ON THE DISTRIBUTION OF HIGH ENERGY STARS IN SPHERICAL STELLAR SYSTEMS

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Summary

A distribution function $f(r, v, \mu; t)$ for isolated spherical stellar systems is obtained from the Boltzmann equation with encounters described by the Fokker-Planck equation. The distribution function is believed to correspond rather closely to actual stellar systems since it is obtained from the Boltzmann equation, the potential is obtained from Poisson's equation, and the stellar orbits are not assumed to be isotropic everywhere but rather are more radial at greater distances from the centre. The paper emphasizes the importance of a careful analysis in the region of phase space at and near the energy of escape. In this region it is shown that the velocity space flux vector is constant, and it is this constancy which allows a solution for f. The distribution of high energy stars is depopulated for (i) those stars whose mass is small compared to the average stellar mass, (ii) regions close to the centre of the system, and (iii) large values of the model parameter C. It is proposed that the method of analysis presented in this study may be used for obtaining a distribution function for rotating stellar systems.

1. *Introduction.*—The purpose of this paper is to obtain a distribution function for spherical stellar systems, using a method of analysis which does not require the full solution of the Boltzmann equation for all values of the independent variables. The basic theoretical observation concerns the flux of representative points in phase space, for energies at and slightly less than the energy of escape. The analysis will include a range of masses for the individual stars, an increasing velocity space anisotropy at greater distances from the centre of the system, and a "smoothed out" potential consistent with the distribution function for all values of energy and angular momentum. Before beginning the theoretical development, it will be advantageous to review certain topics in the dynamics of spherical stellar systems.

The total distribution function f for an isolated system can be written as a sum,

$$f = \sum_i f_i(\mathbf{r}, \mathbf{v};\ t), \tag{1.0}$$

where each f_i is the distribution of representative points in phase space for stars of mass m_i. Each distribution function corresponding to a certain stellar mass must satisfy the Boltzmann equation,

$$\frac{\partial f_i}{\partial t} + \mathbf{v}.\frac{\partial f_i}{\partial \mathbf{r}} - \nabla\Phi.\frac{\partial f_i}{\partial \mathbf{v}} = \sum_j \left(\frac{\partial f_i}{\partial t}\right)_{\text{ENC}}. \tag{1.1}$$

The term on the right hand side gives the time rate of change of f_i due to encounters, and must be summed for encounters with the various masses m_j. On the left

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side of equation (1.1) there occurs the gradient of the total potential Φ which is obtained from Poisson's equation,

$$\nabla^2\Phi = 4\pi G \sum m_j \int f_j \, d\mathbf{v} = 4\pi G \sum_j \rho_j. \qquad (1.2)$$

Again, there must be a sum of the mass densities ρ_j corresponding to mass m_j. For a spherical system we may write $f_i = f_i(r, v, \mu; t)$ with $\mu = \cos\theta$, and θ the angle between the radius vector and velocity vector. Equation (1.1) now simplifies to

$$\frac{\partial f_i}{\partial t} + \mu v \frac{\partial f_i}{\partial r} + \mu a_r \frac{\partial f_i}{\partial v} + (1-\mu^2)\left(\frac{a_r}{v} + \frac{v}{r}\right)\frac{\partial f_i}{\partial \mu} = \sum_j \left(\frac{\partial f_i}{\partial t}\right)_{\text{ENC}}, \qquad (1.3)$$

where $a_r = -|\nabla\Phi|$ is the force per unit mass which is in the radial direction. The encounter term involves second order derivatives with respect to μ and v. But the essential complication is that it is non-linear in the dependent variable f_i. Another difficulty, aside from the encounter term just mentioned, is that the non-linearity in f_i also occurs through equation (1.2). It certainly is clear that the above equations (1.2) and (1.3) which must be solved simultaneously, are very complex. Present day computers have storage capacities sufficiently large and machine-cycle times sufficiently short almost to allow an attempt at solving these non-linear integrodifferential equations.

