THE DYNAMICAL INSTABILITY OF GASEOUS MASSES APPROACHING THE SCHWARZSCHILD LIMIT IN GENERAL RELATIVITY

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ABSTRACT

In this paper the theory of the infinitesimal, baryon-number conserving, adiabatic, radial oscillations of a gas sphere is developed in the framework of general relativity. A variational base for determining the characteristic frequencies of oscillation is established. It provides a convenient method for obtaining sufficient conditions for the occurrence of dynamical instability. The principal result of the analysis is the demonstration that the Newtonian lower limit $\frac{4}{3}$, for the ratio of the specific heats γ, for insuring dynamical stability is increased by effects arising from general relativity; indeed, is increased to an extent that, so long as γ is finite, dynamical instability will intervene before a mass contracts to the limiting radius ($\geq 2.25\ GM/c^2$) compatible with hydrostatic equilibrium. Moreover, if γ should exceed $\frac{4}{3}$ only by a small amount, then dynamical instability will occur if the mass should contract to the radius

$$R_c = \frac{K}{\gamma - \frac{4}{3}} \frac{2GM}{c^2} \qquad (\gamma \rightarrow \tfrac{4}{3}),$$

where K is a constant depending, principally, on the density distribution in the configuration. The value of the constant K is explicitly evaluated for the homogeneous sphere of constant energy density and the polytropes of indices $n = 1$, 2, and 3.

I. INTRODUCTION

It is well known that the gravitational field external to a spherical distribution of matter is described by Schwarzschild's metric

$$-ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)(dx^0)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{dr^2}{1 - 2GM/rc^2}, \tag{1}$$

where M is the inertial mass of the source of the gravitation. The metric (1) obtains, whether or not the matter which gives rise to the field is endowed with radial motions, so long as the spherical symmetry is preserved.

From the form of the metric (1), it is apparent that the spacelike and the timelike characters of the coordinates x^0 and r are retained only for

$$r > \frac{2GM}{c^2} = R_S(\text{say}). \tag{2}$$

The radius R_S defined by this inequality is the *Schwarzschild limit* appropriate to the mass M. If we should contemplate a situation in which the matter giving rise to the field is confined within a radius $R < R_S$, then the character of the coordinates x^0 and r must change as we cross the Schwarzschild limit; and the nature of space-time under these circumstances has been discussed widely in the literature (cf. Landau and Lifshitz [1962], p. 329). But these questions do not arise if we suppose that the matter, which gives rise to the external metric (1), is in a state of hydrostatic equilibrium and is further described by a pressure p and an energy density ϵ ($= -T_0{}^0$); for it can then be shown that the radius R of the configuration is necessarily restricted by the inequality (Buchdahl 1959),

$$R \geq \tfrac{9}{8} \frac{2GM}{c^2} = \tfrac{9}{8} R_S. \tag{3}$$

In other words, we are not called upon to consider what might happen if $r < R_S$: it simply does not happen!

The question which now occurs is whether the existence of the limit (3) has any bearing on the dynamical stability of the mass. It is known that, in the framework of the Newtonian theory of gravitation, the condition for dynamical stability is that the "ratio of the specific heats" γ (in case it should be a constant) must exceed $\frac{4}{3}$. The question is now essentially this: is the Newtonian lower limit $\frac{4}{3}$ on γ affected by general relativity? We shall show that general relativity does affect the condition for dynamical stability in a very striking way: for any finite γ, dynamical instability *always* intervenes before the limit on the radius set by the inequality (3) is reached; and, more particularly, if γ should exceed $\frac{4}{3}$ only by a small amount, then for dynamical stability

$$R > \frac{K}{\gamma - \frac{4}{3}} \frac{2GM}{c^2} \qquad (\gamma \rightarrow \tfrac{4}{3}), \tag{4}$$

where K is a constant which depends, principally, on the density distribution. According to the asymptotic requirement (4), *dynamical instability can intervene long before the Schwarzschild limit is reached.*

The fact that instability may arise under certain circumstances, by virtue of the effects of general relativity, has been considered by Iben (1963) and Fowler (1964). These authors introduce the concept of a "binding energy" and argue in terms of it. While such arguments are physically plausible (and sometimes give the correct results), it is known that they are not always conclusive. In any event, it is clear that the question of the dynamical stability of a gaseous mass can be answered without any ambiguity by an analysis of its normal modes of radial oscillation. In this paper such an analysis will be provided; and on its basis the condition for dynamical stability will be established in a number of special cases.

The principal results of this paper have been announced in preliminary Letters (Chandrasekhar 1964*a*, *b;* see also Chandrasekhar and Tooper 1964).

After the present paper was mostly completed, Dr. A. H. Taub drew my attention to an earlier paper of his (1962) in which the problem considered in this paper has been formulated by him in one special case. But the variational base for discriminating dynamical stability and the fact that a relation of the form (4) emerges in general relativity do not appear to have been considered before.

II. THE FIELD EQUATIONS

We consider a spherically symmetric system with motions, if any, only in the radial directions. There is then no loss of generality in supposing that the metric is of the form

$$-ds^2 = -e^{\nu}(dx^0)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\lambda} dr^2, \tag{5}$$

where ν and λ are functions of r and x^0 only. The Einstein field equations appropriate to the metric (5) are well known (see, e.g., Landau and Lifshitz [1962], p. 325); they can be written in the forms

$$R_0{}^0 - \tfrac{1}{2}R = -\frac{8\pi G}{c^4} T_0{}^0 = +e^{-\lambda}\left(\frac{1}{r}\frac{\partial\lambda}{\partial r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r e^{-\lambda}) + \frac{1}{r^2}, \tag{6}$$

$$R_1{}^1 - \tfrac{1}{2}R = -\frac{8\pi G}{c^4} T_1{}^1 = -e^{-\lambda}\left(\frac{1}{r}\frac{\partial\nu}{\partial r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{7}$$

$$R_2^2 - \tfrac{1}{2}R = R_3^3 - \tfrac{1}{2}R = -\frac{8\pi G}{c^4} T_2^2 = -\frac{8\pi G}{c^4} T_3^3$$

$$= -e^{-\lambda}\left[\tfrac{1}{2}\frac{\partial^2 \nu}{\partial r^2} - \tfrac{1}{4}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} + \tfrac{1}{4}\left(\frac{\partial \nu}{\partial r}\right)^2 + \frac{1}{2r}\frac{\partial}{\partial r}(\nu - \lambda)\right] \tag{8}$$

$$+ e^{-\nu}\left[-\tfrac{1}{4}\frac{\partial \lambda}{\partial x^0}\frac{\partial \nu}{\partial x^0} + \tfrac{1}{2}\frac{\partial}{\partial x^0}\left(\frac{\partial \lambda}{\partial x^0}\right) + \tfrac{1}{4}\left(\frac{\partial \lambda}{\partial x^0}\right)^2\right],$$

and

$$R_0^1 = -\frac{8\pi G}{c^4} T_0^1 = -\frac{e^{-\lambda}}{r}\frac{\partial \lambda}{\partial x^0}, \tag{9}$$

where R_j^i denotes the contracted Riemann-Christoffel tensor and T_j^i the energy-momentum tensor. We shall assume that T_j^i is of the form

$$T_j^i = (p + \epsilon)u^i u_j + p\delta_j^i, \tag{10}$$

where

$$u^i = \frac{dx^i}{ds} \tag{11}$$

is the contravariant four-velocity, p is the pressure, and ϵ is the energy density (arising from all causes).

