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# TURBULENCE OF A CONDUCTING FLUID IN A STRONG MAGNETIC FIELD

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The spectral method is applied to turbulence of a conducting fluid in a strong magnetic field, the presence of which implies that turbulent flow of a fairly fine-scale nature is a system of weakly interacting plane traveling waves. It is shown that the interaction of the waves is due in the main to the compressibility of the gas. The interaction of waves of vastly different scales is described by the method of adiabatic invariants, and waves of roughly the same scale (local interaction) are described by means of the transport coefficient, which is similar to the diffusion coefficient. It is assumed that the energy flux through a hierarchy of eddies is proportional to the square of the spectral energy density. A nonlinear equation in partial derivatives describing homogeneous turbulence under stationary external conditions is derived. The spectrum of stationary turbulence is obtained with account taken of viscous dissipation. The stationary turbulence spectrum is obtained with viscous forces ignored (i.e., similar to the Kolmogorov spectrum), and without resorting to the local-interaction approximation.

## Introduction

The problem of the intensification of an initially weak magnetic field frozen into an ideally conducting incompressible turbulent fluid, and the effect of the field on turbulence, are fascinating questions. The initial magnetic field H may be due to noise phenomena (see [1], for example). Light must be shed on the ultimate behavior of a magnetic field varying in response to the tangling of magnetic lines of force and dissipation of magnetic energy through Joule losses [2, 3].

A paper by Batchelor [2, 3] dealt with this problem as early as 1950. This author treated the evolution of a magnetic field so weak that its countereffect on turbulence could be neglected, while assuming that at any moment **H** = const · curl **v**, where **v** is the velocity of the fluid flow. Then, when the conventional viscosity $\nu$ and the magnetic viscosity $\nu_m$ are equal, the magnitude $|\mathbf{H}|$ of the field will be stationary on the average. On the other hand, the distance between two arbitrarily selected elements of the fluid will clearly increase without bound with time on the average (the average being taken over the infinite volume occupied by the fluid). In the region of the Kolmogorov spectrum [4, 5] (where the dissipative terms in the equations for **v** and **H** are negligibly small compared to the remaining terms), the magnetic field is frozen into the fluid, so that it must also grow without bound. Hence, clearly, the solution presented by Batchelor is that rare particular case of flow where the mechanism responsible for tangling up the field **H** is not operative. This becomes particularly clear in the case of two-dimensional (planar) turbulence: the field **H** is perpendicular to the plane in which flow occurs – there is no tangling of the magnetic field.

To be sure, the most popular view holds that when conductivity is high the magnetic field becomes increasingly stronger and of larger scale with the passage of time; there exists a characteristic scale $\lambda_0(t)$ at which equipartition of the kinetic and magnetic energy occurs; at finer scales, the magnetic field suppresses turbulence, and at larger scales, the field has virtually no effect on turbulence. However no correct calculations of this model were provided.

Chandrasekhar [6] and S. A. Kaplan [7] solved systems of equations describing the behavior of the spectral functions of the kinetic and magnetic energy. In constructing the equations, a priori and physically unjustified assumptions were made to the effect that the presence of the magnetic field did not alter the mode of interaction between eddies of different scales.

A. Ya. Kipper [8] poses the possible existence of a completely tangled stationary magnetic field, with the spectrum of the field in effect dependent on the same parameters as the spectrum of ordinary turbulence. The

Kolmogorov spectrum is naturally derived as a corollary, and unambiguously.

Schluter and Biermann [9] calculated the time variation of the growth of the magnetic field at those scales where the field would exert no countereffect on the turbulence.

However, it is apparently of greatest interest to obtain the turbulence spectrum in the region of those scales where the effect of the magnetic field is too significant to be neglected.

This region of the spectrum may be broken up into two parts. First, a comparatively restricted portion of the spectrum to which correspond flows on the maximal turbulence scale. This portion of the spectrum presents the greatest difficulties for exact calculations.

