^4
G	549	. . .	78	22		+ 6 9×10^7	6×10^5	−1 8×10^4
H	565	0 174	43	72	He shell	+ 8 8×10^7	3×10^5	−4 0×10^6
I	585	287	77	96		+10 9×10^7	3×10^5	−1 6×10^7
J ..	0 594	0 379	122	118		+11 7×10^7	3×10^5	+1 2×10^6

* q_{Hy} = mass fraction interior to hydrogen-burning shell; q_{He} = mass fraction interior to helium-burning shell; L_{Hy} = flux produced by hydrogen burning; L_{He} = flux produced by helium burning; t = evolution time counted from peak of helium flash (backward and forward); Δt = time step between consecutive computed models (total about 400); τ = characteristic time of instability (minus sign = stable; plus sign = unstable).

spond to phases well before the helium flash when the whole luminosity of the star arises from hydrogen burning in a shell just outside the helium core. The cores of these models are degenerate; however, the hydrogen-burning shell is well outside the degenerate region. No positive eigenvalues were found for these models, indicating that they are stable against thermal perturbations of the kind we are investigating.

Model D of Table 1 represents the phase when the helium flash is just starting (the helium burning in the core has risen to not quite 1 $L_{\odot}$ while the hydrogen-burning shell provides 2600 $L_{\odot}$). This model turns out thermally unstable, just as we should have expected, with an e-folding time of less than 300000 years. To make entirely certain that this instability really represents the onset of the helium flash we have computed through Model E, which differs from Model D only in the fact that the helium burning was turned off, a very minor modification as far as the equilibrium model is concerned. This modified model turned out fully as stable as all the preceding models in which helium burning did not play any role. The instability of Model D is then clearly nothing but the well-known thermal instability of a degenerate core in which a new nuclear source is starting.

Models F and G represent phases, well after the helium flash, when part of the lumi-

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nosity of the star is provided by a non-degenerate convective helium-burning core while the rest of the luminosity is, as before, generated in the hydrogen-burning shell. These models turned out stable. The three remaining Models (H–J) represent a still later phase in which the star contains two separate energy-producing shells, an inner helium-burning shell and an outer hydrogen-burning shell. Even the centers of these models are at most incipiently degenerate. The first two of these three models were found to have no positive eigenvalues, i.e., to be thermally stable.

The last model corresponds approximately to that evolution phase in which the apparent instabilities were first noticed, as described in the first section of this paper. For this model a positive eigenvalue was found, corresponding to an e-folding time of approximately 2000000 years (the evolution sequence computations described in the first section suggest that the e-folding time rapidly decreases as the perturbation grows). The solution for the perturbations corresponding to this eigenvalue fits roughly the description for a single burning shell in the preceding two sections. The two burning shells of this model have temperature perturbations with opposite signs, in agreement with the results of the evolution computations described in § I.

V. DISCUSSION

The numerical results just reported appear to show that thermal instabilities do occur in some non-degenerate stellar models. The circumstances under which such instabilities occur, however, appear fairly restrictive. In particular, models tested that correspond to evolution phases prior to any helium burning appear to be thermally stable. On the other hand, the last model tested with a helium-burning shell, in addition to a hydrogen-burning shell, turned out to be unstable. How far these results can be generalized, particularly when a wider range of stellar masses is considered, is entirely unclear at present.

The question arises whether the instability, whenever it occurs, should be automatically noticed in ordinary stellar-evolution computations. For the particular form of the implicit method nowadays usually employed to handle the time derivative term in the basic equation (2), it can be shown with fair certainty that a possible thermal instability will be passed without any notice as long as the time steps employed between successive models are larger than twice the e-folding time of the instability. Now the e-folding time of the thermal instability is by order of magnitude related to the Kelvin time scale of the relevant shell, while for nuclear-burning phases the time interval between consecutive models will generally be chosen in relation to the much longer nuclear-burning time scale. Thus, in general, the condition for not noticing any thermal instability during ordinary evolution computations will be fulfilled. The fact that in our specific recent evolution computations we nevertheless were led to notice the thermal instability was, it seems, caused solely by the accidental circumstance that technical difficulties connected with the fine structure of the helium-burning shell forced us to take unusually small time steps, just small enough to get a hint of the instability. In general, then, we apparently have to conclude that the thermal stability of stellar models, particularly those representing advanced evolution phases, derived by us and others can not be taken for granted without specific investigation.

