1964ApJ...140.1482F

THE PERIOD-RADIUS RELATION FOR PULSATING VARIABLE STARS

J. D. Fernie
David Dunlap Observatory, University of Toronto
Received February 12, 1964; revised June 29, 1964

ABSTRACT

A collection of radius determinations by Wesselink's method and others for twenty-five pulsating variables is made from the literature. The majority of classes of pulsating variables are represented, and the range in period extends from DQ Herculis of period 1 minute to Mira of period almost 1 year. It is found that all these variables closely obey a relation of the form $P \propto R^2$. The variables fall clearly into two groups, those still in the pre-red-giant stage of evolution (Group I), and those in the post-red-giant stage of evolution (Group II). The latter group includes the RR Lyrae and W Virginis stars as well as old Population I stars such as the δ Scuti stars and DQ Herculis. In Group II the constant of proportionality in the $P \propto R^2$ relation is 10 ± 2 times as great as for Group I.

As a consequence of these findings it is shown that $P\sqrt{\rho} = Q$ cannot be a universal number, in agreement with Kraft's earlier suggestion, but increases with period and mass. The reason that classical Cepheids and long-period variables have nearly the same value of Q is explained.

Substitution of the period-radius relation into the $L = 4\pi R^2 \sigma T_e^4$ relation allows the prediction of detailed period-luminosity-color relations for all classes of pulsating variables for which the relations between bolometric correction, effective temperature, and color are known. These predicted relations are made for the classical Cepheids, β Cephei stars, δ Scuti stars, and RR Lyrae stars. Satisfactory agreement with observation is obtained in all cases. In particular, the predicted relation for classical Cepheids is tested on nine Cepheids with periods between 5 and 45 days, and the average residual between observed and predicted absolute magnitude is 0.1 mag., compared to 0 2 mag. obtained using Kraft's P-L-C law. Evidence is given that the ratio of total to selective absorption for Cepheids is more nearly 3.4 than 3.0.

Among the RR Lyrae stars the Bailey type-c variables obey the same P-L-C relation as types a and b. Further partial tests of absolute magnitude predictions are carried out on Mira and a W Virginis star in the globular cluster M5. Satisfactory agreement with observation is obtained.

I. INTRODUCTION

A brief glance at the positions of the various classes of pulsating variables on the H-R diagram establishes the well-known fact that there is a general increase in period in going from left to right across the diagram. There is, therefore, a rough inverse relation between period and surface temperature. Closer examination, however, reveals that the β Cephei and δ Scuti stars, while having very similar periods have dissimilar spectral types (early B and F, respectively). The lower luminosities of the δ Scuti stars, however, combined with their later spectral types, indicate that they must have radii similar to the β Cephei stars. Likewise, in comparing a W Virginis star and a classical Cepheid of equal period, it is found that the former tends to lie below and to the right in the H-R diagram of the latter, indicative of similar radii. Finally, the lines of constant period within the classical Cepheid domain of the H-R diagram are also sloped very nearly along lines of equal radius. There is therefore a strong suggestion that a much better correlation obtains between period and radius than between period and surface temperature.

A preliminary plot of radius against period for a few variables indicated that the correlation might be very good indeed, and furthermore that it probably is not restricted to individual classes of variables. The importance of such a relation is several-fold: it presumably offers a strong clue in the theory of stellar pulsation, especially since a wide variety of variables appear to obey it, while observationally it may be used in conjunction with the $L/L_\odot = (R/R_\odot)^2 (T_e/T_{e_\odot})^4$ relation either to predict absolute magnitudes or to examine the scales of bolometric corrections and effective temperatures. In view of this, I have searched the literature fairly extensively for independent radius determinations of variable stars, which are discussed in the next section.

II. THE DATA

Throughout the paper all radii, magnitudes, colors, etc., refer to their cyclic averages.

SV Vul.—Classical Cepheid, period 45.145^d. The radius has been determined by Sanford (1956) using Wesselink's method. Sanford gives twice the formal probable error of his result, and there is also a typographical error of a factor of 10 in his published error. $R = 142 \pm 5$ ⊙. The given error assumes that the light-curve is exact. In fact, however, the derived radius is very sensitive to small errors in the photometry, a point which should be borne in mind throughout the paper.

l Car.—Classical Cepheid, period 35.532^d. Rodgers (1957) has applied Wesselink's method and finds $R = 138$ ⊙. He gives no probable error, but from an examination of his paper I conclude that it is probably about ± 10 ⊙.

T Mon.—Classical Cepheid, period 27.018^d. Sanford (1956) applied Wesselink's method and finds $R = 119 \pm 5$ ⊙. His published probable error is twice the formal probable error given here.

β Dor.—Classical Cepheid, period 9.843^d. Rodgers (1957), applying Wesselink's method, finds $R = 105$ ⊙. No probable error is given, but it is probably about ± 10 ⊙.