Second, a region of finer-scale pulsations, where turbulence is suppressed by a strong externally applied (external with respect to these pulsations) magnetic field. This region is the one we shall discuss in this paper.

Indeed, let there be a random flow of the conducting fluid in a strong uniform external magnetic field $\mathbf{H}_0$.

The presence of a strong (stronger than the random) externally applied magnetic field results in any flow being decomposed into a sum of weakly interacting waves [1, 3] with different wave vectors **k**. The term "weak interaction" used here means that the characteristic time of deformation of the waves is much longer than their periods.

According to the virial theorem, the densities of the potential and kinetic energy are equal in a weak wave; in an Alfvén wave, for example:

$$\frac{\rho v^2}{2} = \frac{H_\sim^2}{8\pi}. \quad (1)$$

Here $\rho$ is the density of the medium, **v** is the velocity of flow, $\mathbf{H}_\sim$ is a perturbation of the magnetic field. Hence, we need not introduce the spectral densities of the kinetic and magnetic energy separately, as has been the practice up to the present [7], but instead we shall determine the spectral density of the total energy: $E_k(k)dk$ is the energy density per unit mass of the medium, concentrated in flows with absolute values of the wave numbers falling in the interval from $k$ to $k + dk$ [10]

$$E_k(k) = 2F(k)$$

$$= \frac{1}{2\pi^2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\sum_{i=1}^{3} Q_{ii}(\mathbf{r})\sin(\mathbf{kr})(\mathbf{kr})\sin\theta\, d\theta\, d\varphi\, d\mathbf{r}, \quad (2)$$

where F(k) is the spectral density of the kinetic energy; **k** and $k$ are the wave vector and its modulus; $r$, $\theta$, and $\varphi$ are spherical coordinates in geometrical space; $Q_{ij}$ is the tensor correlating the velocity and the magnetic field:

$$Q_{ij} = \overline{\mathbf{v}_i\mathbf{v}_j'}. \quad (3)$$

In the case of an intense external magnetic field there will be no spherical symmetry, but we can average over both angles $\varphi$ and $\theta$. Taking asphericity into account would greatly complicate the problem, since refraction of the waves occurs in the case of turbulent flow, on any twists of the field $\mathbf{H}_0$.

The problem consists in constructing an equation for the function $E_k(k)$.

## 1. The Role of Compressibility in a Weak Interaction between Magnetohydrodynamic Waves

In this section, it will be shown that a weak interaction between Alfvén waves in an incompressible fluid is very specific, and that interaction between waves is therefore governed primarily by the compressibility.

The general solution of the linearized equations of motion of an infinitely conducting inviscid incompressible fluid in a strong magnetic field is a superposition of transverse plane waves:

$$\mathbf{v} = \mathbf{v}^+ + \mathbf{v}^-,$$

$$\mathbf{h} \equiv \frac{\mathbf{H}-\mathbf{H}_0}{\sqrt{4\pi\rho}} \equiv \frac{H}{\sqrt{4\pi\rho}} - \mathbf{u}_0 = \mathbf{v}^+ - \mathbf{v}^-, \quad (4)$$

$$\mathbf{v}^\pm = \int \mathbf{a}_\mathbf{k}^\pm e^{i\mathbf{k}(\mathbf{r}\mp\mathbf{u}_0 t)}\, d\mathbf{k}, \quad (5)$$

$$(\mathbf{k}\mathbf{a}_\mathbf{k}^\pm) = 0. \quad (6)$$

Here **v** is the flow vector of the fluid, **H** is the magnetic field vector, $\rho$ is the density of the fluid, $\mathbf{a}_\mathbf{k}^\pm$ are the complex amplitudes of the aves, **k** is the wave vector.