In the case under consideration only the components u^0 and u^1 are different from zero; therefore,

$$T_2^2 = T_3^3 \equiv p, \tag{12}$$

whether or not motions are present.

A useful relation which follows from equations (6) and (7) is

$$\frac{e^{-\lambda}}{r}\frac{\partial}{\partial r}(\lambda + \nu) = \frac{8\pi G}{c^4}(T_1^1 - T_0^0). \tag{13}$$

Equations (6)–(9) are not all independent since the covariant divergence of $R_j^i - R\delta_j^i$ vanishes identically; and this last fact leads to the relation

$$(T_j^i)_{;i} = 0 \tag{14}$$

as a necessary identity. In the case under consideration equation (14) leads to the two relations

$$\frac{\partial T_0^0}{\partial x^0} + \frac{\partial T_0^1}{\partial r} + \tfrac{1}{2}(T_0^0 - T_1^1)\frac{\partial \lambda}{\partial x^0} + T_0^1\left[\tfrac{1}{2}\frac{\partial}{\partial r}(\lambda + \nu) + \frac{2}{r}\right] = 0 \tag{15}$$

and

$$\frac{\partial T_1^0}{\partial x^0} + \frac{\partial T_1^1}{\partial r} + \tfrac{1}{2}T_1^0\frac{\partial}{\partial x^0}(\lambda + \nu) + \tfrac{1}{2}(T_1^1 - T_0^0)\frac{\partial \nu}{\partial r} + \frac{2}{r}(T_1^1 - p) = 0, \tag{16}$$

where it may be noted that

$$T_0^1 = -e^{\nu - \lambda}T_1^0. \tag{17}$$

III. THE EQUATIONS GOVERNING EQUILIBRIUM

In a state of hydrostatic equilibrium none of the quantities depends on x^0; and u^1 is moreover zero. Distinguishing the quantities describing this equilibrium state by a subscript zero, we have

$$T_1^1 = T_2^2 = T_3^3 = p_0, \quad \text{and} \quad T_0^0 = -\epsilon_0; \tag{18}$$

and as the independent field equations we shall take (cf. eqs. [6], [7], and [16])

$$\frac{d}{dr}\left(r e^{-\lambda_0}\right)=1-\frac{8\pi G}{c^4} r^2\epsilon_0\,, \tag{19}$$

$$\frac{e^{-\lambda_0}}{r}\frac{d\nu_0}{dr}=\frac{1}{r^2}\left(1-e^{-\lambda_0}\right)+\frac{8\pi G}{c^4} p_0\,, \tag{20}$$

and

$$\frac{dp_0}{dr}=-\tfrac{1}{2}\left(p_0+\epsilon_0\right)\frac{d\nu_0}{dr}. \tag{21}$$

We also have the identity (cf. eq. [13])

$$\frac{e^{-\lambda_0}}{r}\frac{d}{dr}\left(\lambda_0+\nu_0\right)=\frac{8\pi G}{c^4}\left(p_0+\epsilon_0\right). \tag{22}$$

Eliminating ν_0 from equations (20) and (21), we obtain the following pair of equations which are usually taken to describe hydrostatic equilibrium in general relativity:

$$\left(1-\frac{2GM_r}{r c^2}\right)\frac{dp_0}{dr}=-\frac{1}{c^2}\left(p_0+\epsilon_0\right)\left(\frac{GM_r}{r^2}+\frac{4\pi G}{c^2} p r\right) \tag{23}$$

and

$$M_r=\frac{4\pi}{c^2}\int_0^r \epsilon_0 r^2 dr\,. \tag{24}$$

IV. EQUATIONS GOVERNING INFINITESIMAL RADIAL OSCILLATIONS

We shall now suppose that an equilibrium configuration governed by the equations of Section III is perturbed in a manner that its spherical symmetry is not violated. As a consequence of such a perturbation, motions in the radial directions will ensue.

In obtaining the equations governing the perturbed state, we shall neglect all quantities which are of the second and higher orders in the motions and retain only the terms which are linear in them. Then

$$u^1=e^{-\nu_0/2}v\,,\qquad u_1=e^{\lambda_0-\nu_0/2}v\,,$$
$$u^0=e^{-\nu_0/2}\,,\quad\text{and}\quad u_0=-e^{\nu_0/2}\,, \tag{25}$$

where

$$v=\frac{dr}{dx^0} \tag{26}$$

is the velocity in the radial direction in units of the velocity of light. To the same order as equations (25),

$$T_1{}^1=T_2{}^2=T_3{}^3=p\,,\qquad T_0{}^0=-\epsilon\,; \tag{27}$$

and

$$T_0{}^1=(p_0+\epsilon_0)u^1u_0=-(p_0+\epsilon_0)v\,,$$
$$T_1{}^0=(p_0+\epsilon_0)u^0u_1=e^{\lambda_0-\nu_0}(p_0+\epsilon_0)v\,. \tag{28}$$

(Notice the absence of the subscripts zero for p and ϵ in eq. [27].)

Now let

$$\lambda=\lambda_0+\delta\lambda\,,\qquad \nu=\nu_0+\delta\nu\,,\qquad p=p_0+\delta p\,,\qquad\text{and}\qquad \epsilon=\epsilon_0+\delta\epsilon \tag{29}$$

be the values of the various quantities in the perturbed state. In view of equations (27), equations (19) and (20) are valid as they stand if λ_0 and ν_0 are replaced by λ and ν. The corresponding linearized equations governing the perturbations are, therefore,

$$\frac{\partial}{\partial r}\left(r e^{-\lambda_0}\delta\lambda\right)=\frac{8\pi G}{c^4}r^2\delta\epsilon \tag{30}$$

and

$$\frac{e^{-\lambda_0}}{r}\left(\frac{\partial}{\partial r}\delta\nu-\frac{d\nu_0}{dr}\delta\lambda\right)=\frac{e^{-\lambda_0}}{r^2}\delta\lambda+\frac{8\pi G}{c^4}\delta p\,. \tag{31}$$

As the remaining two field equations, we may use the appropriately linearized forms of equations (9) and (16). We have (cf. eqs. [28])

$$\frac{e^{-\lambda_0}}{r}\frac{\partial}{\partial x^0}\delta\lambda=-\frac{8\pi G}{c^4}(p_0+\epsilon_0)v \tag{32}$$

and

$$e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial v}{\partial x^0}+\frac{\partial}{\partial r}\delta p+\tfrac{1}{2}(p_0+\epsilon_0)\frac{\partial}{\partial r}\delta\nu+\tfrac{1}{2}(\delta p+\delta\epsilon)\frac{d\nu_0}{dr}=0\,. \tag{33}$$