TABLE 1

CLASSICAL CEPHEIDS IN GALACTIC CLUSTERS

Star	log P	M_{vis}	$(B-V)$	log $R/R_\odot$
EV Sct .	0 490	−2 58	0 61	1 427
CF Cas ...	687	−3 39	70	1 627
U Sgr .	.828	−3 87	65	1.703
DL Cas	.908	−3 79	77	1 738
S Nor	0 989	−3 94	0 80	1 782

η Aql.—Classical Cepheid, period 7.177^d. Stebbins (1953) by Wesselink's method obtained $R = 68 \pm 3$ ⊙; Whitney (1955) by Wesselink's method found 67 ± 4 ⊙, and by a photometric method 56 ± 2 ⊙. Oke (1961*a*), by a modification of Wesselink's method, found 63.6 ± 2.6 ⊙. Assigning weights inversely as the squares of the probable errors, the weighted mean is 61.4 ± 2.0 ⊙.

δ Cep.—Classical Cepheid, period 5.366^d. Stebbins (1953) by Wesselink's method obtained $R = 53 \pm 2$ ⊙. Whitney (1955) found by Wesselink's method 53 ± 3 ⊙, and by a photometric method 47 ± 2 ⊙. Oke (1961*b*) by a modified Wesselink's method found 40.3 ± 0.8 ⊙. The weighted mean is 44.2 ± 1.8 ⊙.

Y Oph.—Classical Cepheid, period 17.119^d. Abt (1954*a*) applied Wesselink's method to obtain $R = 71.5 \pm 4.2$ ⊙. Becker (1955) rediscussed Abt's work and revised his result to 118 ⊙. Neither of these results is very likely, although Becker's is probably the better, and since Y Oph is anomalous in a number of properties, I have not included it in the later discussion.

χ Cyg.—Classical Cepheid, period 16.387^d. Becker (1955) obtained $R = 138$ ⊙, but in view of the results for Y Oph obtained in the same paper, and the fact that the result seems much too high (it would predict M_V similar to l Car and SV Vul, which have periods 2–3 times as great), I have disregarded this determination.

Cluster Cepheids.—The five classical Cepheids which occur in galactic clusters have had absolute magnitudes and colors established for them (Kraft 1963). Since the relations between color and bolometric correction and effective temperature are well established (Kraft 1963) for classical Cepheids, reliable radii may be computed for these stars. Kraft's absolute magnitudes and colors for these stars, however, have first been slightly revised for reasons discussed elsewhere (Fernie 1963). The results are given in Table 1.

β Cep.—β Cep star, period 0.190^d. Stebbins and Kron (1954) applied Wesselink's method to obtain $R = 9.0$ ⊙. No probable error is given, but it is possibly about ±3 ⊙.

Mira.—Population I long-period variable, period 337^d. The angular diameter of Mira has been fairly well established as 0.054″ (Pease 1925; Pettit and Nicholson 1933; Scott 1942). The parallax has been determined by Burns (1941), Scott (1942), Jenkins (1952), and Feast (1963). I have adopted 0.014″. Then the radius of Mira is 420 ± 75 ⊙.

M5, No. 42.—W Vir star, period 25.74^d. Wallerstein (1959), from a combination of Wesselink's method and a direct calculation involving an assumed absolute magnitude (based on $M_V = 0.0$ for RR Lyrae stars) and effective temperature, obtained $R = 57$ ⊙. From a rediscussion of the absolute magnitudes and temperatures of W Virginis stars (Fernie 1964) I obtain $R = 41 \pm 10$ ⊙.

W Vir.—W Vir star, period 17.277^d. From a rediscussion of Abt's (1945*b*) work Becker (1955) obtained $R = 52$ ⊙. Wallerstein (1959), proceeding as for M5, No. 42, obtained 53 ⊙. Again, revising Wallerstein's calculation with more recent data for this class of stars (Fernie 1964), I obtain 45 ⊙. I have adopted $R = 48 \pm 5$ ⊙.

κ Pav.—W Vir star, period 9.073^d. From an application of Wesselink's method Rodgers (1957) finds $R = 21$ ⊙. I estimate the uncertainty to be about ±8 ⊙.

BL Her.—Population II variable, period 1.307^d. Abt and Hardie (1960) from Wesselink's method find $R = 10.1 \pm 1.1$ ⊙. From a modification of Wesselink's method they obtain 8.3 ± 0.6 ⊙. I have adopted $R = 9 \pm 1$ ⊙.

SU Dra.—RR Lyr star, period 0.660^d. This star has been the subject of a detailed study by Oke, Giver, and Searle (1962). From a sophisticated application of Wesselink's method they obtain $R = 5.2 \pm 1$ ⊙.

RR Lyr.—RR Lyr star, period 0.567^d. Stebbins (1953) from Wesselink's method obtained $R = 7.2 \pm 0.9$ ⊙. Abt (1959) from a modification of Wesselink's method obtained 6.0 ⊙, while Oke and Bonsack (1960) from a sophisticated application of Wesselink's method obtained 7.8 ± 0.7 ⊙. I have adopted $R = 7 \pm 1$ ⊙.

VZ Cnc.—δ Scu star (?), period 0.178^d. Smith (1955) finds from Wesselink's method $R = 3.8$ ⊙. I estimate the uncertainty as possibly ±0.5 ⊙. This star is listed as an RR Lyrae star in the Russian *Variable Star Catalogue*, but Smith gives evidence that it and CY Aqr belong to an old disk population rather than a halo population.

DY Her.—RR Lyr star, period 0.149^d. Wesselink's method fails for this star, but Hardie and Lott (1961) give evidence that its radius is about 2.2 ± 1.5 ⊙.