In the case of an asymptotically weak interaction of waves, the general solution will be represented, as earlier, by formulas (4) to (6), but the complex amplitudes $\mathbf{a}_\mathbf{k}^\pm$ will vary slowly (as compared to the periods of the waves) with time. Their rate of change is determined by the nonlinear terms in the equations of motion:

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\text{nonlin}} = -(\mathbf{v}\nabla)\mathbf{v} + (\mathbf{h}\nabla)\mathbf{h},$$

$$\left(\frac{\partial \mathbf{h}}{\partial t}\right)_{\text{nonlin}} = (\mathbf{h}\nabla)\mathbf{v} - (\mathbf{v}\nabla)\mathbf{h}. \quad (7)$$

From Eqs. (7) and (4) we have

$$\left(\frac{\partial \mathbf{v}^+}{\partial t}\right)_{\text{nonlin}} = -2(\mathbf{v}^-\nabla)\mathbf{v}^+,$$

$$\left(\frac{\partial \mathbf{v}^-}{\partial t}\right)_{\text{nonlin}} = -2(\mathbf{v}^+\nabla)\mathbf{v}^-. \quad (8)$$

Clearly, from these formulas, the Alfvén waves of the same group velocity will not interact mutually.

From formulas (5) and (8) we infer

$$\int \frac{\partial \mathbf{a}_\mathbf{k}^\pm}{\partial t} e^{i\mathbf{k}(\mathbf{r}\pm\mathbf{u}_0 t)}\, d\mathbf{k}$$

$$= -2i\int (\vec{\varkappa}\mathbf{a}_{\vec{\mathbf{k}}-\vec{\varkappa}}^\mp)\,\mathbf{a}_{\vec{\varkappa}}^\pm e^{i\mathbf{kr}\pm i(2\vec{\varkappa}-\mathbf{k})\mathbf{u}_0 t}\, d\mathbf{k}\, d\vec{\varkappa} \quad (9)$$

or

$$\frac{\partial a_{\mathbf{k}}^{\pm}}{\partial t} = -2i \int (\varkappa a_{\mathbf{k}-\varkappa}^{\mp}) a_{\varkappa}^{\pm} e^{-i(2\varkappa-\mathbf{k})u_0 t} d\varkappa, \quad (10)$$

such that

$$(a_{\varkappa}^{\pm} \mathbf{k}) = 0. \quad (11)$$

Formula (10) reveals that the "dissipative" (i.e., aperiodic) contribution to $\partial a_{\mathbf{k}}^{\pm}/\partial t$ will yield interactions of only those waves for which

$$|(\vec{\varkappa} - \mathbf{k}) \mathbf{u}_0| \approx \frac{\pi}{2T}, \quad (12)$$

where T is the characteristic time of the variation of $a_{\mathbf{k}}^{\pm}$. Because of the weakness of the interaction, this time will be much longer than the periods $\tau = 2\pi/\omega$ of the interacting waves, so that

$$|(\vec{\varkappa} - \mathbf{k}) \mathbf{u}_0|_{\text{eff}} \approx \frac{\pi}{2T_k} \ll \frac{\pi}{2\tau}$$
$$\approx \frac{1}{4} \overline{|\mathbf{k}\mathbf{u}_0|} \approx \frac{1}{4} \overline{|(\vec{\varkappa} - \mathbf{k}) \mathbf{u}_0|}. \quad (13)$$

The line above denotes averaging with respect to the direction of wave travel.

We infer from Eq. (13) that only a small fraction of the interacting pairs are effective from the standpoint of "dissipativity."

Since the Alfvén velocity is close to the speed of sound in a plasma in cases of practical interest, it is evident that the energy flow relative to the hierarchy of eddies will be governed chiefly by the compressibility. The magnetic field is important as the force suppressing conventional turbulence and leading to the weakness of the interactions between waves.

## 2. Interaction of Waves of Essentially Distinct Scales

First consider interaction between waves of essentially different wavelengths. For our crudest estimate of the interaction, we may assume that the long-wave oscillation slowly alters the parameters of the short-wave oscillation, and we may apply the theory of adiabatic invariants to the purpose [11].