It is convenient to introduce a "Lagrangian displacement" ξ with respect to the world-time x^0 defined by

$$v=\frac{\partial\xi}{\partial x^0}. \tag{34}$$

Equation (32) then directly integrates to give

$$\frac{e^{-\lambda_0}}{r}\delta\lambda=-\frac{8\pi G}{c^4}(p_0+\epsilon_0)\xi\,, \tag{35}$$

or in view of equation (22),

$$\delta\lambda=-\xi\frac{d}{dr}(\lambda_0+\nu_0). \tag{36}$$

Equations (30) and (35) now give

$$\delta\epsilon=-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(p_0+\epsilon_0)\xi\right], \tag{37}$$

or alternatively,

$$\delta\epsilon=-\xi\frac{d\epsilon_0}{dr}-\xi\frac{dp_0}{dr}-(p_0+\epsilon_0)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\xi). \tag{38}$$

Substituting for dp_0/dr in this last equation from equation (21), we can also write

$$\delta\epsilon=-\xi\frac{d\epsilon_0}{dr}-(p_0+\epsilon_0)\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left(r^2e^{-\nu_0/2}\xi\right). \tag{39}$$

Considering next equation (31) and substituting for $\delta\lambda$ in accordance with equation (35), we obtain

$$\frac{e^{-\lambda_0}}{r}\frac{\partial}{\partial r}\delta\nu=\frac{8\pi G}{c^4}\left[\delta p-(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\xi\right], \tag{40}$$

or in view of equation (22),

$$(p_0+\epsilon_0)\frac{\partial}{\partial r}\delta\nu=\left[\delta p-(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\xi\right]\frac{d}{dr}(\lambda_0+\nu_0). \tag{41}$$

We shall now suppose that all the perturbations have a dependence on x^0 of the form

$$e^{i\sigma x^0}, \tag{42}$$

where $c\sigma$ is a characteristic frequency to be determined. Letting $\delta\lambda$, $\delta\nu$, δp, and $\delta\epsilon$ now stand for the *amplitudes* of the respective quantities with the time-dependence (42), we can rewrite equation (33) in the form

$$\begin{aligned}\sigma^2 e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\,\xi &= \frac{d}{dr}\,\delta p+\delta p\,\frac{d}{dr}\,(\tfrac{1}{2}\lambda_0+\nu_0)+\tfrac{1}{2}\delta\epsilon\,\frac{d\nu_0}{dr}\\ &\quad -\tfrac{1}{2}(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\left(\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi\,,\end{aligned}\tag{43}$$

where we have substituted for $(p_0+\epsilon_0)\partial\delta\nu/\partial r$ in accordance with equation (41). Also, it should be recalled that $\delta\epsilon$ is expressed in terms of ξ and the unperturbed variables by equations (38) or (39).

V. THE CONSERVATION OF THE BARYON NUMBER

The field equations have enabled us to express the perturbations in all the variables, except the pressure, in terms of the Lagrangian displacement ξ and the unperturbed variables. To express δp similarly in terms of ξ, we need an additional assumption which will make the physical content of the present theory equivalent to the physical content of the corresponding non-relativistic theory of the adiabatic radial oscillations of a gaseous mass. Since the energy-momentum tensor (10) which we have assumed includes no dissipative terms, the first law of thermodynamics is, in effect, already included in the theory (cf. Landau and Lifshitz [1962], p. 382). What requires to be added to the field equations is, therefore, some statement which will be equivalent to the separation of ϵ, in the classical theory, into a part which is related to the *rest mass* of the constituent particles of the system and a part which is related to what is conventionally the *internal energy* of the system. While the additional assumption by which we must supplement the field equations can be formulated in a variety of ways, it is perhaps simplest (particularly for the contexts in which the present theory may find application) to state the supplementary condition as that required by the *conservation of the baryon number*.[1]

If N is the number of baryons (anti-baryons counted negatively) per unit volume, its conservation in general relativity will require that

$$(Nu^k)_{;k}=0 \tag{44}$$

or

$$\frac{\partial}{\partial x^k}\,(Nu^k)+Nu^k\,\frac{\partial}{\partial x^k}\log\sqrt{-g}=0\,. \tag{45}$$

In the framework of the present linearized theory in which the non-vanishing velocity components are given by equations (25), equation (45) becomes

$$\begin{aligned}&\frac{\partial}{\partial x^0}\,(Ne^{-\nu_0/2})+\frac{\partial}{\partial r}\,(Nv\,e^{-\nu_0/2})+\tfrac{1}{2}Ne^{-\nu_0/2}\,\frac{\partial}{\partial x^0}\,(\lambda+\nu)\\ &\quad +Ne^{-\nu_0/2}v\,\frac{\partial}{\partial r}\left[\tfrac{1}{2}(\lambda+\nu)+\frac{2}{r}\right]=0\,.\end{aligned}\tag{46}$$

Now writing

$$N=N_0(r)+\delta N(r,x^0) \tag{47}$$

[1] The procedure we follow here is essentially the same as that of Landau and Lifshitz (1959, p. 501) in their formulation of the equations of relativistic fluid dynamics.

in an obvious notation, we obtain from equation (46) (again neglecting all terms which are quadratic and higher in v)

$$e^{-\nu_0/2}\frac{\partial}{\partial x^0}\,\delta N+\frac{1}{r^2}\frac{\partial}{\partial r}\left(N_0 r^2 v\, e^{-\nu_0/2}\right)+\tfrac{1}{2}N_0 e^{-\nu_0/2}\frac{\partial}{\partial x^0}\,\delta\lambda + \tfrac{1}{2}N_0 e^{-\nu_0/2} v\,\frac{d}{dr}\left(\lambda_0+\nu_0\right)=0\,. \tag{48}$$

With v expressed in terms of the Lagrangian displacement ξ, equation (48) integrates to give

$$\delta N+\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left(N_0 r^2 \xi\, e^{-\nu_0/2}\right)+\tfrac{1}{2}N_0\left[\delta\lambda+\xi\frac{d}{dr}\left(\lambda_0+\nu_0\right)\right]=0\,. \tag{49}$$

The last term on the right-hand side of equation (49) vanishes on account of equation (36). We are thus left with

$$\delta N=-\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left(N_0 r^2 \xi\, e^{-\nu_0/2}\right), \tag{50}$$

or alternatively,

$$\delta N=-\xi\frac{dN_0}{dr}-N_0\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left(r^2 e^{-\nu_0/2}\xi\right). \tag{51}$$

Now if

$$N\equiv N(\epsilon,\,p) \tag{52}$$

is the *equation of state*, then, it follows from equations (39) and (51) that

$$\delta p=-\xi\frac{dp_0}{dr}-\gamma p_0\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}\left(r^2 e^{-\nu_0/2}\xi\right), \tag{53}$$

where γ is the "ratio of the specific heats" defined in the manner

$$\gamma=\frac{1}{p\,\partial N/\partial p}\left[N-(p+\epsilon)\frac{\partial N}{\partial\epsilon}\right]. \tag{54}$$ [2]