CY Aqr.—δ Scu star (?), period 0.0610^d. Smith (1955) finds from Wesselink's method $R = 1.4$ ⊙. From data on absolute magnitude and effective temperature given by McNamara, Augason, Huerta, and Murri (1961), I compute $R = 1.5$ ⊙. I have adopted $R = 1.45 \pm 0.05$ ⊙.

DQ Her.—Former nova, period 0.000823^d. This period is the 1.18-min. period of "flickering" discovered by Walker (1956). The radius is quite uncertain, but from Kraft's (1959) discussion I have adopted 0.03 ⊙.

All of the above data have been summarized in Tables 2 and 3. Because of previous indications that stars which have already passed through the red-giant stage of evolution might obey a different period-radius relation to those which have not, the stars are divided between Tables 2 and 3 according to this criterion.

III. THE PERIOD-RADIUS RELATION

Figure 1 shows a plot of log R against log P. The division of the stars into two groups is clearly evident. One group is composed of stars which have not yet passed beyond the red-giant stage of evolution; the other consists of those that have. Hereinafter these are referred to as Group I and Group II, respectively. The reason for the division is not clear. The fact that Group II contains both metal-poor Population II objects and old Population I objects would seem to indicate that chemical composition plays little if any part. Group I objects have hydrogen burning as their energy source, while Group II

TABLE 2

PERIODS AND RADII FOR GROUP I STARS

Star	log P	log $R/R_{\odot}$	Star	log P	log $R/R_{\odot}$
SV Vul	+1 655	2 15±0 01	EV Sct	0 490	1 43±0 05
T Mon	+1 432	2 08± 01	CF Cas	0 687	1 63± 05
δ Cep	+0 730	1 65± 01	U Sgr	0 828	1 70± 05
η Aql	+0 856	1 79± 01	DL Cas	0 908	1 74± 05
β Cep	−0 721	0 95± 12	S Nor	0 989	1 78± 05
β Dor	+0 993	2 02± 03	Mira	2 521	2 62±0 07
l Car	+1 551	2 14±0 02			

TABLE 3

PERIODS AND RADII FOR GROUP II STARS

Star	log P	log $R/R_{\odot}$	Star	log P	log $R/R_{\odot}$
DQ Her	−3 086	−1 5 ±0 7	VZ Cnc	−0 750	0 58±0 05
BL Her	+0 116	+0 95± 05	W Vir	+1 237	1 68± 04
RR Lyr .	−0 246	+0 85± 05	M5 No. 42	+1 411	1 6 ± 1
DY Her .	−0 827	+0 3 ± 3	κ Pav	+0 958	1 3 ± 1
CY Aqr	−1 215	+0 16±0 03	SU Dra	−0 180	0 72±0 07

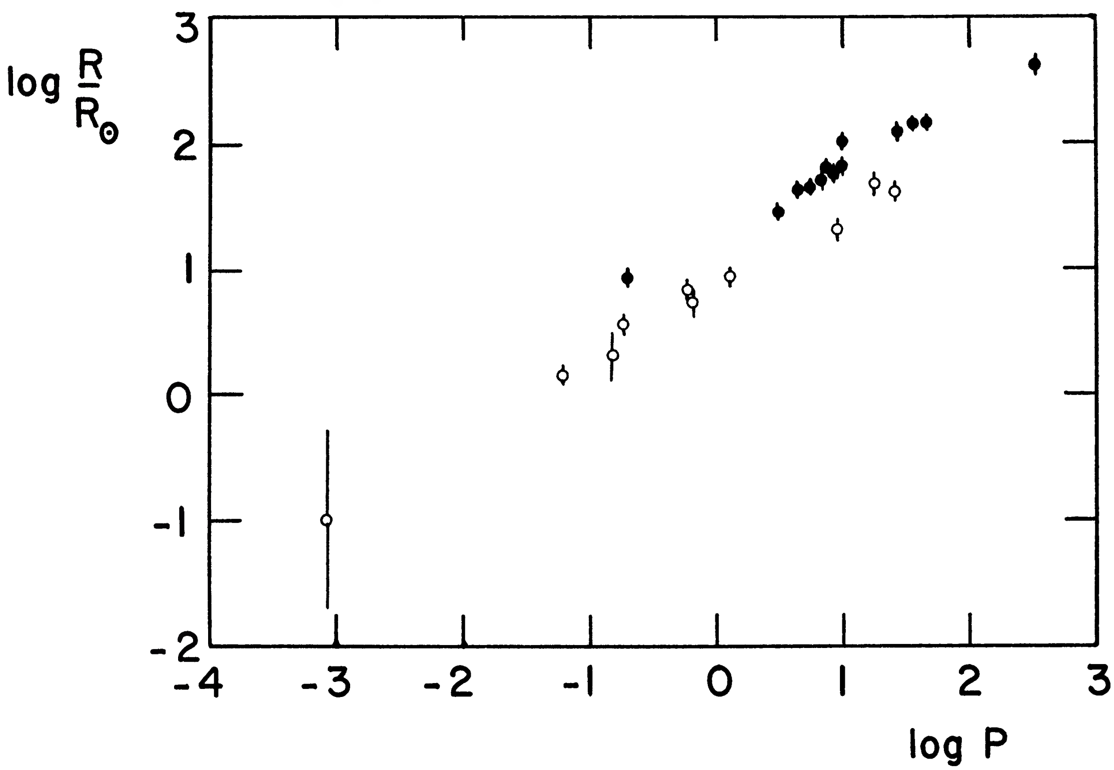

FIG. 1.—The observed period–radius relation. Filled circles are Group I stars, which have not yet passed beyond the red-giant stage of evolution; open circles are Group II stars, which have.

objects have higher-element burning as theirs, but whether it is the change in nuclear fuel per se which brings about different pulsation properties, or whether it is the attendant alteration in the interior structure which brings it about is uncertain. One may also note the general rule that Group I objects are evolving from left to right across the H-R diagram, while Group II objects are presumably evolving from right to left.