Until such computing facilities are available, we must be content with certain approximations which usually fall into three categories; (1) the field stars are often approximated by a Maxwellian velocity distribution; (2) a particular form for the potential is sometimes assumed; (3) a specific form for the distribution function may be chosen, usually in terms of E and $\mathbf{J}$. The first type of approximation is easily justified by the small value for the energy relaxation time in the inner portions of the system. Where the actual distribution function differs significantly from

$$f = A \exp[-(jv)^2] \qquad (1.4)$$

the effects of encounters are negligible. The type (2) approximation has been employed by Spitzer and Härm (1958) and also by King (1958a, b, 1960). Spitzer and Härm used a square-well potential, and King used polytropes of index 3 and 5 as well as a constant potential. Both Oort and van Herk (1959) and Michie (1961) assumed a specific form for $f(E, \mathbf{J})$ and used a truncated Maxwellian (field star) velocity distribution. They then found the density consistent with the chosen distribution function. Other authors have studied particular regions of the system; the distribution function for stars in the outer regions of a cluster has been studied by Woolley and Robertson (1956) and also by von Hoerner (1957). These authors recognized the problem of populating the outer regions by stars suffering encounters near the centre and being thrown into orbits of high energy and low angular momentum.

It is evident that from these studies a great deal has been learned concerning the structure of spherical stellar systems. But the basic problem is yet to be solved; obtaining the total distribution function $f(r, v, \mu; t)$ by simultaneously solving the Boltzmann equation and Poisson's equation. The only available solutions of these equations employ a constant potential which reduces the problem to solving the Boltzmann equation alone for $f(v; t)$. Nevertheless, the use of a square-well potential by Spitzer and Härm (1958) gave important information concerning the high energy tail of the distribution function for stars moving in a constant potential. King, also using a square-well potential (1960), allowed the stars to encounter each other instead of a Maxwellian distribution, and obtained $f(v; t)$ similar to the results just mentioned; but he found a somewhat greater rate of mass loss.

A simple time-scale comparison gives (qualitatively) some information concerning the dependence of f upon energy. As we look at stars with progressively higher energies, the distribution function will tend toward zero, and it will become very small at the energy of escape. In the inner regions of a globular cluster the escape velocity is of the order 10–20 km/sec, the exact value depending on the total mass and degree of central concentration. In 10^7 years, a star with sufficient energy to escape may travel a distance well beyond the "radius" R, and is therefore effectively gone from the system. But this "escape time" is at least an order of magnitude less than the relaxation time in the regions where the stars suffer the greater rate of encounters. Therefore a star, having obtained through encounters sufficient energy to escape, can be considered immediately gone on the dynamical time scale; from these considerations, we expect the actual distribution function to exhibit a strong depopulation of high energy stars as compared to an equilibrium distribution. In their analysis using a square-well potential, Spitzer and Härm obtained the energy cut-off function for various masses, with the depopulation of high energy stars being greater. the smaller the mass.

Aside from the intrinsic theoretical interest in the distribution of high energy stars, there are two reasons why such a knowledge of this distribution is important in stellar dynamics. First, for an isolated system, the cutoff has little effect on the space density in the inner regions. At larger distances from the centre, certainly beyond the sphere containing half the total mass, the cutoff does become important in determining the space density; and therefore it is necessary to have a realistic distribution of high energy stars in the outer regions. Secondly, the rate of escape of stars depends on $\partial f/\partial v$ evaluated at the escape velocity. Since most of the stars leave the system as a result of encounters in the inner regions, it is necessary to have knowledge of the distribution of high velocity stars close to the centre. Throughout then, it is important to know the distribution function for energy at and slightly less than the escape energy. In this article, a distribution function for spherical systems will be obtained which does not include any approximation of the potential; will give increasingly radial orbits at larger distances from the centre; will include a range of stellar masses; and will satisfy the Boltzmann equation for large and small values of the total energy E.