Finally, it is tempting to speculate regarding the possible consequence of a thermal instability in non-degenerate regions of a star when it actually occurs. The computational test runs carried out to date with the full non-linearized equations suggest that a thermal perturbation in an unstable model will be likely to run far enough to increase the helium-burning rate by more than a factor 10 over its normal value. By then appreciable convective mixing of the layers around the shell have occurred. It appears possible, although perhaps not so likely, that the thermal perturbation could reach a level strong enough to cause dynamical effects. It would seem probably more likely that in due course the thermal perturbation will spread over a sufficiently wide layer in the star to permit relaxation of the perturbation by a general adiabatic expansion. Since such a cycle would

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probably not alter the star basically and hence may be repetitive, one might guess that a thermal instability of the type here considered may lead to a kind of relaxation oscillation, a thermal flicker. Whether the convective mixing caused by such a flicker could reach an extent sufficient to have substantial evolutionary consequences seems at present an entirely open question.

We happily record that discussions with L. Spitzer played a key role in the pursuit of this investigation. This work made use of computer facilities supported in part by National Science Foundation grant NSF-GP579. This investigation has been carried out under Air Force Office of Scientific Research Contract AF 49(638)-1555.

APPENDIX

This Appendix presents equations applicable for the construction of stellar models by the Henyey method as well as for thermal stability testing.

Basic Variables:

$S =$ log (mass interior to r, M_r) $- 30$

$R =$ log (distance from center, r)

$P =$ log (total pressure)

$T =$ log (temperature)

$B =$ luminosity at r/luminosity of Sun.

The subtraction of 30 in the equation for S gains a factor of 10 in round-off accuracy over the relevant range in M_r, which can be critical in advanced evolution phases. The luminosity L_r is not represented in logarithmic form since L_r becomes negative under some circumstances.

Basic Derivatives:

$QR = +dR/dS\ [= 1/U]$

$QP = -dP/dS\ [= V/U]$

$QT = +dT/dP\ [= 1/(n+1)]$

$QB = +dB/dS.$

The symbols in brackets refer to the usual homology invariants. The choice of P instead of S in the third equation leads to a derivative which does not depend explicitly on R and varies conveniently slowly.

Auxiliary Variables:

$FNAT = \log e$

$GAM = \frac{5}{3}$

$G = \log(\Delta t)$ ($\Delta t =$ time step between models)

$D =$ log (density)

$QDP = +\partial D/\partial P$ at constant T

$QDT = -\partial D/\partial T$ at constant P

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A = absorption coefficient, including conduction

$QAP = +(FNAT/A)*(\partial A/\partial P)$ at constant T

$QAT = -(FNAT/A)*(\partial A/\partial T)$ at constant P

C = log (energy generation, ϵ) $-$ log $\epsilon_0(X, Z)$

$QCP = +\partial C/\partial P$ at constant T

$QCT = \partial C/\partial T$ at constant P

$E = P - GAM*D$ (linear function of entropy)

$EP = E$ for preceding model at same S

$QBN = QB$ from nuclear processes

$QBE*(-\Delta t*dE/dt) = QB$ from gravitation

$QTB = +\partial QT/\partial B$ at constant P and T.

The simple formula for E is not applicable in important ionization zones or where radiation pressure or relativistic degeneracy plays a role.

