50 MAGNETIC FIELD ANNIHILATION*

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Sweet's mechanism for the rate of annihilation of the magnetic field at the boundary between two regions of plasma containing oppositely directed field lines is re-examined. It is pointed out that previous analyses overlooked standing magneto-hydrodynamic waves as a possible mechanism for converting magnetic energy to plasma energy. An estimate of the annihilation rate including such waves is made. Using this rate, it is found that the energy required for a flare can be released in 10^2 sec. This time is short enough to account for the observed solar flare times if the source of the flare energy is stored magnetic energy.

One of the principal objections which has been raised to the suggestion that solar flares result from the rapid release of magnetically stored energy has been that the rate at which magnetic energy can be released is too slow. The model of the boundary between regions in which oppositely directed field lines exist, which was originally suggested by Sweet (References 1 and 2) and evaluated quantitatively by Parker (Reference 3), leads to times for the release of energy which are too large by a factor of 10 to 100. This is true even when it is assumed that the gas remains partially ionized and the resulting decrease in the effective conductivity due to ambipolar diffusion is included. It has been pointed out by Jaggi (Reference 4) that on the basis of the resistive instability analysis of Furth, Killeen, and Rosenbluth (Reference 5) the boundary in the Parker-Sweet model would be unstable. It is thus clear that the Parker-Sweet annihilation rates underestimate the actual rate of magnetic field annihilation. However, the linearized instability analysis does not lead immediately to an estimate of what the actual rate would be.

The existence of this instability leads to the suggestion that the flow becomes turbulent and that the increased annihilation rate results from the increased dissipation due to turbulence (References 6 and 7). The purpose of the present paper is to point to another possibility. This alternate possibility is based on the fact that Parker's analysis has overlooked a significant mechanism for the dissipation of magnetic field energy. If this mechanism is also included, it is possible to construct a steady flow configuration with much faster annihilation rates. Parker's solution should be regarded as a special case in a series of possible solutions. The existence of the instability of this particular solution should probably be considered largely as an indication that the particular solution underestimates the actual rate at which annihilation will occur.

To illustrate the mechanisms by which magnetic energy can be converted into plasma energy, let us consider for the moment a completely one-dimensional time-dependent situation in an incompressible fluid in which two regions with oppositely directed magnetic field lines are placed in contact at zero time (the actual situation to be discussed in the body of this paper will not be one-dimensional or time-dependent). In the present case, two mechanisms exist for the annihilation of magnetic field energy: (1) dissipation due to to the finite conductivity of the plasma which may be regarded as diffusion of the magnetic field, and (2) annihilation of the magnetic energy by the propagation of an Alfvén wave.

Figure 50-1(a) illustrates the diffusion case. The thickness of the region of reduced magnetic field is given by the ordinary skin depth formula. If, on the other hand, there is a component of magnetic field normal to the boundary, waves will travel outward in both directions and give

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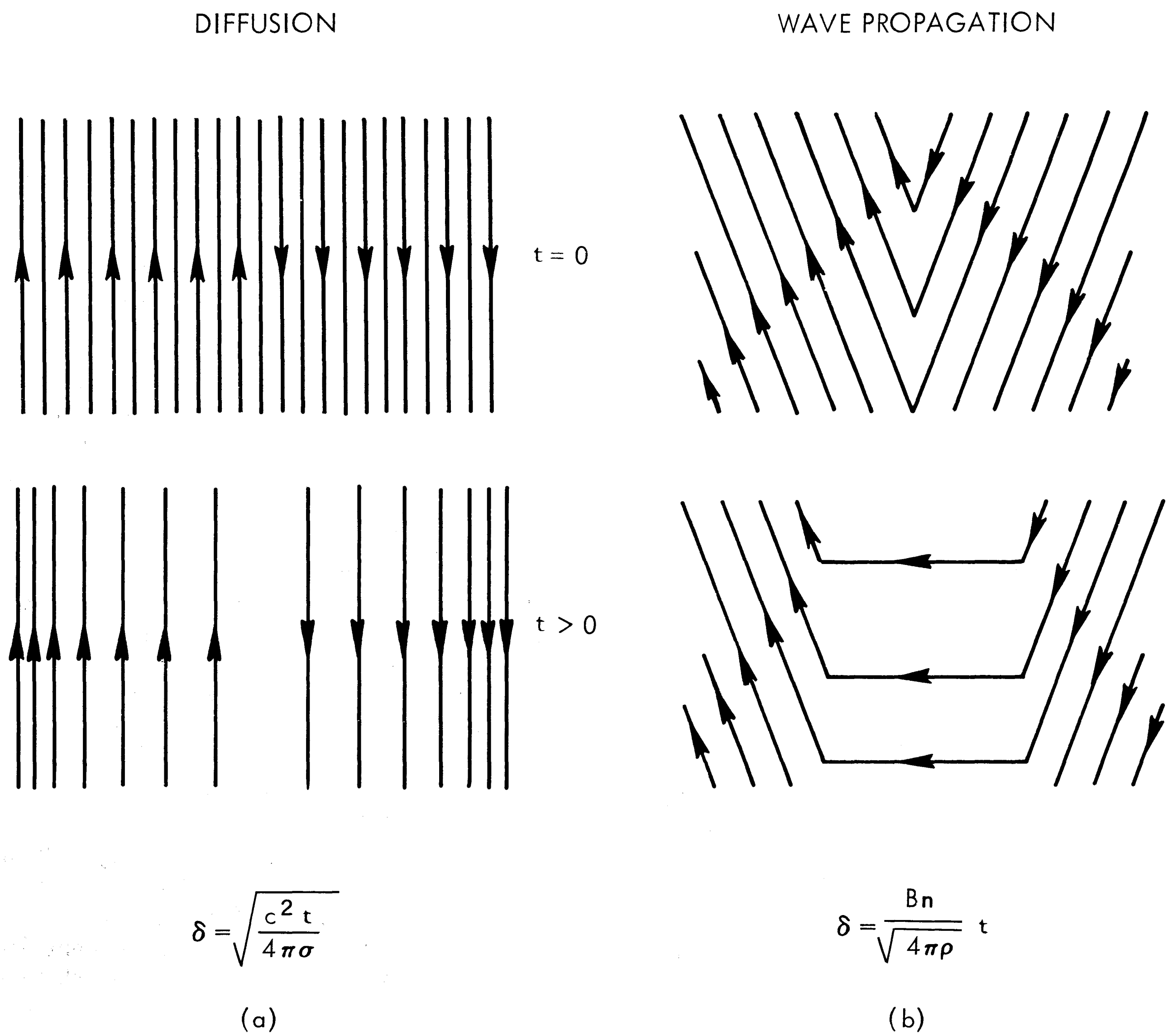