The propagation of any weak plane wave is described by the linear wave equation [1]

$$\frac{\partial^2 \psi}{\partial t^2} = u_0^2 \frac{\partial^2 \psi}{\partial x^2}, \quad (14)$$

where the $\underline{x}$ axis lies parallel to the wave vector $\mathbf{k}$, $u_0$ is the speed of transmission of the signal, $\psi$ characterizes perturbations of the velocity $\mathbf{v}$, of the magnetic field $\mathbf{H}$, and of the density $\rho$ of the fluid.

We must find out how the energy associated with the wave varies as the parameter $u_0$ changes slowly. The slowness of this change means that the characteristic time $\tau$ of the characteristic dimension L of any essential change experienced by $u_0$ is much less than the period T or, correspondingly, the wavelength $\lambda$.

Since Eq. (14) describes the propagation of any plane waves, it is obvious that the solution of the problem posed is independent on the concrete wave form. We therefore proceed to consider longitudinal sound waves.

The idea underlying the solution consists in finding a canonically conjugate coordinate $\xi$ and momentum $\pi$ pair describing the collective wave motion [12]. The wave motion then reduces to the motion of a quasiparticle in phase space $(\xi, \pi)$. Adiabatic invariants are indeed applicable to such motion of one "particle."

Since we may learn how to find a pair of collective coordinates for longitudinal waves by consulting ter Haar's review [12], we need not make a presentation of that theory here, and now cite only the result: if we consider collective motion of a system of N bodies ($N \gg 1$), then the motion corresponding to the wave vector $\pm\mathbf{k}$ will be described by the coordinates

$$\bar{\xi}_{\pm\mathbf{k}} = (Nk)^{-1} \sum_j e^{\mp i(\mathbf{k}\mathbf{x}_j)}$$
$$\bar{\pi}_{\pm\mathbf{k}} = \pm k^{-1} \sum_j (\mathbf{k}\mathbf{p}_j)\, e^{\pm i(\mathbf{k}\mathbf{x}_j)}. \quad (15)$$

The time dependence is

$$\xi_{\pm\mathbf{k}}(t) = \xi_{\pm\mathbf{k}}(0)\, e^{i\omega_k t}, \quad (16)$$

where $\omega_{\mathbf{k}}$ is the frequency of oscillation, $\mathbf{x}_j$, $\mathbf{p}_j$ are the coordinate and momentum of the j-th particle in Cartesian space.

The Hamiltonian of the entire system in the case of collective motion in the space $(\xi, \bar{\pi})$ has the form

$$E = H = T + U = (Nm)^{-1} \bar{\pi}_{+\mathbf{k}} \bar{\pi}_{-\mathbf{k}} + Nm\omega^2 \bar{\xi}_{+\mathbf{k}} \bar{\xi}_{-\mathbf{k}}, \quad (17)$$

where $\underline{m}$ is the mass of a single particle, and E is the energy associated with the wave. Hence the phase trajectory of the quasi-particle is an ellipse of semiaxes $\sqrt{NmH}$ and $\sqrt{H/Nm\omega^2}$, so that the adiabatic invariant is

$$I_k = \oint \bar{\pi}_{\mathbf{k}}\, d\bar{\xi}_{\mathbf{k}} = \pi \frac{E}{\omega} = \text{const}. \quad (18)$$

A continuous medium is arrived at by proceeding to the limit $N \to \infty$, with $\rho = Nm/V = \text{const}$ (V is the volume of the system).

Summarizing, for slow changes of the parameters, $E \propto \omega$.

We see from formula (18) that the adiabatic interaction lacks a dissipative character. The oscillations of the parameters due to turbulence need not be taken into account, then, and as a consequence we infer that only a systematic change in the parameters governing the maximum turbulence scale (i.e., external velocity field, density, pressure, magnetic field) will be important. The frequency $\omega$ of weak oscillations may vary in a different manner depending on the wave form and on the direction of the wave from and on the direction of the wave vector **k**.