VI. THE PULSATION EQUATION AND THE VARIATIONAL PRINCIPLE

With $\delta\epsilon$ and δp given by equations (37) and (53), equation (43) becomes

$$\begin{aligned}\sigma^2 e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\,\xi &= -\frac{d}{dr}\left(\xi\frac{dp_0}{dr}\right)-\left(\tfrac{1}{2}\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi\frac{dp_0}{dr}\\ &-\tfrac{1}{2}(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\left(\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi-\tfrac{1}{2}\frac{d\nu_0}{dr}\left\{\frac{d}{dr}\left[(p_0+\epsilon_0)\,\xi\right]+\frac{2}{r}(p_0+\epsilon_0)\,\xi\right\}\\ &-e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left[e^{(\lambda_0+2\nu_0)/2}\frac{\gamma p_0}{r^2}e^{\nu_0/2}\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right].\end{aligned} \tag{55}$$

Substituting for dp_0/dr its value given by equation (21) in the first two terms on the right-hand side of equation (55) and combining them with the two following terms, we find

$$\tfrac{1}{2}(p_0+\epsilon_0)\left(\frac{d^2\nu_0}{dr^2}-\tfrac{1}{2}\frac{d\lambda_0}{dr}\frac{d\nu_0}{dr}-\frac{1}{r}\frac{d\lambda_0}{dr}-\frac{3}{r}\frac{d\nu_0}{dr}\right)\xi\,. \tag{56}$$

[2] For an inclosure containing matter and radiation, γ defined by this equation is identical with the coefficient Γ_1 as customarily defined in stellar structure (cf. Chandrasekhar [1939], pp. 55–59).

On the other hand, according to equation (8), under conditions of equilibrium,

$$\frac{16\pi G}{c^4} p_0 e^{\lambda_0} = \frac{d^2\nu_0}{dr^2} - \tfrac{1}{2}\frac{d\lambda_0}{dr}\frac{d\nu_0}{dr} + \tfrac{1}{2}\left(\frac{d\nu_0}{dr}\right)^2 + \frac{1}{r}\frac{d}{dr}(\nu_0 - \lambda_0). \tag{57}$$

Using this relation, we find that expression (56) becomes

$$\begin{aligned}&\frac{8\pi G}{c^4} e^{\lambda_0} p_0(p_0+\epsilon_0)\xi - \tfrac{1}{4}(p_0+\epsilon_0)\frac{d\nu_0}{dr}\left(\frac{8}{r}+\frac{d\nu_0}{dr}\right)\xi \\ &= \frac{8\pi G}{c^4} e^{\lambda_0} p_0(p_0+\epsilon_0)\xi + \tfrac{1}{2}\frac{dp_0}{dr}\left(\frac{8}{r} - \frac{2}{p_0+\epsilon_0}\frac{dp_0}{dr}\right)\xi \\ &= \frac{4}{r}\frac{dp_0}{dr}\xi + \frac{8\pi G}{c^4} e^{\lambda_0} p_0(p_0+\epsilon_0)\xi - \frac{1}{p_0+\epsilon_0}\left(\frac{dp_0}{dr}\right)^2\xi,\end{aligned} \tag{58}$$

where use has again been made of equation (21). With this reduction of the four terms, equation (55) takes the form

$$\begin{aligned}\sigma^2 e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\xi &= \frac{4}{r}\frac{dp_0}{dr}\xi - e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left[e^{(\lambda_0+3\nu_0)/2}\frac{\gamma p_0}{r^2}\frac{d}{dr}(r^2 e^{-\nu_0/2}\xi)\right] \\ &\quad + \frac{8\pi G}{c^4} e^{\lambda_0} p_0(p_0+\epsilon_0)\xi - \frac{1}{p_0+\epsilon_0}\left(\frac{dp_0}{dr}\right)^2\xi.\end{aligned} \tag{59}$$

This is the required "pulsation equation." Solutions of this equation must be sought which satisfy the boundary conditions

$$\xi = 0 \text{ at } r = 0 \qquad \text{and} \qquad \delta p = 0 \text{ at } r = R. \tag{60}$$

Equation (59) together with the boundary conditions (60) constitute a characteristic value problem for σ^2. It is apparent that this problem is self-adjoint and that a variational base for determining σ^2 is provided by multiplying equation (59) by $r^2\xi \exp[\frac{1}{2}(\lambda_0 + \nu_0)]$ and integrating over the range of r. We thus obtain

$$\begin{aligned}\sigma^2\int_0^R e^{(3\lambda-\nu)/2}(p+\epsilon)r^2\xi^2 dr &= 4\int_0^R e^{(\lambda+\nu)/2} r\frac{dp}{dr}\xi^2 dr \\ &+ \int_0^R e^{(\lambda+3\nu)/2}\frac{\gamma p}{r^2}\left[\frac{d}{dr}(r^2 e^{-\nu/2}\xi)\right]^2 dr - \int_0^R e^{(\lambda+\nu)/2}\left(\frac{dp}{dr}\right)^2\frac{r^2\xi^2}{p+\epsilon}dr \\ &+ \frac{8\pi G}{c^4}\int_0^R e^{(3\lambda+\nu)/2} p(p+\epsilon)r^2\xi^2 dr,\end{aligned} \tag{61}$$

where we have now suppressed the distinguishing subscript zero as no longer necessary. Associated with the variational equation (61) is the orthogonality relation

$$\int_0^R e^{(3\lambda-\nu)/2}(p+\epsilon)r^2\xi^{(i)}\xi^{(j)}dr = 0 \qquad (i \neq j), \tag{62}$$

where $\xi^{(i)}$ and $\xi^{(j)}$ are the proper solutions belonging to different characteristic values of σ^2.

Since equation (61) expresses a minimal (and not merely an extremal) principle, it is clear that *a sufficient condition for the dynamical instability of a mass is that the right-hand side of equation (61) vanishes for some chosen "trial function" ξ which satisfies the required boundary conditions.* We shall investigate in the following sections the conditions for the occurrence of dynamical instability for a number of special models.

VII. THE HOMOGENEOUS MODEL

We shall first apply the condition for dynamical instability derived in Section VI to the homogeneous sphere of constant energy-density ϵ and constant ratio of specific heats γ. As is well known, equations (23) and (24) governing hydrostatic equilibrium allow of immediate integration in this case. Thus, writing

$$y^2 = 1 - \frac{r^2}{a^2} \quad \text{and} \quad y_1^2 = 1 - \frac{R^2}{a^2}, \tag{63}$$

where

$$a^2 = \frac{3\,c^4}{8\pi G\epsilon}, \tag{64}$$

we find that the solutions for the relevant physical quantities can be expressed in the forms

$$p = \epsilon\,\frac{y - y_1}{3\,y_1 - y}, \qquad e^{\lambda} = \frac{1}{y^2}, \qquad \text{and} \qquad e^{\nu} = \tfrac{1}{4}(3\,y_1 - y)^2. \tag{65}$$

We may parenthetically note here that according to the present solution for p, a necessary condition for the pressure to be positive everywhere is

$$3y_1 > 1 \qquad \text{or} \qquad R^2 < \tfrac{8}{9}a^2. \tag{66}$$

Expressing this last inequality in terms of the inertial mass

$$M = \tfrac{4}{3}\pi R^3 \epsilon / c^2, \tag{67}$$

we have

$$\frac{2GM}{Rc^2} = \frac{R^2}{a^2} < \tfrac{8}{9} \tag{68}$$

or (cf. eq. [3])

$$R > \tfrac{9}{8}\,\frac{2GM}{c^2} = \tfrac{9}{8} R_S. \tag{69}$$

As we have already stated in Section I, Buchdal (1959) has shown that the lower limit to R set by the inequality (69) is *absolute* in the sense that it obtains independently of any constitutive relations between p and ϵ that may exist.