FIGURE 50–1. Illustration of the conversion of magnetic to plasma energy by: (a) diffusion, and (b) wave propagation. The initial magnetic field configuration and the configuration at a later time are sketched for a hypothetical one-dimensional time-dependent situation.

rise to a magnetic field configuration as shown in Fig. 50–1(b). The waves, or bends in the magnetic field lines, propagate at a constant velocity equal to the Alfvén speed. It is apparent from the spacing between field lines that the magnetic energy is considerably reduced between the waves. This energy has been converted into directed kinetic energy of the plasma which is moving upward. Since the Alfvén speed is independent of conductivity, the rate at which magnetic energy is converted to plasma energy is in this case independent of the conductivity. Since the diffusion rate decreases with increasing conductivity, we may expect the wave propagation mechanism to be important at high conductivities. We may also note that, since the diffusion velocity due to finite conductivity decreases as the square root of time, diffusion would be the faster and therefore the dominant effect at early times. However, at later times the wave propagation velocity will be faster, and therefore this effect should predominate. In this paper we will show that, in the steady flow situation, both the diffusion and the wave propagation mechanism are important and that this leads to a much more rapid rate of annihilation and reconnection of field lines in a high conductivity medium than is obtained when the wave propagation mechanism is overlooked.

Since the analysis to be given in this paper is in many respects similar to Parker's, we will review his analysis briefly in the next section for the case of incompressible flow. In the section titled "Incompressible" we will present the analysis including the effects of wave propagation again for the incompressible case. Since incompressible corresponds to high gas pressure as compared with magnetic pressure, the compressible case is of more interest for solar flares. Thus, in the section "Compressible" the analysis will be extended to the case of a compressible fluid. In the last section, "Application to Solar Flares," we will discuss briefly the application of this analysis to solar flares.

DIFFUSION MODEL

It was recognized by Sweet that the strictly one-dimensional situation illustrated in Figure 50–1(a) is not applicable, since the fluid will be ejected outward along the boundary. The region in which the magnetic field goes through zero must have a higher pressure than the surrounding medium by an amount equal to the magnetic pressure. If we imagine that the boundary has a finite length determined by the general dimensions of the flow field which is being considered and that beyond this length the pressure is reduced to the ambient pressure, then the pressure gradient along the boundary will accelerate fluid outward. As this fluid moves outward, the distance between oppositely directed field lines decreases and thus a more rapid rate of dissipation is obtained than in the purely one-dimensional time-dependent case.

Parker recognized that this effect could lead to the steady flow situation sketched in Figure 50–2. The fluid moves toward the boundary from both sides and is ejected along the boundary. The magnetic field lines are brought toward the boundary by the fluid. Within the boundary an x-type neutral point exists at which the field lines are reconnected by finite conductivity and the magnetic energy is dissipated.

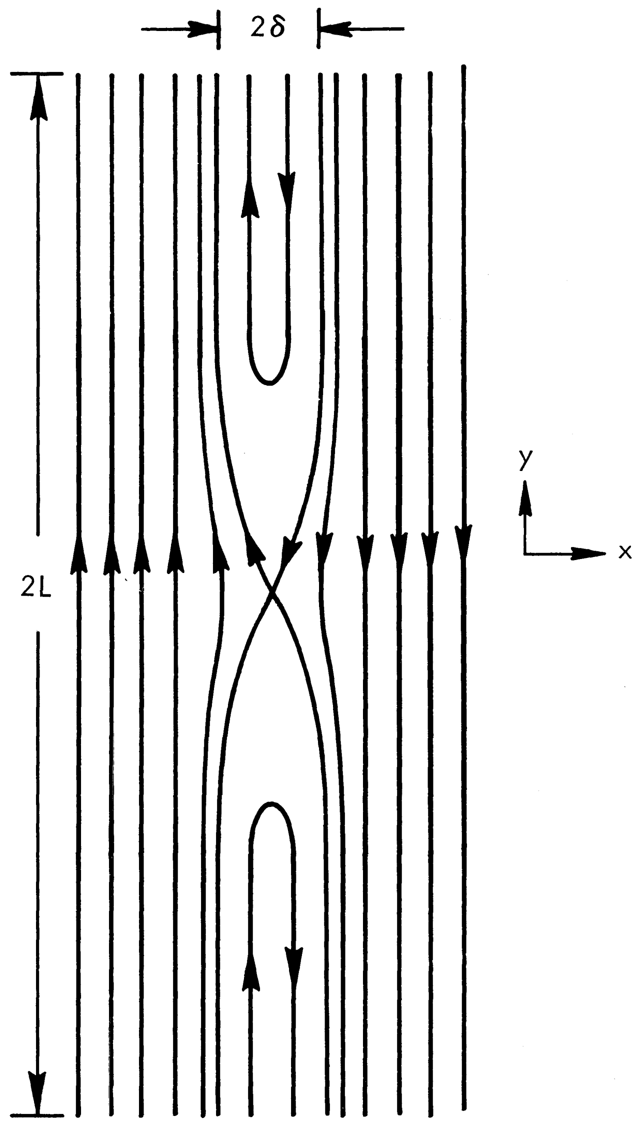

FIGURE 50–2. Magnetic field configuration for Parker's analysis of the Sweet mechanism. The fluid moves toward the boundary from both sides and is ejected along it.

For a steady flow, the rate at which fluid moves into the boundary is balanced by the rate at which fluid is expelled along the boundary. Thus,

$$u_{xo}L=v\delta, \quad (1)$$

where u_{xo} is the velocity of the fluid toward the boundary, v is the flow velocity along the boundary, and $2L$ and 2δ are respectively the length and thickness of the boundary. The velocity along the boundary was obtained from Bernoulli's law:

$$\frac{\rho v^2}{2}=p-p_0, \qquad (2)$$

where ρ is the density, p is the pressure in the middle of the boundary, and p_o is the gas pressure at large distances from the boundary. We may note that this relation neglects the magnetic forces which, as can be seen from the sketch of the magnetic field pattern, also tend to pull the fluid out along the boundary. Actually, the magnetic and pressure forces are comparable, so that there is an error of the order of a factor of 2 in v^2. The pressure balance across the boundary is essentially a hydrostatic condition, since the flow velocity into the boundary is expected to be very small. Thus,

$$p-p_0=\frac{B_{yo}{}^2}{8\pi}, \qquad (3)$$

where B_{yo} is the field outside of the boundary.