Differential Equations:

QR = antil. $(-3*R - D + S + 28.9008)$, where $+28.9008 = 30 - \log(4\pi)$

QP = antil. $(-4*R - P + 2*S + 51.7249)$, where $+51.7249 = 2*30 + \log(G/4\pi)$

$QT = B*QTB$ for radiative equilibrium with $QTB = A*$antil. $(+P - 4*T - S + 13.1746)$, where $+13.1746 = -30 + \log(3L_\odot/16ac\pi G)$

$QT = 0.4$ for convective equilibrium under the same restrictions as for E

$QB = QBN + QBE*(-\Delta t*dE/dt)$ with $QBN = \epsilon_0(X, Z)*$antil.$(+S + C - 3.2153)$,
where $-3.2153 = +30 - \log(FNAT*L_\odot)$
where $-3.2153 = +30 - \log(FNAT*L_\odot)$ and QBE = antil. $(P - D + S - G - 2.6770)$, where $-2.6770 = +30 - \log(FNAT^2*L_\odot/1.5)$

Difference Equations:

$$(R2 - R1)/(S2 - S1) = +0.5*(QR1 + QR2)$$

$$(P2 - P1)/(S2 - S1) = -0.5*(QP1 + QP2)$$

$$(T2 - T1)/(P2 - P1) = +0.5*(QT1 + QT2)$$

$$(B2 - B1)/(S2 - S1) = +0.5*(QB1 + QB2).$$

The indices "1" and "2" distinguish the two points between which the difference equations are applied.

Additional Auxiliary Quantities:

$$GS = 2*FNAT/(S2 - S1)$$

$$GP = 2*FNAT/(P1 - P2)$$

$$GT = GP*(T1 - T2)/(P1 - P2).$$

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For application of the Henyey method:

$$UR = GS*(R2 - R1) - FNAT*(QR1 + QR2)$$

$$UP = GS*(P1 - P2) - FNAT*(QP1 + QP2)$$

$$UT = GP*(T1 - T2) - FNAT*(QT1 + QT2)$$

$$UB = GS*(B2 - B1) - FNAT*(QB1 + QB2).$$

For thermal stability testing:

$$UR = 0,\ UP = 0,\ UT = 0,\ UB = 0\ .$$

The U-quantities measure the accuracy with which the difference equations are fulfilled.

Corrections or Perturbations:

The four dependent basic variables are replaced by

$$R + DR,\ P + DP,\ T + DT,\ B + DB.$$

Linearization of Entropy Derivative:

For application of the Henyey method

$$-\Delta t*dE/dt = -\Delta t*(E + DE - EP)/\Delta t = EP - E - DE.$$

For thermal stability testing:

$$-\Delta t*dE/dt - \Delta t*[(E - EP)/\Delta t + DE/\tau] = EP - E - (\Delta t/\tau)*DE,$$

where τ stands for the e-folding time.

Hence for both applications:

$$-\Delta t*dE/dt = EP - E - EIG*DE$$ with $EIG = 1$ for Henyey method but $EIG = \Delta t/\tau$ for stability testing.

Linearization of Difference Equations:

$$+DR1*(+GS - 3*QR1) + DP1*(-QR1*QDP1) + DT1*(+QR1*QDT1)$$
$$+DR2*(-GS - 3*QR2) + DP2*(-QR2*QDP2) + DT2*(+QR2*QDT2) = UR;$$

$$+DR1*(-4*QP1) + DP1*(-GS - QP1)$$
$$+DR2*(-4*QP2) + DP2*(+GS - QP2) = UP;$$

$$+DB1*(+FNAT*QTB1)$$
$$+DP1*[+GT + QT1*(1 + QAP1)] + DT1*[-GP - QT1*(4 + QAT1)]$$
$$+DB2*(+FNAT*QTB2)$$
$$+DP2*[-GT + QT2*(1 + QAP2)] + DT2*[+GP - QT2*(4 + QAT2)] = UT;$$

$$+DB1*(+GS)$$
$$+DP1*\{+QBN1*QCP1 + QBE1*[(EP1 - E1)*(1 - QDP1) + EIG*FNAT*(GAM*QDP1 - 1)]\}$$

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$$+DT1*[+QBN1*QCT1 + QBE1*QDT1*(EP1 - E1 - EIG*FNAT*GAM)]$$