2. *Theory*.—For a cluster with (say) 10^5 stars, approximately one percent is lost during a relaxation time which gives a "production rate" of about one star in 10^6 years. Since the "escape time" is of the order 10^6–10^7 years, the expected number of stars with escape energy is of the order 10 or fewer. Evidently it is quite valid to put

$$f(r, v, \mu; t) \approx 0, \tag{2.0}$$

for stars with $v \geqslant v_e$ (the velocity of escape). Furthermore, a star with escape energy has little chance of suffering a significant number of encounters while leaving the system. For stars with escape energy, the mean-free-path is at least three orders of magnitude greater than the "size" of the system

(Chandrasekhar 1942). Therefore, we can safely neglect any "back" diffusion of representative points in phase space with speeds $v \geqslant v_e$. For a boundary condition at v_e, we may equate the distribution function to zero, and this should be quite adequate. But the object in this study is not to make any definite statement about the value of f at and near v_e (except to say that it is quite small), but to study the derivative of f with respect to v at this boundary in phase space. We shall later impose equation (2.0), but it must be stressed that it is $\partial f/\partial v$ at v_e which is of primary physical interest, and not f itself.

Now the distribution function for stars at (r, v_e, μ) at time $t+\delta t$ is related to $f(r, v_e-\delta v, \mu-\delta\mu; t)$ by

$$f(r, v_e, \mu; t+\delta t) = \int\int f(r, v_e-\delta v, \mu-\delta\mu; t)\, \Psi\, (v_e-\delta v, \mu-\delta\mu; \delta v, \delta\mu)\, d(\delta v)\, d(\delta\mu), \tag{2.1}$$

and the integration is extended over the region $0 \leqslant v \leqslant v_e$. We must insure that, δt is sufficiently large compared to the time interval of the force fluctuations, but short enough so that δv and $\delta\mu$ are small compared to v_e and μ. In equation (2.1), Ψ is the transition probability that $\mathbf{v}$ suffers an increment $\boldsymbol{\delta}\mathbf{v}$ in time δt due to encounters. If we expand the functions under the integral sign in Taylor series, there results

$$f(r, v_e, \mu; t+\delta t) =$$
$$\int\int\left\{\left[f-\delta v\frac{\partial f}{\partial v}-\delta\mu\frac{\partial f}{\partial\mu}+\dots\right]\left[\Psi-\delta v\frac{\partial\Psi}{\partial v}-\delta\mu\frac{\partial\Psi}{\partial\mu}+\dots\right]\right\}_{|v_e,\mu} d(\delta v)\, d(\delta\mu). \tag{2.2}$$

On the right hand side we will get, among other terms involving derivatives, the term

$$\int\int f(r, v_e, \mu; t)\Psi(v_e, \mu; \delta v, \delta\mu)\, d(\delta v)\, d(\delta\mu) = f(r, v_e, \mu; t). \tag{2.3}$$

Can we equate f identically to zero at the escape boundary in phase space for all time t? If so, then obviously $(\partial f/\partial t)_{\mathrm{ENC}}$ is zero for all t, and this would impose certain conditions on the various derivatives of f at v_e. It is not entirely obvious that this would lead to correct results during (say) the later stages of evolution when the evolutionary rate may be rather high. If during some time of evolution there is a significant divergence of representative points at the escape boundary, then $(\partial f/\partial v)_{|v_e}$ will be affected, and this, in turn, will affect the rate of escape of stars from the system. To see this, let us write the encounter term in the form

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{ENC}} = -\nabla_v\,.\,\mathbf{F} = -\frac{1}{v^2}\frac{\partial}{\partial v}(v^2F_v) + \frac{1}{v}\frac{\partial}{\partial\mu}(\sqrt{1-\mu^2}F_\mu), \tag{2.4}$$

which defines the flux vector $\mathbf{F}$. For an isotropic distribution of field stars we find using (2.0),

$$\mathbf{F}_{|v_e} = F_{v|v_e}\,\mathbf{e}_v = -\mathbf{e}_v\frac{1}{2}\Gamma\left(\frac{d^2g}{dv^2}\frac{\partial f}{\partial v}\right)_{|v_e}, \tag{2.5}$$

where Γ is a constant, and $(d^2g/dv^2)_{|v_e}$ is a function of r (cf. equations (2.8), (2.9) and (2.12)). The number of stars leaving the system per unit volume and per unit time now follows directly from the above:

$$-\frac{dm}{dt} = \int\nabla_v\,.\,\mathbf{F}d\mathbf{v} = \int\mathbf{F}\,.\,d\mathbf{S}_v = \int_{-1}^{1} 2\pi v_e^2 F_{v|v_e} d\mu. \tag{2.6}$$