Since both the magnetic field and the velocity are small in the neighborhood of the neutral point, Ohm's law may be written as

$$j_z=\sigma E_z, \qquad (4)$$

where j_z and E_z are the current density and electric field and σ is the electrical conductivity assumed to be a scalar. For a steady flow $\nabla\times E=0$, and therefore E_z is a constant throughout the flow field. The constant may be evaluated from the fact that outside the boundary layer the current density is zero, and therefore

$$E_z=-\frac{u_{xo}B_{yo}}{c}. \qquad (5)$$

Since the change in magnetic field across the boundary must be equal to $2B_{yo}$, the total current in the boundary layer is specified; and

$$\frac{4\pi j_z}{c}\,(2\delta)=2B_{yo}. \qquad (6)$$

Equations 4 through 6 may be combined in the form

$$u_{xo}=\frac{c^2}{4\pi\sigma\delta}. \qquad (7)$$

For future reference we may note that in this form the equation can be interpreted as the statement that the flow velocity into the boundary layer u_{xo} must be equal to the diffusion velocity of the magnetic field relative to the fluid as given by the right-hand side. In other words, in order to maintain a steady state, the rate at which the magnetic field diffuses relative to the fluid must just equal the rate at which it is blown back by the fluid velocity.

We may now combine Equations 1 through 3 and 7 to obtain a complete description of the basic parameters of the flow configuration. In particular, if we solve for the velocity u_{xo} at which the fluid and magnetic field lines approach the boundary, we obtain

$$u_{xo}=\left(\frac{B_{yo}}{\sqrt{4\pi\rho}}\,\frac{c^2}{4\pi\sigma L}\right)^{1/2}. \qquad (8)$$

If we define the Alfvén speed in terms of the magnitude of the field away from the boundary,

$$V_A=\frac{B_{yo}}{\sqrt{4\pi\rho}}, \qquad (9)$$

and introduce the nondimensional flow velocity

$$M_o=\frac{u_{xo}}{V_A} \qquad (10)$$

and also a magnetic Reynolds number based on Alfvén speed as

$$R_m=\frac{4\pi\sigma V_A L}{c^2}, \qquad (11)$$

Equation 8 may be rewritten as

$$M_o=R_m{}^{-1/2}, \qquad (12)$$

which expresses the rate of annihilation in nondimensional terms. We may note that, since in most astronomical applications the magnetic Reynolds number is very large, the above relation leads to a very slow rate at which field lines approach each other and annihilate.

INCOMPRESSIBLE

If we now modify the picture to take into account the effects of wave propagation, we expect a magnetic field pattern as shown in Figure 50–3. The general topology of this field

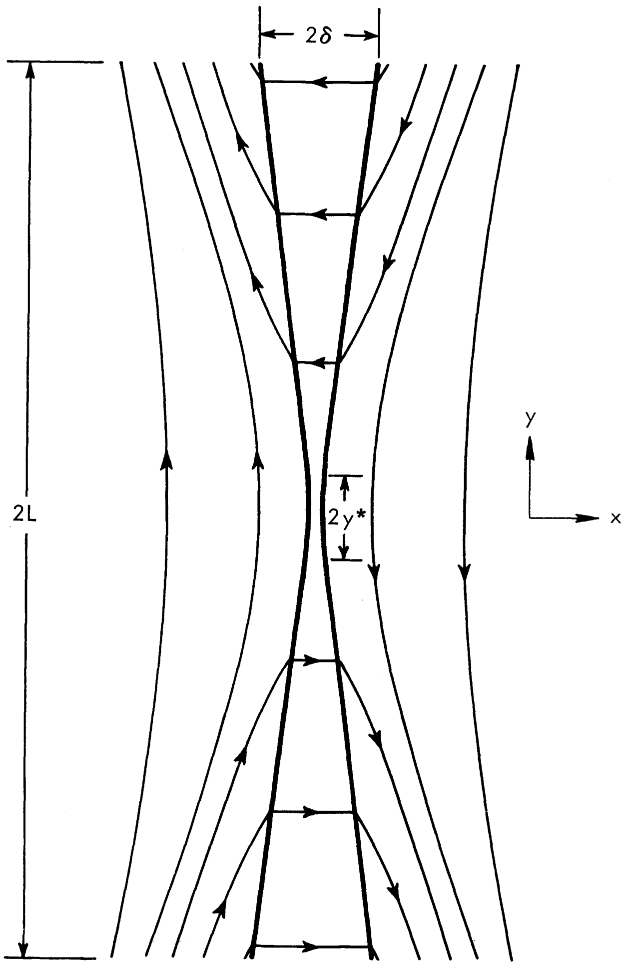

FIGURE 50–3. Flow configuration including standing waves. The magnetic field is indicated by light lines. The heavy lines indicate the edge of the boundary layer. For $y > y^*$, the edge of the boundary layer is determined by magneto-hydrodynamic waves and is therefore rather sharply defined. The fluid again moves toward the boundary and is ejected along it.

configuration is the same as in the previous section in that it is consistent with an x-type neutral point. In the neighborhood of the origin, the normal component of magnetic field must be zero by the symmetry of the problem. Since the propagation speed of waves is proportional to the normal component of magnetic field, the wave propagation speed must vanish in this neighborhood. Locally, the flow is therefore controlled by diffusion and will be quite similar to the flow discussed in the previous section. However, at some distance from the neutral point the normal component of magnetic field can be large enough so that wave propagation becomes faster than diffusion. At large distances from neutral point, we therefore expect that waves will be predominant. In this region the field lines bend sharply at the wave as opposed to turning gradually throughout the boundary layer.

It will be convenient to subdivide the flow into two regions which we will refer to as the *boundary layer* and the *external flow*. The *boundary layer* may be defined as the region near the boundary in which the magnetic field has an appreciably different magnitude than it has far from the boundary. Within the boundary layer the flow velocities will be high, of the order of V_A. In the *external flow* the changes in magnetic field are by definition small. Conditions are, however, not completely uniform in this region. As is illustrated in Figure 50–3, the presence of a normal component of the magnetic field at the boundary requires the magnetic field lines in the external flow to be bowed toward the boundary. The external flow may be regarded as being distorted because the plasma is pushed out along the boundary. In the analysis which follows, we will find it necessary to consider both the external flow field and the boundary layer. In fact, we will see that the distortion of the external flow by the boundary layer will ultimately limit the rate of field annihilation.