A least-squares solution for the Group I objects gave the result:

$$\log \frac{R}{R_{\odot}} = (0.536 \pm 0.025) \log P + (1.291 \pm 0.031).$$

(The errors are mean errors.) The solution is made in this way since the period is virtually error-free compared to the radius determinations. Physically, however, it is presumably the radius which is the independent variable, and we write

$$P = k\left(\frac{R}{R_{\odot}}\right)^{n}.$$

Then the above result implies

$$n = 1.9 \pm 0.1 ,$$

$$k = 0.0039 \pm 0.0004 \text{ days} .$$

The value for n suggests the likelihood that the period is proportional to the square of the radius, and observational tests described below strongly indicate that $n = 2$ is the correct value. I have therefore adopted the latter. In this case one obtains

$$k_{\mathrm{I}} = 0.0022 \pm 0.0004 \text{ days},$$

the subscript referring to the group.

It is evident from Figure 1 that the two groups have the same slope in the log R–log P plane, i.e., $n = 2$ for them both. One then finds

$$k_{\mathrm{II}} = 0.022 \pm 0.004 \text{ days} .$$

Thus for a Group I star and a Group II star of equal radius, the latter has a period 10 ± 2 times that of the former. This factor seems rather large if the physical difference between the two groups is to be explained merely in terms of pulsation in different overtones.

In seeking the physical meaning of k, it may be noted that in absolute units k has the dimensions of (angular momentum per unit mass)$^{-1}$, a reflection of the fact that, if a rotating star of constant mass is allowed to alter its radius while conserving angular momentum, its period of rotation will be proportional to the square of its radius. How this could be related to the periods of pulsation of such a diversity of stars as are contained in Figure 1, however, is not clear. It is perhaps just this fact that such diverse stars should obey a single relation that is most interesting theoretically.

If the Group II relation is applicable to white dwarfs, then such stars, if they pulsate at all, would have periods of only a few seconds. Their possible variability may thus have been overlooked because of practical difficulties.

We consider now the implications and observational uses of the period-radius relation. In doing so we adopt the following practical relations:

$$\log \frac{R}{R_{\odot}} = 0.500 \log P + 1.33 \qquad \text{(Group I stars)} ,$$

$$\log \frac{R}{R_{\odot}} = 0.500 \log P + 0.83 \qquad \text{(Group II stars)} .$$

IV. THE $P\sqrt{\rho}$ RELATION

One of the most often quoted relations in connection with variable stars is the $P\sqrt{\rho/\rho_{\odot}} = Q$ relation. The question has recently arisen as to whether Q is a universal number or not. Abt (1957) concluded observationally that Q was the same for all variables to within a factor of 2 or 3, set by the reliability of his data. Feast (1963) has concluded that classical cepheids and long-period variables have very nearly the same Q. Kraft (1963), on the other hand, has given evidence that even within a restricted body of variables such as the classical cepheids, Q is not a constant. For Q to be a universal number requires all variables to be homologous transformations of one another, which certainly does not seem to be true.

The period-density relation may be written

$$P\, \mathfrak{M}^{1/2} R^{-3/2} = Q .$$

For this to be simultaneously true with the relation

$$P = kR^2$$

one must have $\mathfrak{M}R =$ constant. Since this is not universally true, and since $P = kR^2$ does seem to be true, we conclude that Q cannot be a universal number. This conclusion may be tested directly by comparing β Cephei stars and classical Cepheids. Reliable data for the former stars have been compiled by Schmalberger (1960). From these one obtains $Q = 0.028 \pm 0.005$ days. Kraft (1963), for a classical Cepheid of period 10 days, obtained $Q = 0.0535^{\mathrm{d}}$. The two numbers differ significantly. The above equations may be arranged to give

$$Q = k^{3/4} P^{1/4} \mathfrak{M}^{1/2} .$$

This is in agreement with Kraft's finding that Q slowly increases with period.

Feast's result is also explainable. The mass of a long-period variable is of order one solar mass (Fernie and Brooker 1961; Feast 1963), which is about one-tenth the mass of a classical Cepheid (Kraft 1963), while the radius of a long-period variable is about ten times that of a classical Cepheid (see Table 2). Hence the product of mass times radius for a long-period variable is nearly the same as that for a classical Cepheid, and from the equations given above, they must therefore have the same value of Q.