As an illustration, suppose there were a negative flux divergence at v_e. Then $(\partial f/\partial v)_{|v_e}$ would increase, and so would $|dn/dt|$. All this again illustrates the fact that it is critical to obtain accurately the *shape* of f at high energies, while the "exact" value of f is of less concern. So, if we should consider f and $\nabla_v \cdot \mathbf{F}$ as strictly zero at v_e, we might possibly commit an error during some part of the *evolution* of the system. The only way to answer the question is to look at the kinematical and dynamical terms in the Boltzmann equation. At the energy of escape, the dominant term is $(\partial f/\partial E)(\partial \Phi/\partial t)$. Hence, if we can be assured that for all r and during all time t this term is negligible when compared to the corresponding term in equation (2.4), then we could expect little error by putting $f \nabla_v \cdot \mathbf{F}$ equal to zero at the v_e boundary in phase space. In Sections 3 and 4, an estimate of this error will be made; for the time being we will write,

$$\left(\frac{\partial f}{\partial t}\right)_{\text{ENC}} |_{v_e} = \epsilon(r, \mu; t), \tag{2.7}$$

and proceed in the development of the necessary equations.

With a Maxwellian distribution of field star velocities (equation (1.4)) truncated at v_e, the Fokker-Planck equation has been put into the following form by Michie (1961):

$$\frac{1}{\Gamma}\left(\frac{\partial f}{\partial t}\right)_{\text{ENC}} = -\frac{1}{v^2}\frac{\partial}{\partial v}\left[f\left(v^2\frac{dh}{dv}+\frac{dg}{dv}\right)\right] + \frac{1}{2v^2}\frac{\partial^2}{\partial v^2}\left(fv^2\frac{d^2g}{dv^2}\right) + \frac{1}{2v^3}\frac{dg}{dv}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right], \tag{2.8}$$

where

$$\Gamma = 4\pi G^2 m_0^2 \log_e\left(\frac{D\langle v^2\rangle}{2m_0 G}\right), \tag{2.9}$$

G is the gravitational constant, and D is an upper limit to the impact parameter, necessary to avoid the well known divergence. For stars of mass m encountering stars of average mass m_0 and number density n_0,

$$\frac{dh}{dv} = 2\frac{n_0}{\sqrt{\pi}}\left(1+\frac{m}{m_0}\right)\left[\frac{j}{v}\,e^{-(jv)^2} - \frac{1}{v^2}F_1(jv)\right], \tag{2.10}$$

$$\frac{dg}{dv} = \frac{n_0}{\sqrt{\pi}}\left[-\frac{4}{3}jv\,e^{-(jv_e)^2} + F_1(jv)\left(2-\frac{1}{j^2v^2}\right) + \frac{1}{jv}\,e^{-(jv)^2}\right], \tag{2.11}$$

$$\frac{d^2g}{dv^2} = \frac{n_0}{\sqrt{\pi}}\left[-\frac{4}{3}je^{-(jv_e)^2} - \frac{2}{jv^2}\,e^{-(jv)^2} + \frac{2F_1(jv)}{j^2v^3}\right] \tag{2.12}$$

and

$$F_1(x) = \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\,erf(x). \tag{2.13}$$

We immediately see how complicated the problem is if we allow the stars to encounter each other. If this were mathematically allowed, then the resulting partial differential equation would be non-linear in the dependent variable f. To avoid this, some sort of assumption must be made concerning the field star velocity distribution; in this study we shall use a truncated Maxwellian distribution, and the equations just presented. A fuller discussion of this approximation and its restrictive consequences will be given in Section 4. But before turning to the analysis of a spherical stellar system with a varying potential, it is worth while to illustrate these ideas by working a simple problem—one which does not have the complications of a varying potential nor an anisotropic velocity distribution.