Before proceeding with the analysis, we may point out that—whereas the analysis in the previous section yielded only a single value for the rate of approach of the field lines—in the present analysis we will find that consistent solutions exist for a range of values. This would seem to be more reasonable, since one can imagine that the rate of flow could be controlled by a throttle placed elsewhere in the flow field (for example, motions of the *feet* of the magnetic field lines in the solar surface). Thus, all flow velocities up to a maximum one, allowed by the rate of field cutting, should be possible. In proceeding with the analysis, we will therefore assume that the flow velocity at infinity has a predetermined value which will appear as a parameter when describing the flow field. The actual value of this flow velocity which would occur in situations where the flow rate is controlled only by the allowed rate of field cutting will then be the maximum value of the parameter for which a consistent solution can be obtained.

In view of the above discussion, we may subdivide the analysis into three parts: (1) the boundary layer, (2) the external flow field, and (3) the evaluation of the maximum flow velocity for which a consistent flow pattern exists.

Boundary Layer

Since in the limit of infinite conductivity the boundary layer reduces to a single line, we would expect that for large conductivities the boundary layer will still be thin. In the analysis that follows, we will assume this to be the case.

It was pointed out just above that the external flow field will be slightly distorted. In considering the boundary layer itself, we will assume that to lowest order the flow at the edge of the boundary layer is the same as the flow at infinity. The validity of this assumption can be checked after we have examined the external flow field.

Within the above approximations, the requirement of conservation of mass leads to the relation

$$u_{xo}y = v\delta, \tag{13}$$

where both the flow velocity along the boundary layer v and the boundary layer thickness δ are functions of y.

The momentum equation in the y-direction may be written as

$$\frac{d}{dy}(\rho v^2 \delta) = -\frac{B_{yo}B_x}{4\pi}, \tag{14}$$

where B_x is the x-component of the magnetic field within the boundary layer and is also a function of y. The above equation has equated the rate of change of momentum flux within the boundary layer to the magnetic forces. The pressure gradient in the y-direction has been neglected. This would correspond to a situation in which the pressure is maintained constant along the boundary and all of the pressure drop which was used in discussing the diffusion model occurs at the end of the boundary. Actually the pressure gradient is probably a linear function of y whose magnitude becomes comparable to the magnetic force near the end of the boundary, $y \approx L$. It is therefore reasonably small for small values of y, and the net effect of the pressure gradient is probably to reduce the effective length of the boundary. Since the annihilation rate is insensitive to the length of the boundary, the error introduced is quite small. It is convenient to combine Equations 13 and 14 in the form

$$M_o^2 \frac{d}{dy}\left(\frac{y^2}{\delta}\right) = -b_x, \tag{15}$$

where we have introduced some of the notations from the previous section and where $b_x = B_x/B_{yo}$.

At appreciable distances from the neutral point we expect wave propagation to be important. To maintain a steady flow, the speed at which the wave propagates relative to the fluid must equal the rate at which it is blown back by the flow. Equating the flow velocity into the boundary layer with the wave propagation speed yields

$$M_o = |b_x|. \tag{16}$$

It is important to remember that the wave propagation speed depends only on the normal component of magnetic field and not on its magnitude. The absolute value of b_x is required in the above equation, since a wave can propagate in either direction along the magnetic field. This equation requires that, within the approximation of only small distortions of the external flow, the absolute value of b_x be a constant along the boundary layer. Substituting this result into Equation 15 gives

$$\delta = M_o|y|, \tag{17}$$

or the thickness of the boundary increases linearly with y.

Since the x-type neutral point requires that b_x be an odd function of y, the constancy of the absolute value of b_x implies a discontinuous jump of b_x at the neutral point. This is, of course, unreasonable and indicates that near the neutral point diffusion—and not wave propagation—must be considered.

For the immediate vicinity of the neutral point we will therefore return to essentially the same situation as discussed in the previous section. Equating the flow velocity with the diffusion velocity, we obtain

$$M_o=\frac{c^2}{4\pi\sigma V_A\delta}, \tag{18}$$

which replaces Equation 16 for the neighborhood of the neutral point and may be regarded as specifying the thickness of the boundary layer at the origin.* From the symmetry of the flow pattern, the boundary layer thickness must be an even function of y and therefore constant in the neighborhood of the origin, thus substituting the above value of δ into Equation 15. We obtain

$$b_x=-\frac{M_o^3 8\pi\sigma V_A}{c^2}y, \tag{19}$$

which states that the neighborhood of the neutral point b_x varies linearly with y.

To estimate the extent of the diffusion region, we may ask when the value of b_x, as obtained from its linear slope at the origin, becomes equal to the value required in the wave-dominated region. The length of the diffusion region is therefore given by

$$y^*=\frac{c^2}{8\pi\sigma V_A M_o^2}. \tag{20}$$

Roughly speaking, the boundary is diffusion-dominated for $y<y^*$ and Equations 18 and 19 apply; and it is wave-dominated for $y>y^*$ and Equations 16 and 17 apply. We may note that in terms of the length y^* the flow in the diffusion region is quite similar to the flow calculated in the previous section; that is, apart from a factor of 2, Equations 20 and 12 are identical if y^* and L are interchanged. (This difference is probably related to the use of the magnetic force rather than the pressure gradient.)

The principal effect of including the wave propagation mechanism is to reduce the length over which the diffusion mechanism must operate. Therefore, if the flow velocity is higher than the value given by Equation 12, based on the length of the entire boundary, wave propagation becomes important and the length of the diffusion region is reduced to the value required to accommodate this much flow.