V. P-L-C RELATION FOR CLASSICAL CEPHEIDS

Substitution of the period-radius relation into the relation

$$\frac{L}{L_{\odot}} = \left(\frac{R}{R_{\odot}}\right)^2 \left(\frac{T_e}{T_{e_{\odot}}}\right)^4$$

leads to

$$M_{\mathrm{bol}} = M_{\mathrm{bol}_{\odot}} - 2.500 \log P - 6.65 - 10 \log T_e + 10 \log T_{e_{\odot}} .$$

I have adopted $M_{\mathrm{bol}\,\odot} = +4.72$, $T_{e_{\odot}} = 5800^{\circ}$ K (Allen 1963). Based on results given by Kraft (1963) I have adopted for the classical Cepheids

$$\log T_e = 3.888 - 0.175(B - V)$$

$$M_{\mathrm{bol}} - M_V = -0.122 + 0.597(B - V) - 0.704(B - V)^2 .$$

Thus,

$$M_V = -3.05 - 2.500 \log P + 1.15(B - V) + 0.704(B - V)^2 .$$

Finally, it is found that the two terms in $(B - V)$ may be replaced by $(2.06[B - V] - 0.28)$ with an error of no more than 0.01 or 0.02 mag. in M_V over the range of interest. Hence one finally obtains

$$M_V = -3.33 - 2.500 \log P + 2.06(B - V) .$$

This relation has been tested on the nine Cepheids listed in Table 4. Among these are the the Cepheids in galactic clusters, except EV Sct. The observational results for this star seem definitely inconsistent with those for the others, a fact apparently noted by Kraft (1963), who assigns it low weight in his work. It seems about 0.5 mag. too faint. It may be noted also that the observational absolute magnitudes given in Table 4 for the remaining cluster Cepheids are slightly brighter than those given in Table 1. This comes about as follows. Application of the predicted P-L-C relation for the cepheids of Table 4 not in clusters gives good results, with residuals showing no trend with period. For the cluster Cepheids, however, the predictions are consistently about 0.2 mag. too bright for the observational values of Table 1. A possible explanation is found when it is recalled

TABLE 4

OBSERVED AND CALCULATED ABSOLUTE MAGNITUDES OF CLASSICAL CEPHEIDS

Star	log P	$(B-V)$	$M_{V\ \mathrm{obs}}$	$M_{V\ \mathrm{calc}}$	$M_{V\ \mathrm{calc}}$(Kraft)
SV Vul	1 655	0 94	−5 45	−5 53	−5 87
T Mon	1 432	90	−5 21	−5 06	−5 24
δ Cep	0 730	63	−3 65	−3 85	−3 48
η Aql	0 856	67	−4 27	−4 09	−3 86
l Car	1 551	93*	−5 43	−5 29	−5 53
CF Cas	0 687	70	−3 61	−3 61	−3 26
U Sgr	0 828	65	−4 07	−4 06	−3 84
DL Cas	0 908	77	−3 98	−4 01	−3 85
S Nor	0 989	0 80	−4 02	−4 15	−4 02

* Fernie (1960).

that the observed absolute magnitudes for the cluster Cepheids depend on corrections involving the ratio of total to selective interstellar absorption, but that this is not so for the remaining Cepheids, whose absolute magnitudes come from applications of Wesselink's method. A number of recent theoretical investigations (Blanco and Lennon 1961; Schmidt-Kaler 1961) have indicated that the ratio of total to selective absorption is more nearly 3.4 than 3.0 for stars such as Cepheids. From an unpublished observational investigation I have found a value close to 3.3 for Cepheids. I have adopted 3.4. The absolute magnitudes in Table 1 were obtained using 3.0. Therefore, since these stars have color excesses of about 0.5 mag., their observed absolute magnitudes should be brightened by (3.4 − 3.0) (0.5) = 0.2 mag., which is just that required for agreement with the predictions of the P-L-C relation. In listing observational absolute magnitudes in Table 4, therefore, I have adjusted the cluster Cepheids individually for this effect.

The agreement between predicted and observed absolute magnitude in Table 4 is very satisfactory, the average residual being 0.10 mag. If the period-radius relation originally found by least squares in Section III is used, the agreement with observation becomes significantly poorer. In fact, if the observed absolute magnitudes for the cluster Cepheids had been taken from Table 4 originally, the slope in Figure 1 would have been decreased from 0.536 toward 0.500.

For comparison, Kraft's predicted absolute magnitudes are also shown in Table 4. The average residual in this case is 0.20 mag. However, while no great improvement in accuracy has been gained over Kraft's (1963) P-L-C relation, what is perhaps significant is that a P-L-C relation has been established more simply and without any of the assumptions such as the slope of the mean P-L relation, the evolutionary tracks in the H-R diagram, etc., which Kraft had to make, except that we assume his effective temperature–bolometric correction–color relations to be correct. As a corollary it follows from the essential agreement between the two predicted P-L-C relations that Kraft's assumptions were basically correct.