$$+DB2*(-GS)$$

$$+DP2*\{+QBN2*QCP2 + QBE2*[(EP2 - E2)*(1 - QDP2) + EIG*FNAT* (GAM*QDP2 - 1)]\}$$

$$+DT2*[+QBN2*QCT2 + QBE2*QDT2*(EP2 - E2 - EIG*FNAT*GAM)] = UB.$$

The third of these four equations holds directly only if radiative equilibrium obtains at both points involved. The same equation can be made to apply also to the case that convective equilibrium obtains (at only one or at both points) by the following trick. Whenever the equilibrium at a point is found to be convective, i.e., if $QT = 0.4$, then for this point set $QTB = 0$, $QAP = -1$, $QAT = -4$, irrespective of preceding definitions.

Boundary Conditions at Center:

The first point (near center) is so chosen that the variation of P and T between the center and the first point can be neglected. Accordingly the two boundary conditions at this point are

$$S2 = 3*R2 + D2 - 29.3779, \text{ where } -29.3779 = -30 + \log (4\pi/3),$$

$$B2 = FNAT*QB2\,.$$

Linearized boundary conditions:

$$+DR2*(+3) + DP2*(+QDP2) + DT2*(-QDT2) = UR;$$

$$+DB2*(-1)$$

$$+DP2*\{+QBN2*QCP2 + QBE2*[(EP2 - E2)*(1 - QDP2) + EIG*FNAT* (GAM*QDP2 - 1)]\}$$

$$+DT2*[+QBN2*QCT2 + QBE2*QDT2*(EP2 - E2 - EIG*FNAT*GAM)] = UB;$$

with $UR = S2 - 3*R2 - D2 + 29.3779$, $UB = B2 - FNAT*QB2$ for application of Henyey method; but $UR = 0$, $UB = 0$ for thermal stability testing.

Boundary Conditions at Surface:

The last point (near surface) is so chosen that in the layers between this point and the surface the flux divergence can be taken to be zero, i.e., that nuclear-energy generation and gravitational-energy release can be neglected. The solution for these layers will then in general be available either in analytical or in tabular form. For a star of given mass and outer composition, this solution can depend only on two parameters, the star's luminosity and radius. By eliminating these two parameters, one can derive two relations between the four basic dependent variables at the last point. These relations, or boundary conditions, can be given in the form:

$R2 = RSU$, $B2 = BSU$, where RSU and BSU are functions of $P2$ and $T2$.

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Linearized boundary conditions:

$$DR2 + DP2*(-QRSUP) + DT2*(-QRSUT) = UR$$

$$DB2 + DP2*(-QBSUP) + DT2*(-QBSUT) = UB$$

with $QRSUP = \partial RSU/\partial P$ at constant T

$QRSUT = \partial RSU/\partial T$ at constant P

$QBSUP = \partial BSU/\partial P$ at constant T

$QBSUT = \partial BSU/\partial T$ at constant P

and $UR = RSU - R2$, $UB = BSU - B2$ for application of Henyey method; but $UR = 0$, $UB = 0$ for thermal stability testing.

Total Set of Linearized Equations:

The two center boundary conditions (applied to the first point), the two surface boundary conditions (applied to the last point), and the four difference equations (applied to every pair of points in between) add up to a set of equations equal in number to that of the unknowns (DR, DP, DT and DB at every point).

For the application of the Henyey method:

The equations are inhomogeneous (U-values not generally zero) and do not contain an eigenparameter ($EIG = 1$). They will therefore have in general a unique solution for the four corrections at all points.

For the stability testing:

The equations are homogeneous (U-values are all zero by the assumption that the model to be tested fulfils all the basic equations) and contain an eigenparameter (the e-folding time or EIG). They will therefore have a unique solution for the four perturbations at all points (except for a scaling factor), if the determinant of the equation matrix vanishes, i.e., if the eigenparameter is equal to an eigenvalue.

For both applications the equations are nearly identical and the solutions require a Gaussian elimination scheme. Hence large parts of the machine codes for the two problems are identical.

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