External Flow Field

To analyze the external flow field quantitatively, we will first show that both the magnetic field and the flow field can be solutions of Laplace's equation in this region—in other words, that the flow is irrotational and the magnetic field is current-free. Since we are dealing with a high conductivity medium and are outside of the region of high current concentration (boundary layer), we may neglect j/σ in Ohm's law and obtain the usual frozen flux relation:

$$\mathbf{E}+\frac{\mathbf{u}\times\mathbf{B}}{c}=0. \tag{21}$$

As was mentioned earlier we will make the assumption, to be checked *a posteriori*, that the flow in this region departs only slightly from a uniform flow. Thus we may write

$$\begin{aligned}\mathbf{u}&=u_{xo}(\mathbf{i}+\mathbf{u}'),\\ \mathbf{B}&=B_{yo}(\mathbf{j}+\mathbf{B}'),\end{aligned} \tag{22}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions and $\mathbf{u}'$ and $\mathbf{B}'$ are small compared with unity. If we now take the curl of Equation 21 twice, remembering that $\nabla\times E=0$ and using Equation 22 to first order in the primed quantities, we obtain

$$\frac{\partial}{\partial x}(\nabla\times\mathbf{B}')=-\frac{\partial}{\partial y}(\nabla\times\mathbf{u}'). \tag{23}$$

This particular equation is obviously satisfied if both $\mathbf{B}'$ and $\mathbf{u}'$ are curl-free. We must still show that this is consistent with the remaining magneto hydrodynamic equations. For

*We have tacitly assumed that the only dissipation mechanism present is due to finite electrical conductivity. Actually, other mechanisms such as viscosity may also be present. If the kinematic viscosity is larger than $\frac{c^2}{4\pi\sigma}$ viscosity would be the controlling diffusion mechanism at the origin and $\frac{c^2}{4\pi\sigma}$ should be replaced by the kinematic viscosity. The formulas in the remainder of the paper may be regarded as applying directly if the electrical conductivity gives rise to the fastest diffusion. In other cases, the formulas can probably also be applied if $\frac{c^2}{4\pi\sigma}$ is reinterpreted as the appropriate diffusion coefficient.

$\nabla \times \mathbf{B}'=0$ there are no magnetic forces; and therefore the fluid flow equations and the magnetic field equations are decoupled. For the ordinary fluid equations in an incompressible, inviscid two-dimensional flow it is well known that vorticity ($\nabla \times \mathbf{u}'$) is conserved. Therefore, since the vorticity is zero at infinity, it will remain zero throughout the flow field external to the boundary layer. It is therefore consistent with the complete set of equations to assume that both $\mathbf{B}'$ and $\mathbf{u}'$ are curl-free.

If we returned to Equation 21, we would find that this specifies a relation between the potential for $\mathbf{B}'$ and the one for $\mathbf{u}'$. In the present discussion we will consider only the calculation of $\mathbf{B}'$. In the boundary layer analysis Equations 16 and 19 specified the x-component of magnetic field along the y-axis in terms of conditions far from the boundary. This in turn determines $\mathbf{B}'$ in the external flow. Alternatively we may say that, since there are no currents in the external flow, $\mathbf{B}'$ must be related to a change in the current distribution in the boundary. Instead of a boundary layer current per unit length which is independent of y, the boundary layer current will be somewhat reduced near $y=0$ to produce the required x-component of magnetic field at larger values of y. This change in boundary layer current will be small in the same sense that $\mathbf{B}'$ is small. (In order to satisfy the velocity potential required by Equation 21, the boundary layer thickness and velocity would have to depart slightly from the values calculated in the discussion titled "Boundary Layer.")

Since we consider the boundary layer to be thin, we may regard the boundary layer analysis as specifying B_x' along the y-axis. Since $\mathbf{B}'$ satisfies Laplace's equation, this is sufficient to determine B′ everywhere. In order to determine B_x' immediately outside the boundary layer, we note that across the edge of the boundary layer the normal component of magnetic field is conserved; therefore,

$$B_x'+\frac{d\delta}{dy}=b_x. \qquad (24)$$

The term $d\delta/dy$ arises from the fact that the zero order magnetic field has a component normal to the edge of the boundary layer unless $d\delta/dy=0$. In the wave-dominated region ($y>y^*$) Equations 16 and 17 specify b_x and $d\delta/dy$ as having equal magnitudes M_o; and we obtain

$$B_x'=-2M_o\frac{y}{|y|}, \qquad \text{for } y>y^*. \qquad (25)$$

In the diffusion-dominated region b_x varies linearly with y. Although $d\delta/dy$ was not specified in this region, it goes from zero at $y=0$ to a value equal to b_x at $y=y^*$. Therefore, it cannot produce a serious error to take $d\delta/dy$ as equal to b_x within the diffusion region also. Therefore,

$$B_x'=-2M_o\frac{y}{y^*}, \qquad \text{for } y<y^*. \qquad (26)$$

To determine $\mathbf{B}'$, we may now regard the B_x' as specifying a distribution of magnetic sources along the y-axis; and therefore

$$\mathbf{B}'(\mathbf{r})=\frac{1}{\pi}\int_{-L}^{L}\frac{B_x'(\eta)(\mathbf{r}-\boldsymbol{\eta})}{|\mathbf{r}-\boldsymbol{\eta}|^2}\,d\eta, \qquad (27)$$

where $\mathbf{B}_x'$ is given by Equations 25 and 26, $\mathbf{r}$ is the position vector, and η is a dummy integration variable along the boundary. (This formula gives rise to logarithmically divergent values of $\mathbf{B}'$ in the vicinity of the boundary for $r=L$. This is probably not significant, since L is intended only as a measure of the scale of the overall flow field. A more detailed consideration of the flow at very large distances would presumably eliminate this singularity.)

Equation 27 specifies completely the magnetic field in the external flow in terms of the flow velocity at infinity M_o. Coupled with the relations of the discussion titled "Boundary Layer," which define the boundary layer, we therefore have a complete description of the flow in terms of M_o.

In carrying out the analysis, we linearized the external flow. The justification of this assumption depends on $\mathbf{B}'$ being small compared with unity. The largest value of $\mathbf{B}'$ occurs just outside the boundary layer at the origin. Using Equations 25 and 26 to evaluate the integral in Equation 27 at this point, we obtain

$$B_y'(0)=-\frac{2M_0}{\pi}\,\ln\left(\frac{L}{y^*}\right), \qquad (28)$$

where we have assumed that the logarithm is large compared with unity. The linearization is therefore valid for M_0 sufficiently small.

Maximum Annihilation Rate

The results of "Boundary Layer" and "External Flow Field" state that a consistent solution of the flow field can be obtained for all values of M_0, provided that $B'(0)$ as given by Equation 28 remains small compared with unity. It remains to be seen whether or not larger flow velocities are possible if nonlinear terms in the external flow field are considered. This has not been done in any rigorous sense; however, a rough argument can be given to show that these nonlinear terms would not allow larger flow velocities.