TABLE 5

OBSERVED AND CALCULATED ABSOLUTE MAGNITUDES OF β CEPHEI STARS

STAR	LOG P	$(B-V)$	$M_{V\,\mathrm{obs}}$						$M_{V\,\mathrm{calc}}$
			(1)*	(2)	(3)	(4)	(5)	(6)	
β CMa	−0 602	−0 280	−4 9	−3 85	−4 9				−4 4
β Cru	− 602:	− 28	−4 5	...			...	...	−4 4
σ Sco.	− 609	− .26	−4 3				...	−4 5	−4 2
ξ^1 CMa.	− 660	− 280	−4 2	...	...	...			−4 2
BW Vul	− .697	− 270	−4 5	−3 4			...		−4 1
HD 21803	− 699	− .24	−4 0	...				...	−3 8
12 Lac	− .705	− 265	−4 3	−2 9	...	...	...	−3 4	−4 0
β Cep	− .719	− 275	−4 5	−3 25	−2 8	−3 4	−4 2		−4 0
15 CMa	− .735	− 25	−3 5						−3 8
ν Eri	− 750	− 255	−4 1	...					−3 8
τ^1 Lup	− 752	− 24	−2 9	...				...	−3 7
16 Lac .	− 767	− 260	−4 0	−2 7				−3 2	−3 8
δ Cet	− 793	− 245	−3 1	−2 1		...			−3 6
γ Peg	− 821	− 240	−3 8	−2 5		−3 6			−3 5
θ Oph	−0 851	−0 22	−3 0			...			−3 3

* References: (1) Schmalberger (1960); (2) Petrie (1954); (3) Blaauw and Savedoff (1953); (4) McNamara (1953); (5) Stebbins and Kron (1954); (6) Blaauw and Morgan (1953)

VI. P-L-C RELATION FOR β CEPHEI STARS

Proceeding as in the previous section, but with different bolometric correction–effective temperature–color relations, the period-luminosity-color law for β Cephei stars may be predicted. The new color relations are based on data given by Harris (1963). From the latter, I find for $(B - V) < -0.2$:

$$\log T_e = 3.812 - 2.24(B - V) ,$$

$$M_{\mathrm{bol}} - M_V = 1.50 + 15.5(B - V) .$$

The result is

$$M_V = -3.92 - 2.500 \log P + 6.9(B - V) .$$

The absolute magnitudes predicted by this relation are compared with various observational determinations in Table 5. Of the latter, probably the compilation by Schmalberger (1960) is the most accurate, being based principally on observed equivalent widths of Hγ converted to absolute magnitudes by the calibration of Johnson and Iriarte (1958). Most of the other observed values depend on membership in associations, parallactic motions, binary companions, etc. It is seen that in general the predicted absolute magnitudes agree with the observed values to within the probable errors of the latter. Once again use of a slope of 0.536 instead of 0.500 in the log R–log P relation leads to significantly poorer agreement with observation—by about 0.5–1.0 mag.

For those β Cephei stars which show two nearly equal periods, it matters little which of the two periods is chosen in predicting M_V. The difference in the predictions amounts to only a few thousandths of a magnitude in M_V.

VII. P-L-C RELATION FOR δ SCUTI STARS

Since the δ Scuti stars are old Population I objects and Figure 1 indicates that they belong to Group II, the period-radius relation appropriate to this group has been adopted. The effect is to make the zero-point of the P-L-C relation about 2.5 mag. fainter.

TABLE 6

OBSERVED AND CALCULATED ABSOLUTE MAGNITUDES OF δ SCUTI STARS

STAR	LOG P	$(B-V)$	$M_{V\,\mathrm{obs}}$ (1)*	$M_{V\,\mathrm{obs}}$ (2)	$M_{V\,\mathrm{obs}}$ (3)	$M_{V\,\mathrm{calc}}$
δ Scu	−0 712	0 32	+1 1	+1 9		+1 4
ρ Pup	−0 851	39	+0 2	+1 3		+2 0
δ Del.	−0 870	29	−0 9	+1 6		+1 7
CC And	−0 903	29		+2 2		+1 8
DQ Cep	−1 103	29		+1 8		+2 4
SX Phe	−1 259	(30)			+4 2	+2 7
AI Vel	−0 952	(0 30)			+1 9	+1 9

* References: (1) Eggen (1957); (2) McNamara and Augason (1962); (3) Woltjer (1956)

With the following relations based on the discussion by Harris (1963),

$$\log T_e = 3.989 - 0.383(B - V) ,$$

and $(0.15 \leq B - V \leq 0.45)$,

$$M_{\mathrm{bol}} - M_V = -0.186 + 0.326(B - V),$$

one finally obtains

$$M_V = -1.51 - 2.500 \log P + 3.50(B - V) .$$

Absolute magnitudes predicted by this relation are compared with observed values in Table 6. For SX Phe and AI Vel (B − V) has only been guessed at from the colors of the other stars, which may possibly explain the rather poor agreement between theory and observation in the case of SX Phe. For the remaining stars, however, agreement between predicted and observed values of M_V is probably within the probable errors of the observed values. That the choice of the Group II instead of the Group I period-radius relation was justified can be seen by brightening the predicted absolute magnitudes by 2.5 mag., which would be the values given by the Group I relation. This and the successes obtained with the other classes of variables are strong indication that the division into two groups is physically real.

VIII. P-L-C RELATION FOR RR LYRAE STARS

In going to Population II variables, difficulty is encountered in obtaining satisfactory bolometric correction–effective temperature–color relations, since the compilation by Harris refers only to Population I, and chemical composition significantly affects these relations. For the RR Lyrae stars, however, the relation between effective temperature and $(B - V)$ has already been studied in detail by Oke *et al.* (1962). From their results one finds

$$\log T_e = 3.909 - 0.296(B - V) , \qquad 0.00 \leq (B - V) \leq 0.60 .$$

The one bolometric correction (B.C.) given by these authors indicates that the Population II B.C.-T_e relation is probably the same as for Population I, but not the B.C.-$(B - V)$ relation. In any case, since the RR Lyrae stars fall at just those colors where the bolometric correction is a minimum, I have adopted

$$M_{\mathrm{bol}} - M_V = -0.06$$

independent of color, which should be accurate to within a few hundredths of a magnitude.