Now inserting the solutions (65) in equation (61), we obtain

$$\begin{aligned} 4a^2\sigma^2 y_1 \int_0^{\eta_1} \frac{\eta^2 \xi^2 d\eta}{y^3(3\,y_1 - y)^2} &= y_1 \int_0^{\eta_1} \frac{2\,y^2 - 1 - 9\,y_1^2}{y^3(3\,y_1 - y)^2}\,\eta^2\xi^2 d\eta \\ &+ \tfrac{1}{8}\gamma \int_0^{\eta_1} (y - y_1)(3\,y_1 - y)^2 \left[\frac{d}{d\eta}\,(\eta^2 e^{-\nu/2}\xi)\right]^2 \frac{d\eta}{\eta^2 y}, \end{aligned} \tag{70}$$

where

$$\eta = r/a \qquad \text{and} \qquad \eta_1 = R/a\,, \tag{71}$$

and γ (as stated) has been assumed to be a constant throughout the mass.

For the "trial function"

$$\xi = \eta e^{\nu/2} = \tfrac{1}{2}\eta(3y_1 - y)\,, \tag{72}$$

equation (70) becomes

$$\begin{aligned} (a\sigma)^2 y_1 \int_0^{\eta_1} \frac{\eta^4}{y^3}\,d\eta &= \tfrac{1}{4} y_1 \int_0^{\eta_1} (2\,y^2 - 1 - 9\,y_1^2)\,\frac{\eta^4}{y^3}\,d\eta \\ &+ \tfrac{9}{8}\gamma \int_0^{\eta_1} (y - y_1)(3\,y_1 - y)^2\,\frac{\eta^2}{y}\,d\eta\,. \end{aligned} \tag{73}$$

With the substitutions (cf. eqs. [63] and [71])

$$y = \cos\theta \quad \text{and} \quad \eta = \sin\theta\,, \tag{74}$$

equation (73) reduces to

$$(a\sigma)^2\cos\theta_1\int_0^{\theta_1}\frac{\sin^4\theta}{\cos^2\theta}\,d\theta = \tfrac{1}{4}\cos\theta_1\int_0^{\theta_1}(2\cos^2\theta - 1 - 9\cos^2\theta_1)\frac{\sin^4\theta}{\cos^2\theta}\,d\theta$$
$$+\tfrac{9}{8}\gamma\int_0^{\theta_1}(\cos\theta-\cos\theta_1)(3\cos\theta_1-\cos\theta)^2\sin^2\theta\,d\theta\,, \tag{75}$$

where

$$\theta_1 = \sin^{-1}(R/a)\,. \tag{76}$$

The various integrals in equation (75) are elementary and can be evaluated. And we find from the resulting equation (with σ^2 set equal to zero) that for any assigned value of θ_1 ($< \sec^{-1} 3 = 70\overset{\circ}{.}529$) dynamical instability must occur if γ is less than a certain γ_c (which depends on θ_1). The values of γ_c determined in this manner are listed in Table 1. Table 1 also includes the results of similar calculations based on the alternative

TABLE 1

THE LOWER LIMITS FOR γ FOR DYNAMICAL STABILITY OF A HOMOGENEOUS SPHERE OF CONSTANT ENERGY DENSITY ϵ FOR VARIOUS VALUES OF $Rc^2/2GM$ ($= R/R_S$)

θ_1	R/R_S	γ_c Evaluated with the Trial Function		θ_1	R/R_S	γ_c Evaluated with the Trial Function	
		$\xi=\eta e^{\nu/4}$	$\xi=\eta e^{\ /2}$			$\xi=\eta e^{\nu/4}$	$\xi=\eta e^{\nu/2}$
0°	∞	1.3333	1.3333	50°.	1.704	2.0899	2.0627
20.	8.549	1.3940	1.3938	55.	1.490	2.5204	2.4520
30.	4.000	1.4890	1.4879	60.	1.333	3.3703	3.1652
40.	2.420	1.6769	1.6704	65.	1.217	5.5802	4.7146
45.	2.000	1.8375	1.8257	70.529	1.125	∞	∞

trial function $\xi = \eta e^{\nu/4}$; it is seen that this trial function is somewhat "better" than $\xi = \eta e^{\nu/2}$, particularly for the larger values of θ_1.

It is evident that in the limit $\theta_1 \to 0$, the results obtained are *exact:* for, in this limit, the chosen trial functions tend to the true proper solutions. This is the reason for the close agreement of the results derived with the aid of the two trial functions as $\theta_1 \to 0$.

When $\theta_1 \to 0$, we find from equation (75) that

$$(a\sigma)^2 = \tfrac{1}{2}[(3\gamma-4) - \tfrac{1}{14}\theta_1^2(54\gamma-53)] \qquad (\theta_1\to 0)\,. \tag{77}$$

In this limit, the condition for dynamical instability is, therefore,

$$\gamma - \tfrac{4}{3} < \tfrac{19}{42}\theta_1^2 = \tfrac{19}{42}\frac{R^2}{a^2}. \tag{78}$$

Expressed in terms of the mass M, inequality (78) gives

$$\gamma - \tfrac{4}{3} < \tfrac{19}{42}\frac{2GM}{Rc^2} \tag{79}$$

or

$$R<\frac{19}{42(\gamma-\frac{4}{3})}\frac{2GM}{c^2}. \tag{80}$$

The meaning of this last inequality is that if γ should be in excess of $\frac{4}{3}$ by only a small amount, then dynamical instability will intervene when the mass has contracted to the radius given by the right-hand side of the inequality. The inequality (80) is the expression, in the present context, of the relation (4) mentioned in the Introduction.

VIII. RELATIVISTIC POLYTROPES

Tooper (1964) has recently constructed, in the framework of general relativity, models which are generalizations of the classical models of Lane and Emden. In these relativistic models, the pressure is assumed to be proportional to some power of the energy density (rather than the density as in the classical models). Expressing the pressure and the energy density in terms of a single function Θ in the manner

$$p = p_c\Theta^{n+1} \qquad \text{and} \qquad \epsilon = \epsilon_c\Theta^n \tag{81}$$

(where n is the "polytropic index" and the subscript c distinguishes the values at the center), and letting

$$r = \alpha\eta \qquad \text{where} \qquad \alpha = \left[\frac{(n+1)q\,c^4}{4\pi G\epsilon_c}\right]^{1/2} \qquad \text{and} \qquad q = \frac{p_c}{\epsilon_c}, \tag{82}$$ [3]

we may reduce equations (23) and (24) appropriately to obtain differential equations which will determine Θ as a function of η. Tooper has carried out the necessary reductions; and I am grateful to him for providing me with his tables of the function Θ in advance of publication.