The case of a stationary flow field need not be considered apart. Here small oscillations are described by linear equations with time-invariant coefficients, and hence we see readily that $\omega$ = const.

Averaging with respect to wave form and direction of travel of the waves, we have

$$\omega = u_0 k = \text{const}. \tag{19}$$

Here $u_0$ is the average absolute value of the wave velocity. From (18) and (19), we have that the energy of the oscillation referred to a region of the spectrum from $\omega$ to $\omega + d\omega$ or from k to k + dk is constant and equal to

$$E = E_\omega d\omega = E_k dk = \text{const}, \tag{20}$$

where

$$d\omega = u_0 dk. \tag{21}$$

Since we are dealing with a model of an isotropic and homogeneous turbulence, there is no point in taking into account the effect of spatial gradients of the external parameters on the adiabatic interaction. We infer then from Eqs. (18) to (21) nothing more than that for homogeneous turbulence with external conditions stationary, we have

$$\left(\frac{\partial E_\omega}{\partial t}\right)_{\text{ad}} = 0, \tag{22}$$

$$\left(\frac{\partial E_k}{\partial t}\right)_{\text{ad}} = 0. \tag{23}$$

## 3. Interaction between Waves of Approximately the Same Scale

Now consider interaction between waves of approximately the same values of the modulus of the wave vector **k**. This interaction will be known as the local interaction. An equation describing this local interaction will be constructed by analogy with the theory of thermal conductivity or diffusion. The assumption on which this analogy rests is apprently the most dubious of all the assumptions entertained in this paper. Strictly speaking, the method of transport coefficients is applicable only when there is a slight deviation of the spectral function from its equilibrium value. However, one justification for this approach is the fact that we are talking only about a local interaction.

Note from the start that the problem discussed below of the stationary turbulence spectrum, similar to the Kolmogorov spectrum, will not be needed in this assumption.

We shall rely on quantum theory. Instead of waves, let us here introduce "quasi-particles" analogous to phonons [13]. The energy and momentum of the "quasi-particle" are specified by

$$\varepsilon = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}, \qquad \hbar = \frac{h}{2\pi}. \tag{24}$$

The velocity of the particle is equal to the velocity of the wave.

A gas composed of these quasi-particles obeys Bose statistics. Since the number of particles does not remain constant, the chemical potential vanishes and the average number of particles in a given quantum state is determined by Planck's formula

$$n_p = \frac{1}{e^{\varepsilon/\theta} - 1}, \tag{25}$$

where $\theta$ = kT.

Bearing Eq. (24) in mind, we derive the approximate equality

$$n_\omega d\omega \approx \frac{\omega^2 d\omega}{2\pi^2 (u_0)^3} \frac{1}{e^{\frac{\hbar\omega}{\theta}} - 1} \tag{26}$$

by averaging over angles and modes of waves.

What interests us is the case of very low frequencies ($\hbar\omega \ll \theta$), so that

$$n_\omega d\omega \approx \frac{\theta\omega d\omega}{2\pi^2\hbar (u_0)^3}. \tag{27}$$

Hence, at thermal equilibrium

$$\rho E_\omega = \hbar\omega n_\omega \approx \frac{\theta\omega^2}{2\pi^2 (u_0)^3}. \tag{28}$$

A local interaction complies with the continuity equation

$$\left(\frac{\partial E_\omega}{\partial t}\right)_{\text{loc}} + \frac{\partial S_\omega}{\partial \omega} = 0, \tag{29}$$

where $S_\omega$ is the flow of energy in $\omega$ space. At thermal equilibrium, $S_\omega$ = 0, and from formula (28)

$$\frac{\partial}{\partial\omega}(\omega^{-2} E_\omega) = 0.$$

We therefore assume that

$$S_\omega = -D_\omega \frac{\partial}{\partial\omega}(\omega^{-2} E_\omega). \tag{30}$$

Here $D_\omega$ is a coefficient similar to the diffusion and thermal conductivity coefficients. It is dependent solely on the following parameters: $\rho$, $\omega$, $E_\omega$, $u_0$, N. In the quantum-theoretical representation $D_\omega = D_\omega(\rho, \omega, n_\omega, u_0, \hbar, N)$. Here, $N = u_{acoustic}/u_{Alfvén}$.