TABLE 7

OBSERVED AND CALCULATED ABSOLUTE MAGNITUDES OF RR LYRAE STARS

Star	log P	$(B-V)$	$M_{V\,\mathrm{calc}}$	$M_{V\,\mathrm{obs}}$	Reference
RR Lyr	−0 246	0 29	+0 65	+0 65	Eggen and Sandage (1959)
SU Dra	− 180	33	+ 63	{+ 8 {+ 55±0 4	Oke, Giver, and Searle (1962)
M5 stars	− 25:	34	+ 8:	+ 7 ± 2	Arp (1962)
M3 stars	−0 13:	0 31:	+0 4:	+0 3 ±0 3	Arp (1955) Baum, Hiltner, Johnson, and Sandage (1959)

TABLE 8

DISTANCE MODULUS OF M3 FROM RR LYRAE STARS

Star	log P	Bailey type	$(B-V)$	$M_{V\,\mathrm{calc}}$	V	$V-M_{V\,\mathrm{calc}}$
1	−0 283	a, b	0 35	+0 92	15 53	14 6
12	− 497	c	23	+1 08	15 55	14 5
25	− 319	a, b	32	+0 91	15 59	14 7
85	− 448	c	22	+0 94	15 53	14 6
124	−0 124	a, b	0 42	+0 73	15 56	14 8

Once again using the Group II relation one obtains

$$M_V = -0.83 - 2.500 \log P + 2.96(B - V) .$$

This relation is tested on a number of RR Lyrae stars listed in Table 7. For the M5 RR Lyrae's the color refers to the center of the variable gap, as given by Arp (1962). The mean period associated with this color is estimated from Arp (1955). In the case of SU Draconis the alternative observed absolute magnitude comes from alternative absolute energy distributions of α Lyrae, as explained by Oke *et al.* (1962). The agreement between the calculated and the observed absolute magnitudes in Table 7 is satisfactory.

As an example of the use of the relation to obtain the distance of a globular cluster containing RR Lyrae stars we choose M3. Observational data for the variables in this cluster have been given by Roberts and Sandage (1955) and Preston (1961). Five variables have been selected at random from the lengthy lists given by these authors. The data, calculated absolute magnitudes, and resulting distance moduli for these variables are given in Table 8. The average of these moduli is 14.6 mag., which may be compared with an identical value obtained by a detailed main-sequence fitting procedure given by Sandage (1962). It is of considerable interest to note that the Bailey type c variables give the same result as the type-a and b variables. If the former were pulsating in a different overtone with the ratio of periods about 1.5, one would expect them to give moduli differing by about 0.5 mag. Table 8 also illustrates how the color term tends to compensate the period term in the period-luminosity-color relation, resulting in the fact that for the class as a whole the RR Lyrae stars do not show a marked trend of luminosity with period.

With the good agreement between prediction and observation in Tables 7 and 8 the way now seems open to obtaining a reliable distance scale for the globular clusters.

IX. LONG-PERIOD VARIABLES AND W VIRGINIS STARS

The determination of a period-luminosity-color relation for the long-period variables is vitiated, at least in terms of visual magnitude and $(B - V)$ color, by the very large effects of TiO selective absorption and the very poorly determined bolometric corrections. These effects result in $(B - V)$ no longer being a measure of effective temperature, and in causing the absolute visual magnitude to decrease with period. Conceivably a satisfactory period-luminosity-color relation might be established by the use of red and infrared magnitudes, but for the moment the period-radius relation will be used only to compute the absolute bolometric magnitude of Mira. The predicted radius is 393 $\odot$, and assigning an average effective temperature of 2300° K (Ledoux and Walraven 1958), one obtains $M_{\rm bol} = -4.25$. Payne-Gaposchkin and Gaposchkin (1943) concluded from the probable interior and atmospheric structure of such stars that $M_{\rm bol} = -4.3$. Allen (1963) gives $M_{\rm bol} = -4.4$. The agreement is satisfactory.

A period-luminosity-color relation for the W Virginis stars could undoubtedly be obtained if the bolometric correction–effective temperature–color relations for these stars were known. Temporarily, in order to make a rough test, the very approximate relations $\log T_e = 3.871 - 0.2(B - V)$, B.C. $= -0.1$ mag. (Fernie 1964) may be adopted. The only W Virginis star which has both a reliably determined color and absolute magnitude is M5, No. 42, period 25.74^d. For this star $(B - V) = 0.56$, $M_{V\,\rm obs} = -3.0 \pm 0.3$ (Arp 1957, 1962). Applying the Group II period-radius relation and the above rough relations, one obtains $M_{V\,\rm calc} = -2.9$.

It is interesting to note that in principle one may use the variable stars to obtain the color excess of any cluster which contains variables belonging to two different classes. This comes about because the coefficient multiplying the $(B - V)$ term in the P-L-C relation is generally different among the various classes of variables. Thus choosing two stars each belonging to a different class, and computing their absolute magnitudes M_1 and M_2 from their respective P-L-C relations using the observed colors, will in general lead to $(M_1 - M_2)$ being different from $(V_1 - V_2)$, the difference in the apparent magnitudes. By forcing agreement between the two quantities the color excess would be obtained. In practice, however, there are too few clusters containing more than one type of variable for the technique to be useful.