Let us first recapitulate the procedure we have followed thus far. The boundary layer was analyzed assuming that the external flow was completely uniform. This analysis showed that, for large enough flow velocities (such that $y^* < L$), an appreciable x-component of magnetic field was required at the boundary at some distance away from the neutral point. To produce this field, the current in the boundary layer (and what is equivalent, the y-component of magnetic field just outside the boundary layer) had to be reduced at small values of y.

To estimate the effect of the nonlinear terms, we will examine the changes in the boundary layer properties required by the nonuniformity of the external flow. We will find that these changes in turn tend to increase rather than decrease the required nonuniformity.

In the wave-dominated region the decrease in B_y at the edge of the boundary layer leads to a decrease in the flow velocity v along the boundary. The continuity equation therefore leads to a larger value of δ than is given by Equation 17. From Equation 24 this implies that B_x' must be larger than previously and hence that the decrease in boundary layer current near the origin must be larger. On the other hand, since at $y=0$ the flow velocity has increased and the magnetic field has decreased, the length of the diffusion region y^* must decrease. The change in current in the boundary layer therefore tends to be confined to a smaller region. These two facts combine to make the magnetic field just outside boundary layer go to zero more rapidly with increasing M_0 than the linear analysis would indicate.

An absolute limit exists when this field goes to zero, since at this point the boundary layer itself would vanish. The situation may therefore be summarized as follows: (1) solutions exist for $B_y'(0) \ll -1$; (2) $B_y'(0)$ approaches -1 more rapidly than the linear analysis would indicate; and (3) $B_y'(0)$ cannot exceed -1. To determine the maximum allowed flow velocity, we must estimate the largest magnitude of $B_y'(0)$ possible within the above restrictions. We will choose this value somewhat arbitrarily as corresponding to the case in which the linear analysis gives

$$B_y'(0) = -1/2. \tag{29}$$

Substituting this estimate in Equation 28 gives as the limiting flow velocity

$$M_{0_{\max}} = \frac{\pi}{4 \ln\left(\frac{L}{y^*}\right)}. \tag{30}$$

It should be pointed out that the uncertainty in $M_{o_{max}}$ is directly proportional to the uncertainty in the estimate used to obtain Equation 29. Equation 30 is, therefore, probably good to a factor of 2 but not much better.

Using Equation 20, we may rewrite Equation 30 in the form

$$\begin{aligned} M_{o_{max}} &= \frac{\pi}{4 \ln\left(\frac{8\pi\sigma V_A L}{c^2} M_{o_{max}}^2\right)} \\ &= \frac{\pi}{4 \ln\left(2 M_{o_{max}}^2 R_m\right)}. \end{aligned} \tag{31}$$

This equation gives an explicit relation which determines the maximum flow velocity in terms of the macroscopic flow parameters. It is a fortunate circumstance that this answer depends only logarithmically on the magnetic Reynold's number and is therefore rather insensitive to our ignorance of the effective conductivity of the plasma.

We note by comparing this result with Equation 12 that at large Reynolds numbers the allowed flow velocity is appreciably larger than is obtained from the diffusion analysis. Furthermore, since the allowed flow velocity or annihilation rate decreases only logarithmically with Reynolds numbers, rapid annihilation can be achieved even for extreme conductivities and length scale.

COMPRESSIBLE

The incompressible case discussed in the previous section corresponds to the situation in which the gas pressure is large compared with the magnetic pressure. The case which is of most interest for solar flares is one in which the gas pressure is small compared with the magnetic pressure. In this section we will describe the flow for arbitrary values of this pressure ratio.

Neither the general flow configuration nor the annihilation rate change appreciably as compressibility is considered. The compressible case, however, involves waves with whose properties the reader is probably less familiar. A relatively brief but fairly complete discussion of both small amplitude waves and shock waves in a compressible medium in the presence of a magnetic field has been given by Shercliff (Reference 8). For the present discussion, we will simply notice that the magnetic field configuration shown in Figure 51–3 corresponds to the waves being *switch-off* shock waves. A switch-off shock wave is defined as a shock wave behind which the magnetic field is normal to the wave front (the tangential component is switched off across the shock). From symmetry considerations the magnetic field must be purely in the x-direction between the waves. Since the waves make only a small angle with the y-axis, it is clear that this field configuration corresponds to a shock wave which is almost a switch-off shock.

Switch-off shock waves have the special property that they propagate at a speed equal to the Alfvén speed based on the normal component of the magnetic field. The propagation speed of these shocks is thus the same as it was for the Alfvén wave in the incompressible case. This fact is to a large extent responsible for the rates of annihilation being quite similar throughout the whole region of compressible cases.

The principal difference which does arise is that the density increases across a switch-off shock, and thus the density within the boundary layer, is larger than the density in the external flow. The switch-off shock wave also requires dissipation within the shock front; thus the temperature of the gas within the boundary layer, is also increased. The analysis of the compressible case requires keeping track of density changes, which necessitates including both the energy equation and the x-component of the pressure balance. Neither of these equations was required in our analysis of the incompressible case. A rough consideration of these equations will show that the density may be assumed constant inside the boundary layer, unless radiative cooling is important and occurs in a time scale comparable to the flow time along the boundary.

Since B_y just outside the boundary layer is constant to zero order, the pressure within the boundary layer is constant to the same order. The energy dissipated per unit mass is essentially the magnetic energy per unit mass outside the boundary layer and is therefore also independent of position along the boundary layer. If energy is conserved, this implies a constant temperature and, therefore, density along the boundary layer. If radiative cooling is important, the density would be further increased. If the cooling is such that the gas is cooled to a fixed temperature in a time short compared with the flow time, the density would again be constant. In the more general case, density variations would have to be taken into account.

For the present analysis we will assume a constant density in the boundary layer and define the density ratio α by

$$\alpha = \frac{\rho_0}{\rho_{bl}} \qquad (32)$$

where ρ_0 is the density far from the boundary layer and ρ_{bl} is the density within the boundary layer. In the incompressible limit α is, of course, equal to unity. As the ratio of gas to magnetic pressure decreases, α is always less than unity. For a perfect monatomic gas in the limit of zero pressure ahead of the wave,

$\alpha = 2/5$ if energy is conserved. For either more internal degrees of freedom or radiative cooling, α would be smaller.