Since the interaction of "quasi-particles" is a weak interaction, we suppose that only binary "collisions" occur. Hence

$$\left(\frac{\partial E_\omega}{\partial t}\right)_{loc} \propto n_\omega^2. \quad (31)$$

Bearing in mind formulas (28) to (30), we infer that

$$D_\omega = n_\omega \psi(\rho, \omega, u_0, \hbar, N) \quad (32)$$

or

$$D_\omega = \frac{\rho E_\omega}{\hbar\omega} \psi(\rho, \omega, u_0, \hbar, N), \quad (33)$$

with $N \approx 1$ in all cases of practical interest. Since $D_\omega$ depends solely on $\rho$, $\omega$, $E_\omega$, $u_0$, N and since the dimensionality $[D_\omega] = \sec^{-5}$, we arrive at

$$D_\omega = B'(N)\,\omega^6 u_0^{-2} E_\omega, \quad (34)$$

where B'(N) is a dimensionless function, and $B'(1) \approx 1$.

Utilizing formulas (29) and (30), we get

$$\left(\frac{\partial E_\omega}{\partial t}\right)_{loc} = B' u_0^{-2} \frac{\partial}{\partial\omega}\left[\omega^6 E_\omega \frac{\partial}{\partial\omega}(\omega^{-2}E_\omega)\right], \quad (35)$$

$$\left(\frac{\partial E_k}{\partial t}\right)_{loc} = B(N)\,u_0^{-1} \frac{\partial}{\partial k}\left[k^6 E_k \frac{\partial}{\partial k}(k^{-2}E_k)\right], \quad (36)$$

where $B(1) \approx 1$. The indeterminates in functions B(N) and B'(N) are due to the averaging with respect to the directions of the vector **k** and with respect to the mode of the waves, as well as to the use of dimensionality theory.

The total rate of change of the spectral energy density of the waves is determined by the adiabatic, local, and viscous terms:

$$\frac{\partial E_k}{\partial t} = \left(\frac{\partial E_k}{\partial t}\right)_{ad} + \left(\frac{\partial E_k}{\partial t}\right)_{loc} + \left(\frac{\partial E_k}{\partial t}\right)_{vis}. \quad (37)$$

Arriving at the viscous term does not entail any severe difficulties, since that term is linear with respect to $E_k$. The dissipation of the kinetic and magnetic energies is usually treated separately [7]. However, since we are dealing with turbulence as with a system of slowly (compared to the periods of the waves) varying waves, it is sufficient to discuss only the viscous dissipation of the total energy $E_k$. Summing the dissipation of the kinetic and magnetic energies, we have

$$\left(\frac{\partial E_k}{\partial t}\right)_{vis} = -(\nu + \alpha\nu_m)\,k^2 E_k. \quad (38)$$

Here $\nu$ and $\nu_m$ are the conventional and the magnetic viscosity, respectively, $\alpha < 1$ is a coefficient introduced to comply with the fact that the potential energy of the wave is built up not only by the magnetic field, but also by the gas pressure field.

Finally, by using Eqs. (37), (23), (36), and (38), we obtain

$$\frac{\partial E_k}{\partial t} = B(N)\,u_0^{-1} \frac{\partial}{\partial k}\left[k^6 E_k \frac{\partial}{\partial k}(k^{-2}E_k)\right] - (\nu + \alpha\nu_m)\,k^2 E_k. \quad (39)$$

This equation is valid in describing turbulence in a strong magnetic field under stationary external conditions, and then in that region of the spectrum where the characteristic time of deformation of the waves is much longer than the periods of the waves. Clearly, from the derivation of the equation, this idea is applicable to any random wave motions of weak nonlinearity (e.g., in a solid).