In terms of the variables Θ and η, equation (60) becomes

$$\begin{aligned}\frac{\alpha^2\sigma^2}{q}\int_0^{\eta_1} e^{(3\lambda-\nu)/2}\Theta^n(1+q\Theta)\eta^2\xi^2 d\eta &= \gamma\int_0^{\eta_1} e^{(\lambda+3\nu)/2}\Theta^{n+1}\left[\frac{d}{d\eta}(\eta^2 e^{-\nu/2}\xi)\right]^2\frac{d\eta}{\eta^2}\\ &+4(n+1)\int_0^{\eta_1} e^{(\lambda+\nu)/2}\left[1-\frac{(n+1)q\eta}{4(1+q\Theta)}\frac{d\Theta}{d\eta}\right]\Theta^n\frac{d\Theta}{d\eta}\eta\xi^2 d\eta\\ &+2(n+1)q\int_0^{\eta_1} e^{(3\lambda+\nu)/2}\Theta^{2n+1}(1+q\Theta)\eta^2\xi^2 d\eta\,,\end{aligned} \tag{83}$$

where it may be noted that, in the present context, equation (21) takes the form

$$\frac{d\nu}{d\eta} = -\frac{2(n+1)q}{1+q\Theta}\frac{d\Theta}{d\eta}. \tag{84}$$

Also, γ has been assumed to be a constant.

With the trial functions

$$\xi = \eta e^{\nu/4} \qquad \text{and} \qquad \xi = \eta\,, \tag{85}$$

the values of γ at which marginal stability occurs, for some assigned values q, were determined with the aid of equation (83) for the case $n = 3$. The results of the calculation are given in Table 2. We observe that while the trial function $\xi = \eta$ is slightly "better" than $\xi = \eta e^{\nu/4}$, the differences are insignificant in all cases of physical interest ($\gamma \leq \frac{5}{3}$).

[3] The notations used in this paper are slightly different from Tooper's: what he denotes by θ, ξ, and σ are here denoted by Θ, η, and q, respectively.

a) The Post-Newtonian Approximation

In a post-Newtonian approximation, in which the effects of general relativity are treated as first-order corrections, we may expect relations of the forms

$$\gamma_c - \tfrac{4}{3} = C \frac{p_c}{\rho_c c^2} = Cq \tag{86}$$

and

$$R_c = \frac{K}{\gamma - \frac{4}{3}} \frac{2GM}{c^2}, \tag{87}$$

where C and K are constants which depend, principally, on the density distribution in the configuration. The results given in Tables 1 and 2 are in agreement with these expectations; and it appears, moreover, that the relations (86) and (87), which are expected to be valid only asymptotically, actually predict the onset of dynamical instability (for the polytrope $n = 3$) with unexpected precision for $\gamma \leq \frac{5}{3}$. It would, therefore, appear

TABLE 2

THE LOWER LIMITS FOR γ FOR DYNAMICAL INSTABILITY OF RELATIVISTIC POLYTROPES OF INDEX 3 FOR DIFFERENT VALUES OF $q = p_c/\epsilon_c$

q	γ_c EVALUATED WITH THE TRIAL FUNCTION	
	$\xi = \eta e^{\nu/4}$	$\xi = \eta$
0.015	1.3732	. .
040	1.4411	. .
100	1.6088	
0.200	1.8894	1.9144

that most cases of physical interest can be treated adequately in a post-Newtonian approximative treatment. Such a post-Newtonian treatment will now be considered for the polytropes.

It is shown in the Appendix that in the post-Newtonian approximation,

$$\Theta = \theta + q\phi$$
$$e^{-\lambda} = 1 + 2(n+1) q\eta \frac{d\theta}{d\eta}, \qquad \text{and} \qquad e^{\nu} = 1 - 2(n+1) q (\theta + \eta_1 |\theta_1'|), \tag{88}$$

where ϕ is a function which is tabulated in the Appendix for $n = 1, 2,$ and 3; $\theta(\eta)$ is the classical Lane-Emden function; η_1 its first zero; and $|\theta_1'| = -(d\theta/d\eta)_{\eta=\eta_1}$.

Using the relations in the theory of the polytropes which express p_c and ρ_c in terms of M and R (or more immediately from the fact that at the boundary $e^{-\lambda} = 1 - 2GM/Rc^2$), we find that

$$q = \frac{1}{2(n+1)\eta_1 |\theta_1'|} \frac{2GM}{Rc^2}. \tag{89}$$

The constants C and K in equations (86) and (87) are, therefore, related (in the case of the polytropes) by

$$K = \frac{C}{2(n+1)\eta_1 |\theta_1'|}. \tag{90}$$

Inserting now the relations (88) in equation (83), expanding the various quantities in a power series in q, and neglecting all terms which are of order higher than the first in q, we obtain after some reductions

$$\begin{aligned}
&\frac{a^2\sigma^2}{q}\left[\int_0^{\eta_1}\theta^n\eta^2\xi^2 d\eta + q\int_0^{\eta_1}\theta^{n-1}F(\eta)\,\eta^2\xi^2 d\eta\right]\\
&=\gamma\int_0^{\eta_1}\theta^{n+1}\left[\frac{d}{d\eta}(\eta^2\xi)\right]^2\frac{d\eta}{\eta^2}+4(n+1)\int_0^{\eta_1}\theta^n\frac{d\theta}{d\eta}\,\eta\xi^2 d\eta\\
&+(n+1)\gamma q\int_0^{\eta_1}\theta^n\left[\phi-\theta\left(\theta+\eta\frac{d\theta}{d\eta}+\eta_1|\theta_1'|\right)\right]\left[\frac{d}{d\eta}(\eta^2\xi)\right]^2\frac{d\eta}{\eta^2}\\
&+2(n+1)\gamma q\int_0^{\eta_1}\theta^{n+1}\frac{d\theta}{d\eta}\,\xi\,\frac{d}{d\eta}(\eta^2\xi)\,d\eta+(n+1)q\int_0^{\eta_1}\theta^{n-1}S(\eta)\,\eta\xi^2 d\eta\,,
\end{aligned}\tag{91}$$

where

$$F(\eta)=-3(n+1)\eta\theta\frac{d\theta}{d\eta}+n\phi+(n+2)\theta^2+(n+1)\eta_1|\theta_1'|\theta \tag{92}$$

and

$$\begin{aligned}
S(\eta)=&-4(n+1)\theta\frac{d\theta}{d\eta}\left(\theta+\eta\frac{d\theta}{d\eta}+\eta_1|\theta_1'|\right)+4n\phi\frac{d\theta}{d\eta}\\
&+4\theta\frac{d\phi}{d\eta}-(n+1)\eta\theta\left(\frac{d\theta}{d\eta}\right)^2+2\theta^{n+2}\eta\,.
\end{aligned}\tag{93}$$