If we assume that α is a known quantity, only minor modifications of the incompressible analysis are required. In what follows we will indicate the required changes. Corresponding equations will be given the same numbers primed.

Modifying the continuity and y-momentum equation to take into account the difference in density inside and outside the boundary layer, we obtain

$$\alpha u_{xo} y = v\delta, \qquad (13')$$

$$\frac{d}{dy}(\rho_o v^2 \delta) = -\alpha \frac{B_{yo} B_x}{4\pi}. \qquad (14')$$

Since the propagation speed of the switch-off shock wave is the Alfvén speed based on the normal component of magnetic field, Equation 16 is unchanged; however Equation 17 is modified to

$$\delta = \alpha M_o |y|. \qquad (17')$$

In the vicinity of the neutral point where diffusion is important, the diffusion velocity of the magnetic field depends on the actual thickness of the boundary layer and therefore Equation 18 is unchanged for the compressible case. However, the modifications of the continuity and momentum equations change Equation 19 to the form

$$b_x = -\alpha \frac{M_o^3 8\pi\sigma V_A}{c^2} y. \qquad (19')$$

Since b_x increases more slowly with y but must reach the same value at large distances from the neutral point, the length of the diffusion region is somewhat increased:

$$y^* = \frac{1}{\alpha} \frac{c^2}{8\pi\sigma V_A M_o^2}. \qquad (20')$$

In the discussion of the external flow in the incompressible case, we first showed that the magnetic field in this region was current-free and was therefore a solution of Laplace's equation. This conclusion is still valid in the compressible case.

If the ratio of gas pressure to magnetic pressure is of the order of unity or greater, the arguments given in the discussion "External Flow Field" apply directly—since the flow velocities described are then small compared with the ordinary sound speed in the gas, and the flow in this region is then essentially incompressible. If the gas pressure is very small compared with the magnetic pressure, the current must again be small since the gas has no inertia. From Equation 21 it follows that along the stagnation streamline the ratio of the change in dynamic pressure to the change in magnetic pressure is of the order of M_o^2. The change in magnetic pressure must therefore occur without forces, and hence the field is current-free to this order. We have thus shown that the magnetic field satisfies Laplace's equation at low gas pressures as well as at intermediate and high gas pressures. We will assume that it does so through the range of gas pressures. The quantitative evaluation of the external flow again follows straightforwardly; Equation 24 is unchanged except for the fact that δ has been changed. Because of this change in δ, Equations 25 and 26 become

$$B_x' = -(1+\alpha) M_o \frac{y}{|y|}, \text{ for } y > y^*, \qquad (25')$$

$$B_x' = -(1+\alpha) M_o \frac{y}{y^*}, \text{ for } y < y^*. \qquad (26')$$

Equation 27, which defines the magnetic field everywhere in the external flow, is unchanged except for the fact that B_x' has been modified. This modification leads to Equation 28 taking the form

$$B_y'(0) = -\frac{(1+\alpha) M_o}{\pi} \ln\left(\frac{L}{y^*}\right). \qquad (28')$$

The estimate of the maximum annihilation rate is still based on the same arguments: namely, that, if the change in magnetic field outside of the boundary layer but near the origin becomes too large, the rate of annihilation in the neighborhood of the neutral point will tend to limit itself. Using the same choice of the limiting values of change of magnetic field as given in Equation 29, we find that the maximum annihilation rate is defined by

$$M_{o_{max}}=\frac{\pi}{2(1+\alpha)\,\ln\left(\frac{8\pi\sigma V_A L}{c^2}M^2_{o_{max}}\alpha\right)}$$

$$=\frac{\pi}{2(1+\alpha)\,\ln\,(2M^2_{o_{max}}\alpha R_m)}. \qquad (31')$$

We may note that Equations 31 and 31′ differ only slightly. The predominant change is in the factor outside of the logarithm. Since the density ratio α is necessarily between 0 and 1, the difference over the whole range of compressibility is less than a factor of 2. In view of the accuracy of our estimate of the limiting velocity, this difference probably is not very significant.

APPLICATION TO SOLAR FLARES

The rate of annihilation of magnetic energy calculated in the previous section will now be applied to the conditions presumed to exist in solar flares. Parker has estimated that typical conditions might be a magnetic field strength of 500 gauss over a volume corresponding to a length scale of 10^4 km in a region in which the density is roughly 2×10^{11} particles/cm^3. To account for the energy for a flare, all of the magnetic energy within this volume must be converted to plasma energy in a time of the order of 10^2 to 10^3 seconds. From the rate at which field lines approach the boundary, the time to convert all of this energy can be estimated roughly as L/u_{xo}. In Table 50–1, Parker's results are compared with the results obtained from the present calculation using the above plasma conditions.

TABLE 50–1

Results From:	Conductivity Corresponding to:	Annihilation Time (sec)
Parker	$T=10^4$ °K	5×10^4
	Ambipolar diffusion....	6×10^3
Present	$T=10^6$ °K	10^2
Observations		10^2–10^3

The conductivity to be used in estimating these times is somewhat ambiguous. Parker has given cases corresponding to conductivity of a fully ionized plasma at 10^4 °K, as well as an estimate which he considers somewhat dubious in which the conductivity is reduced by ambipolar diffusion under the assumption that the gas is partially ionized. It will be seen that neither of these times is sufficiently rapid to account for the observations. The estimate using the present calculation has been based on a conductivity corresponding to a fully ionized gas at 10^6 °K. This is an overestimate of the actual conductivity and was done merely to illustrate that the annihilation rate is very insensitive to the conductivity and that times which are short enough to account for the observations can be achieved even with extreme assumptions about the electrical conductivity.

The fact that this conductivity is too high is illustrated by the fact that the thickness of the boundary layer in the vicinity of the neutral point as given by Equation 18 for the above conditions is 10^{-4} cm. This is unreasonable, since it is impossible to carry the required current in less than an electron gyro-radius. The electron gyro-radius for an electron having an energy equal to the magnetic energy per particle is about 1 cm. The finiteness of the electron gyro-radius would then decrease the effective electrical conductivity to the point where the boundary layer thickness at the neutral point is at least 1 cm. From Equation 18 this corresponds to a reduction by a factor of 10^4 in the conductivity. The corresponding reduction in annihilation time is only 30 percent. There are probably other effects which reduce the effective conductivity further. However, in view of the insensitivity of the annihilation time to the conductivity, it does not seem fruitful to pursue the subject further in this paper.