## 4. Stationary Hydromagnetic Turbulence

The simplest problem is the case of stationary turbulence in a constant (on the average) magnetic field $|H_0|$ characterized by the modulus $k_0$ of the wave vector. Usually, the acoustic speed is close to the Alfvén velocity, i.e.,

$$N \approx 1, \quad \frac{|H_0|}{\sqrt{4\pi\rho}} \approx u_{ac} \approx u_0.$$

We begin by finding the spectrum analogous to the Kolmogorov spectrum, without relying on the local interaction hypothesis. From the hypothesis dealing with a "paired" interaction, we find

$$S_k(k) = \int_0^\infty\int_0^\infty K(u_0, k, \xi, \eta)\,E_k(\xi)\,E_k(\eta)\,d\xi\,d\eta. \quad (40)$$

Here $S_k(k)$ is the flow of energy through a hierarchy of eddies, $K(u_0, k, \xi, \eta)$ is the kernel of an integral equation describing the contribution to the flux $S_k(k)$ ascribable to interaction between waves lying in the $\xi$ and $\eta$ regions of the spectrum.

Let us introduce the dimensionless variables x and y:

$$x = \frac{\xi}{k}, \quad y = \frac{\eta}{k}. \quad (41)$$

Then

$$S_k(k) = \int_0^\infty\int_0^\infty k^2 K(u_0, k, x, y)\,E_k(kx)\,E_k(ky)\,dx\,dy. \quad (42)$$

Hence, the dimensionality $[K] = [S_k E_k^{-2} k^{-2}] = [ku_0^{-1}]$, and hence

$$K = ku_0^{-1} f(x, y), \tag{43}$$

where $f(x, y)$ is a dimensionless function. Consequently

$$S_k(k) = k^3 u_0^{-1} \int_0^\infty \int_0^\infty f(x, y) E_k(kx) E_k(ky)\, dx dy. \tag{44}$$

In the stationary turbulence case

$$\frac{\partial S_k}{\partial k} = 0, \quad S_k(k) \equiv -\varepsilon. \tag{45}$$

From (44) and (45), we infer

$$E_k(kx) E_k(ky) \sim k^{-3}$$

or $$E_k = E_k(k, u_0, \varepsilon) = k^{-3/2} g(u_0, \varepsilon).$$

From dimensionality considerations, we have

$$E_k \sim \varepsilon^{1/2} u_0^{1/2} k^{-3/2}. \tag{46}$$

The relationship between $u_0$, $\varepsilon$, and the quantity $k_0$ determining the maximum turbulence scale is found from dimensionality considerations:

$$k_0 = k_0(u_0, \varepsilon) \sim \varepsilon u_0^{-3}. \tag{47}$$

We shall now determine the direct effect exerted by the viscosity. From Eq. (39), we have

$$Bu_0^{-1} \frac{d}{dk}\left[k^6 E_k \frac{d}{dk}(k^{-2}E_k)\right] - (\nu + \alpha\nu_m) k^2 E_k = 0. \tag{48}$$

Now we proceed to apply the method of perturbations. For the unperturbed solution we take the spectrum (46)

$$E_k^0 = \sqrt{\frac{2}{7B}}\, \varepsilon^{1/2} u_0^{1/2} k^{-3/2}. \tag{49}$$

The solution of Eq. (48) now takes on the form

$$E_k = E_k^0 + \delta E_k$$

$$= \sqrt{\frac{2}{7B}}\, \varepsilon^{1/2} u_0^{1/2} k^{-3/2} - \frac{4u_0}{33B}(\nu + \alpha\nu_m). \tag{50}$$

This is of course useful only in that region of the spectrum where the first term is much larger than the second.

In concluding, the author would like to express his gratitude to S. A. Kaplan and particularly to S. B. Pikel'ner for their kind and helpful discussion of the problem at hand.

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