It is known that in the Newtonian limit dynamical stability requires that $\gamma > \frac{4}{3}$. Therefore, in the post-Newtonian approximation, dynamical stability will require that $\gamma > \gamma_c = \frac{4}{3} + \epsilon$, where ϵ is a small quantity which will depend linearly on q (in a first-order theory such as the present). It is also known that in the Newtonian limit the proper solution at marginal stability is proportional to η. Consequently, with the trial function $\xi = \eta$ we should obtain from equation (91) the *exact* condition for marginal stability in the post-Newtonian approximation. And equation (91) with $\xi = \eta$ and $\sigma^2 = 0$ gives

$$\begin{aligned}
&9\left(\gamma-\tfrac{4}{3}\right)\int_0^{\eta_1}\theta^{n+1}\eta^2 d\eta\\
&+(n+1)q\left[\int_0^{\eta_1}\theta^{n-1}S(\eta)\,\eta^3 d\eta+\gamma\int_0^{\eta_1}\theta^n R(\eta)\,\eta^2 d\eta\right]=0\,,
\end{aligned}\tag{94}$$

where

$$R(\eta)=9\left[\phi-\theta\left(\theta+\eta\frac{d\theta}{d\eta}+\eta_1|\theta_1'|\right)\right]+6\eta\theta\frac{d\theta}{d\eta}. \tag{95}$$

Equation (94) leads to the following criterion for the onset of dynamical instability:

$$\gamma_c-\tfrac{4}{3}=-q\frac{5-n}{9\eta_1^3|\theta_1'|^2}\left[\int_0^{\eta_1}\theta^{n-1}S(\eta)\,\eta^3 d\eta+\tfrac{4}{3}\int_0^{\eta_1}\theta^n R(\eta)\,\eta^2 d\eta\right], \tag{96}$$

where we have made use of the relation

$$\int_0^{\eta_1}\theta^{n+1}\eta^2 d\eta=\frac{n+1}{5-n}\,\eta_1^3|\theta_1'|^2, \tag{97}$$

well known in the theory of the polytropes.

We observe that equation (96) is in agreement with the expectation expressed in equation (86). Moreover, for the constant C we now have the value

$$C = -\frac{5-n}{9\eta_1^3|\theta_1'|^2}\left[\int_0^{\eta_1}\theta^{n-1}S(\eta)\eta^3 d\eta + \tfrac{4}{3}\int_0^{\eta_1}\theta^n R(\eta)\eta^2 d\eta\right]; \tag{98}$$

and the value of the constant K in the associated relation (87) follows from equation (90).

The values of the constants C and K were determined in accordance with equations (90) and (98) for the cases $n = 1$, 2, and 3; these values are listed in Table 3. The values of the same constants for the homogeneous sphere derived in Section VII are also included in Table 3 under $n = 0$.

IX. CONCLUDING REMARKS

The principal result established in this paper is the demonstration that, in the framework of general relativity, dynamical instability by a mode of radial oscillation will intervene before a gaseous mass contracts to the limiting radius compatible with hydrostatic equilibrium. Under most natural circumstances, when a mass approaches a state where effects arising from general relativity may be expected to play a role, the ratio of the specific heats γ also approaches the value $\frac{4}{3}$ so that dynamical instability (under these circumstances) will, in fact, intervene long before the Schwarzschild limit is reached.

TABLE 3

THE VALUES OF THE CONSTANTS C AND K IN THE ASYMPTOTIC RELATIONS (86) AND (87) FOR DIFFERENT POLYTROPES

n	C	K
0	1 8095	0 4524
1	2.2615	0 5654
2	2 4968	0 7513
3	2 6325	1 1245

As an illustrative example, consider a gaseous configuration with a mass currently assigned to the quasi-stellar radio sources, namely, 10^8 solar masses. For a mass as large as this, the gas pressure can only be a minute fraction ($= 4.3 \times 10^{-4}$) of the total pressure; and this fact entails the consequence that the effective ratio of the specific heats differs from $\frac{4}{3}$ by only 7.2×10^{-5}. If we assume that the configuration is a polytrope of index 3, then dynamical instability will occur when the mass has contracted to the radius (see Table 3)

$$R = \frac{1.1245}{\gamma - \frac{4}{3}}\frac{2GM}{c^2} = 4.7 \times 10^{17}\ \text{cm} = 0.5\ \text{light-year}, \tag{99}$$

and it is suggestive that this is of the same order as the radii presently estimated for the quasi-stellar objects.

A second example which illustrates the theory is the occurrence of dynamical instability along the sequence of the degenerate white-dwarf configurations before the limiting mass is reached and when the mean density is still only 6×10^8 gm/cm^3 (Chandrasekhar and Tooper 1964).

Finally, it should be emphasized that the instability considered in this paper is entirely relativistic in its origin. An unambiguous demonstration that the instability is manifested in nature when the conditions for its occurrence, as required by equation (61), are fulfilled will provide a unique confirmation for the general theory of relativity.

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APPENDIX

THE POST-NEWTONIAN APPROXIMATION FOR POLYTROPIC EQUILIBRIUM

When the pressure p and the energy-density ϵ in a configuration are related in the manner (cf. eq. [81] in the text)

$$p = p_c \Theta^{n+1} \quad \text{and} \quad \epsilon = \epsilon_c \Theta^n , \tag{A1}$$

the equations governing hydrostatic equilibrium (namely, eqs. [23] and [24]) can be reduced to the pair of equations (cf. Tooper 1964)

$$\frac{1 - 2(n+1) q V/\eta}{1 + q\Theta} \eta^2 \frac{d\Theta}{d\eta} + V + q\eta\Theta \frac{dV}{d\eta} = 0 \tag{A2}$$

and

$$\frac{dV}{d\eta} = \eta^2 \Theta^n , \tag{A3}$$

where $q = p_c/\epsilon_c$ and η is the dimensionless radial coordinate defined in equation (82).

In the post-Newtonian approximation, we treat q as a small constant and seek solutions of equations (A2) and (A3) of the forms

$$\Theta = \theta + q\phi + O(q^2) \quad \text{and} \quad V = v + qw + O(q^2) , \tag{A4}$$

where θ is the classical Lane-Emden function satisfying the equations

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\theta}{d\eta} \right) = -\theta^n \tag{A5}$$

and

$$v = -\eta^2 \frac{d\theta}{d\eta} . \tag{A6}$$

Inserting for Θ and V the forms assumed and neglecting all quantities of orders higher than the first in q, we obtain from equations (A2) and (A3)

$$\eta^2 \frac{d\phi}{d\eta} - \left[\theta - 2(n+1)\eta \frac{d\theta}{d\eta} \right] \eta^2 \frac{d\theta}{d\eta} + \eta^3 \theta^{n+1} + w = 0 \tag{A7}$$

and

$$\frac{dw}{d\eta} = n\eta^2 \theta^{n-1} \phi . \tag{A8}$$

Eliminating w between equations (A7) and (A8), we find after some reductions that

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = -n\theta^{n-1}\phi - \left[4\theta^{n+1} - 3(n+1)\eta\theta^n \frac{d\theta}{d\eta} - (2n+3)\left(\frac{d\theta}{d\eta} \right)^2 \right] . \tag{A9}$$

And the boundary conditions with respect to which equation (A9) must be solved are

$$\phi = \frac{d\phi}{d\eta} = 0 \quad \text{at} \quad \eta = 0 . \tag{A10}$$

Solutions of equation (A9) satisfying the boundary conditions (A10) have been obtained by numerical integration for the cases $n = 1$, 2, and 3; these solutions together with their first derivatives are tabulated in Appendix Table 1.