A more serious uncertainty in the annihilation time is probably caused by uncertainties in our knowledge of the magnetic field strength density and length scale of the region involved. No attempt will be made to evaluate the accuracy of these estimates. However, it is worth noticing from Equation 31′ that the annihilation time varies directly as the length scale involved, as the square root of the density, and inversely as the first power of the magnetic field.

In this paper, we have considered only the case in which the magnetic field directions on the two sides of a boundary differ by 180 degrees. This is, of course, a rather special condition which one would not expect to have satisfied in general. There is, however, nothing in the theory which would suggest that the result is particularly sensitive to the angle between the fields. Let us consider briefly the special case in which a magnetic field, perpendicular to the plane of the flow and of equal magnitude on both sides of the boundary, is added. For the case of incompressible flow, the analysis remains completely unchanged and the perpendicular component is uniform throughout the whole flow field. The annihilation rate in that case is therefore unaltered by the addition of this field. It should be remembered, however, that the magnetic energy associated with this field is, of course, not released and that throughout the paper we have used quantities defined in terms of the magnetic field in the plane of the flow. For the compressible case, the perpendicular component of magnetic field would tend to be increased within the boundary layer, thus effectively reducing the compressibility. However, since our result was relatively independent of compressibility, the addition of this magnetic field will probably not have a significant effect in the compressible case either.

In conclusion, the results of this analysis indicate that the rate at which magnetic energy can be converted into plasma energy is sufficiently rapid to account for the observed times in solar flares. The rate at which energy can be released is therefore not a valid criticism of the suggestion that solar flares result from the release of magnetically stored energy.

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DISCUSSION

Dr. Parker: This is a very interesting idea that Petschek has expressed. By introducing the magnetic field ejecting the fluid, this permits him to confine the diffusion to a very narrow strip about the neutral point. And this is—as I see it, at least—the reduction of the width of that diffusion strip which then gives the enormously enhanced diffusion rates, is that right?

Dr. Petschek: It is the reduction of the width and the reduction of the length which goes with it which allows the fields to go through faster there. But I think the important point is not that the magnetic field ejects the fluid but that, instead of having the change in the magnetic field diffuse into the fluid, it propagates as a wave whose propagation velocity is not limited by the conductivity of the medium.

Dr. Parker: The reason I put it the way I did, still, even in this model, is that the rate at which the fields come together is limited by the diffusion at the neutral point, and you have greatly enhanced that diffusion. The other way of looking at it is true, too. I am just trying to contrast what has been said before and what you are saying today.

Dr. Petschek: It is limited by the diffusion. However, the wave mechanism allows you to shrink the diffusion region; and, as the conductivity increases, that region shrinks farther and farther. So that the final answer does depend on conductivity, but only logarithmically. The annihilation rate is, therefore, very insensitive to conductivity.

Dr. Meyer: I must confess that I don't quite understand the difference between your picture and the general picture which Sweet and Parker have discussed.

It seems to me that, in the configuration which you draw, the waves do not travel outward but would travel inward until they are curved in such a way that they are just able to do what has been suggested earlier, namely, to pull the fluid out in the direction of the gap with Alfvén velocity.

This, of course, because of the large extension along the gap, limits the velocity of the flow approaching from the sides to a much slower value; and thereby, it seems to me, one comes back to the old picture.

I don't see how your waves travel outward with the Alfvén velocity, which is the point where your picture differs from the old one. It seems to me that, if you start with a configuration as you draw it, they travel inward; and the field assumes a curvature in the stationary case which accelerates the material from the gap outward until it flows with Alfvén velocity.

Dr. Petschek: The fluid inside the boundary is escaping along the boundary at the Alfvén velocity. I don't see at all why you expect the waves to propagate inward. One can set up a steady configuration where they are propagating outward relative to the fluid but are blown back at an equal velocity by the flow so that they stand still.

In the first example I gave, the time-dependent case, you start with a corner in the magnetic field and waves do travel outward from it.

Dr. Meyer: It seems to me the field configuration will have not as sharp a boundary but will be generally curved, and that was the reason I thought the kink has to wander inward One would have not a sharp kink but just a continuous curve.

Dr. Petschek: In drawing the sharp kink, I really put in some ideas from the compressible case. The wave corresponds to compression of the gas, as the density is higher inside the layer than outside. Therefore, the wave corresponds to a shock wave. It is what is called a slow shock in terms of magnetohydrodynamics. These shock waves do steepen to form a discontinuity, so that it would be a sharp edge in that case. The incompressible case is intended as a limit of the compressible case.

Dr. Sweet: I would like to make two points: One is that I am in favor of your theory, which I thoroughly approve. Dr. Parker and I have been living with this problem for several years and have got the feel of it. Your solution struck me at once as the solution for which we have been seeking.

Secondly, in actual application, a B_z field would be present. It is, therefore, important to know what effect this would have on the rate.

Dr. Petschek: Yes, as far as the B_z field goes, I haven't looked at it in detail, except for one special case mentioned in the written manuscript. In general, I don't see anything in the theory which would make it very sensitive to field orientation.

It was observed by Parker that in the diffusion case you get an appreciably faster rate for compressible, as opposed to incompressible, flow. However, the B_z field tends to make the flow less compressible, so that you lose the gain you had achieved by considering the compressible case. In the present analysis, however, there is not much difference between the compressible and incompressible cases. So that this effect would not be important.

Dr. Wentzel: Most stationary flow theories provide only one time scale, whereas a flare has a rise time and a decay time. It seems to me your model could be well modified to give a rise time equal to the time which it takes to set up your pattern of flow. This time is determined essentially by the hydromagnetic velocity and the size of the region.

Dr. Petschek: I haven't looked at the time it takes to set up this flow. I would imagine that as in most flows it would take a time to establish, which is the time it takes to flow through. In this case, I would take it as the time it takes to go along the boundary layer which, as you said, is essentially the length of the system divided by the Alfvén speed.

Dr. Wentzel: Secondly, I am surprised that the observational people here have not complained of flares which are 100 cm wide theoretically. Somehow one has to adjust a single dissipating plane to the observed, rather than large, flare volume.

Dr. Petschek: This fluid is ejected along the boundary, and where it goes from there I think is completely open. I would expect the gas to spread out and to see luminosity from a much larger region than the boundary. Incidentally, the boundary is only narrow near the neutral point. The standing waves spread at an angle of about one-tenth of a radian, so that the boundary thickness is quite large at the ends of the region.

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