Note added in proof.—Subsequent work has shown that the period depends also on the mass and that the full relation is probably $P \propto R^2M^{-1/2}$. Since, however, Groups I and II are each groups of roughly constant mass, most of the conclusions of this paper are only slightly altered. A detailed revision forms the subject of a paper now in press.

REFERENCES

Abt, H. A. 1954*a*, *Pub. A.S.P.*, **66**, 65.
———. 1954*b*, *Ap. J. Suppl.*, **1**, 63.
———. 1957, *Ap. J.*, **126**, 138.
———. 1959, *ibid.*, **130**, 824.
Abt, H. A., and Hardie, R. H. 1960, *Ap. J.*, **131**, 155.
Allen, C. W. 1963, *Astrophysical Quantities* (London: Athlone Press).

Arp, H. C. 1955, *A.J.*, **60**, 317.
———. 1957, *ibid.*, **62**, 129.
———. 1962, *Ap. J.*, **135**, 311.
Baum, W. A., Hiltner, W. A., Johnson, H. L., and Sandage, A. R. 1960, *Ap. J.*, **130**, 749.
Becker, W. 1955, *Z. f. Ap.*, **35**, 210.
Blaauw, A., and Morgan, W. W. 1953, *Ap. J.*, **117**, 256.
Blaauw, A., and Savedoff, M. 1953, *B.A.N.*, **12**, 69.
Blanco, V. M., and Lennon, C. J. 1961, *A.J.*, **66**, 524.
Burns, K. 1941, *Pub. A.A.S.*, **10**, 120.
Eggen, O. J. 1957, *A.J.*, **62**, 45.
Eggen, O. J., and Sandage, A. R. 1959, *M.N.*, **119**, 255.
Feast, M. W. 1963, *M.N.*, **125**, 367.
Fernie, J. D. 1960, *Monthly Notices Ast. Soc., S. Africa*, **19**, 10
———. 1963, *A.J.*, **68**, 780.
———. 1964, *ibid.*, **69**, 258.
Fernie, J. D., and Brooker, A. A. 1961, *Ap. J.*, **133**, 1088.
Hardie, R. H., and Lott, S. H. 1961, *Ap. J.*, **133**, 71.
Harris, D. L. 1963, *Basic Astronomical Data*, ed. K. Aa. Strand (Chicago: University of Chicago Press), chap. xiv.
Jenkins, L. F. 1952, *General Catalogue of Trigonometrical Parallaxes* (New Haven, Conn.: Yale University Observatory).
Johnson, H. L., and Iriarte, B. 1958, *Lowell Obs. Bull.*, **4**, 47.
Kraft, R. P. 1959, *Ap. J.*, **130**, 110.
———. 1963, *Basic Astronomical Data*, ed. K. Aa. Strand (Chicago: University of Chicago Press), chap. xxi.
Ledoux, P., and Walraven, T. 1958, *Hdb. d. Phys.*, **51**, 402.
McNamara, D. H. 1953, *Pub. A.S.P.*, **65**, 286.
McNamara, D. H., and Augason, G. 1962, *Ap. J.*, **135**, 64.
McNamara, D. H., Augason, G., Huerta, R., and Murri, W. 1961, *Pub. A.S.P.*, **73**, 340.
Oke, J. B. 1961*a*, *Ap. J.*, **133**, 90.
———. 1961*b*, *ibid.*, **134**, 214.
Oke, J. B., and Bonsack, S. J. 1960, *Ap. J.*, **132**, 417.
Oke, J. B., Giver, L. P., and Searle, L. 1962, *Ap. J.*, **136**, 393.
Payne-Gaposchkin, C., and Gaposchkin, S. 1943, Harvard Reprint II, No. 2.
Pease, F. G. 1925, *Pub. A.S.P.*, **37**, 89.
Petrie, R. M. 1954, *J. R. Astr. Soc. Canada*, **47**, 185.
Pettit, E., and Nicholson, S. B. 1933, *Ap. J.*, **78**, 320.
Preston, G. W. 1961, *Ap. J.*, **133**, 29.
Roberts, M. S., and Sandage, A. R. 1955, *A.J.*, **60**, 185.
Rodgers, A. W. 1957, *M.N.*, **117**, 85.
Sandage, A. 1962, *Ap. J.*, **135**, 349.
Sanford, R. F. 1956, *Ap. J.*, **123**, 201.
Schmalberger, D. C. 1960, *Ap. J.*, **132**, 591.
Schmidt-Kaler, T. 1961, *A.N.*, **286**, 113.
Scott, R. M. 1942, *Ap. J.*, **95**, 58.
Smith, H. J. 1955, *A.J.*, **60**, 179.
Stebbins, J. 1953, *Pub. A.S.P.*, **65**, 118.
Stebbins, J., and Kron, G. E. 1954, *Ap. J.*, **120**, 189.
Walker, M. F. 1956, *Ap. J.*, **123**, 68.
Wallerstein, G. 1959, *Ap. J.*,**129**, 356.
Whitney, C. 1955, *Ap. J.*, **121**, 682.
Woltjer, J. 1956, *B.A.N.*, **13**, 62.