APPENDIX TABLE 1

THE FUNCTIONS ϕ AND $d\phi/d\eta$

η	ϕ	$d\phi/d\eta$
	$n=1$	
0 .	0	0
0 1	−0 00666	−0 13322
0 2	−0 02662	−0 26577
0 3	−0 05977	−0 39693
0 4	−0 10593	−0 52591
0 5	−0 16485	−0 65184
0 6	−0 23617	−0 77372
0 7	−0 31942	−0 89038
0 8	−0 41403	−1 00053
0 9	−0 51926	−1 10275
1 0	−0 63426	−1 19547
1 1	−0 75799	−1 27701
1 2	−0 88923	−1 34564
1 3	−1 02662	−1 39960
1 4	−1 16861	−1 43715
1 5	−1 31345	−1 45665
1 6	−1 45928	−1 45659
1 7	−1 60408	−1 43569
1 8	−1 74569	−1 39293
1 9	−1 88191	−1 32762
2 0	−2 01045	−1 23948
2 1	−2 12905	−1 12864
2 2	−2 23544	−0 99570
2 3	−2 32749	−0 84176
2 4	−2 40315	−0 66842
2 5	−2 46059	−0 47776
2 6	−2 49821	−0 27231
2 7	−2 51466	−0 05508
2 8	−2 50894	+0 17058
2 9	−2 48039	+0 40099
3 0	−2 42872	+0 63224
3 1	−2 35405	+0 86025
η_1	−2 31633	+0 95316
	$n=2$	
0 .	0	0
0 1	−0 00666	−0 13311
0 2	−0 02658	−0 26487
0 3	−0 05954	−0 39386
0 4	−0 10521	−0 51854
0 5	−0 16305	−0 63722
0 6	−0 23239	−0 74808
0 7	−0 31234	−0 84916
0 8	−0 40183	−0 93851
0 9	−0 49959	−1 01424
1 0	−0 60417	−1 07463
1 1	−0 71396	−1 11824
1 2	−0 82722	−1 14406
1 3	−0 94215	−1 15150
1 4	−1 05691	−1 14054
1 5	−1 16966	−1 11164
1 6	−1 27867	−1 06581
1 7	−1 38231	−1 00449
1 8	−1 47912	−0 92952
1 9	−1 56783	−0 84304
2 0	−1 64742	−0 74738
2 1	−1 71708	−0 64494
2 2	−1 77626	−0 53815
2 3	−1 82464	−0 42934
2 4	−1 86213	−0 32068
2 5	−1 88885	−0 21413
2 6	−1 90509	−0 11140
2 7	−1 91131	−0 01391
2 8	−1 90808	+0 07717
2 9	−1 89612	+0 16097
3 0	−1 87615	+0 23688
3 1	−1 84901	+0 30454
3 2	−1 81553	+0 36378
3 3	−1 77654	+0 41464
3 4	−1 73287	+0 45730
3 5	−1 68534	+0 49207
3.6	−1 63471	+0 51936
3 7	−1 58170	+0 53966
3 8	−1 52699	+0 55350
3 9	−1 47120	+0 56144
4 0	−1 41488	+0 56404
4 1	−1 35855	+0 56186
4 2	−1 30265	+0 55546
4 3	−1 24758	+0 54535
η_1	−1 21892	+0 53868
	$n=3$	
0 .	0	0
0.1	−0 00666	−0 13300
0 2	−0 02653	−0 26397
0 3	−0 05931	−0 39080
0 4	−0 10448	−0 51123
0 5	−0 16127	−0 62284
0 6	−0 22867	−0 72316
0 7	−0 30544	−0 80985
0 8	−0 39011	−0 88076
0 9	−0 48101	−0 93423
1 0	−0 57634	−0 96918
1 1	−0 67422	−0 98522
1 2	−0 77277	−0 98266
1 3	−0 87017	−0 96253
1 4	−0 96474	−0 92639
1 5	−1 05498	−0 87628
1 6	−1 13960	−0 81454
1 7	−1 21758	−0 74364
1 8	−1 28811	−0 66608
1 9	−1 35065	−0 58424
2 0	−1 40489	−0 50030
2 1	−1 45071	−0 41620
2 2	−1 48818	−0 33356
2 3	−1 51752	−0 25375
2 4	−1 53906	−0 17780
2 5	−1 55323	−0 10650
2 6	−1 56053	−0 04038
2 7	−1 56149	+0 02024
2 8	−1 55667	+0 07524
2 9	−1 54663	+0 12461
3 0	−1 53193	+0 16848
3 1	−1 51311	+0 20706
3 2	−1 49069	+0 24063
3 3	−1 46514	+0 26951
3 4	−1 43693	+0 29403
3 5	−1 40647	+0 31457
3 6	−1 37414	+0 33146
3 7	−1 34028	+0 34508
3 8	−1 30522	+0 35575
3 9	−1 26922	+0 36380
4 0	−1 23254	+0 36952
4 1	−1 19539	+0 37321
4 2	−1 15796	+0 37511
4 3	−1 12042	+0 37546
4 4	−1 08291	+0 37449
4 5	−1 04556	+0 37237
4 6	−1 00847	+0 36929
4 7	−0 97173	+0 36539
4 8	−0 93541	+0 36082
4 9	−0 89958	+0 35570
5 0	−0 86429	+0 35013
5 1	−0 82957	+0 34421
5 2	−0 79545	+0 33802
5 3	−0 76197	+0 33164
5 4	−0 72913	+0 32512
5 5	−0 69695	+0 31851
5 6	−0 66543	+0 31187
5 7	−0 63457	+0 30523
5 8	−0 60438	+0 29862
5 9	−0 57485	+0 29207
6 0	−0 54596	+0 28561
6 1	−0 51772	+0 27925
6 2	−0 49011	+0 27300
6 3	−0 46312	+0 26688
6 4	−0 43673	+0 26091
6 5	−0 41093	+0 25508
6 6	−0 38571	+0 24940
6 7	−0 36105	+0 24386
6 8	−0 33693	+0 23849
η_1	−0 31408	+0 23343

The expressions for the metric coefficients e^{λ} and e^{ν} are, in general, given by (cf. Tooper 1964)

$$e^{-\lambda}=1-2(n+1)q\frac{V(\eta)}{\eta} \quad \text{and} \quad e^{\nu}=\frac{1-2(n+1)qV(\eta_1)/\eta_1}{(1+q\Theta)^{2(n+1)}}, \tag{A11}$$

where η_1 is the first zero of Θ. In the post-Newtonian approximation, the foregoing expressions become

$$e^{-\lambda}=1+2(n+1)q\eta\frac{d\theta}{d\eta} \quad \text{and} \quad e^{\nu}=1-2(n+1)q(\theta+\eta_1|\theta_1'|), \tag{A12}$$

where η_1 is now the first zero of θ and $|\theta_1'| = -(d\theta/d\eta)_{\eta=\eta_1}$.

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