3. *The distribution function for a square-well potential.*—Let us apply equation (2.8) to the simple case of stars moving in a constant potential, and suffering encounters with field stars represented by a Maxwellian velocity distribution. Now we obtain, with $x=jv$,

$$\frac{\sqrt{\pi}x^2}{n_0 j^3\Gamma}\epsilon=f\left(4\frac{m}{m_0}x^2F_1'\right)+\frac{\partial f}{\partial x}\left[2\frac{m}{m_0}\times\left(\frac{F_0-xF_1'}{x}\right)+\frac{d}{dx}\left(\frac{F_1-xF_1'}{x}\right)+\frac{\partial^2 f}{\partial x^2}\left(\frac{F_1-xF_1'}{x}\right)\right]. \tag{3.0}$$

The equilibrium function for this isotropic velocity distribution is the Maxwellian distribution,

$$f=A\,e^{-m/m_0 x^2} \tag{3.1}$$

which obviously satisfies the Boltzmann equation. But this equilibrium distribution cannot be valid for *all* x; for if it were, there would be no diffusion of representative points past the boundary at x_e. For sufficiently small x however, the actual distribution will deviate but slightly from equation (3.1), and we expect the equilibrium distribution to be closely valid over a larger range in x, the larger the mass ratio. For stars with $m/m_0\approx 0$, their large velocity dispersion will produce strong deviations from equilibrium. Accordingly, for these stars, equation (3.1) will closely approximate the true distribution only for x near zero. Let us therefore confine our discussion to mass ratios which are not too small, put

$$f=A(t)\,e^{-m/m_0 x^2}Q(x), \tag{3.2}$$

and require $Q(x_e)\approx 0$. We may also normalize, and put $Q(0)$ equal to unity. Substitution of the above into equation (3.0) produces the following relation between the first and second derivatives of Q:

$$\frac{\epsilon\sqrt{\pi}x^2}{n_0 j^3\Gamma A}\,e^{m/m_0 x^2}=\frac{dQ}{dx}\left[\frac{d}{dx}\left(\frac{F_1-xF_1'}{x}\right)-2\frac{m}{m_0}\times\left(\frac{F_1-xF_1'}{x}\right)\right]+\frac{d^2Q}{dx^2}\left(\frac{F_1-xF_1'}{x}\right). \tag{3.3}$$

For the very small mass stars, all we can obtain is the above relation between dQ/dx and d^2Q/dx^2 at x_e. If we know the value of Q at some x very near to x_e, then the two boundary values would allow a solution for $Q(x)$ in the region near the escape boundary. But to get this information would require the solution of the Boltzmann equation. The situation, however, is more favourable for the more massive stars. If the mass ratio is not too small, then Q will be close to unity over a rather large range of x, and only near x_e will Q start decreasing to zero. Also, for these more massive stars, f itself drops faster to zero as x increases towards x_e. With f and $\partial f/\partial x$ both small at and near the escape boundary, we can expect $\nabla_v\cdot\mathbf{F}\approx 0$, and hence ϵ to be near zero in this region. Therefore, we may apply equation (3.3) with $\epsilon\approx 0$ not only at x_e, but in the neighbourhood of x_e as well. Now as we begin the solution, $Q(x)$ rapidly rises from zero at x_e and approaches unity rather soon. For the second boundary value, if the mass ratio is large enough, it will be sufficient to put

$$Q(0)=1\cdot 0. \tag{3.4}$$

This boundary value, together with

$$Q(x_e)=0\cdot 0, \tag{3.5}$$

will allow a solution for Q in the neighbourhood of x_e. To summarize this discussion, the stars with a small mass ratio show strong deviations from the equilibrium distribution over nearly the whole range of x, and so at x_e we can only obtain a relation between the first and second derivatives of Q (or f); our lack of information allows only one boundary value [equation (3.5)] to be applied. For more massive stars, the significant deviation from equilibrium occurs only in the neighbourhood of x_e, where f is small and slowly varying. Accordingly, we may integrate equation (3.3) over a region near x_e, and use the two boundary values given above to solve for Q itself. If m/m_0 is not too small, we can expect to obtain f near x_e rather accurately, with a greater accuracy the larger the mass ratio.

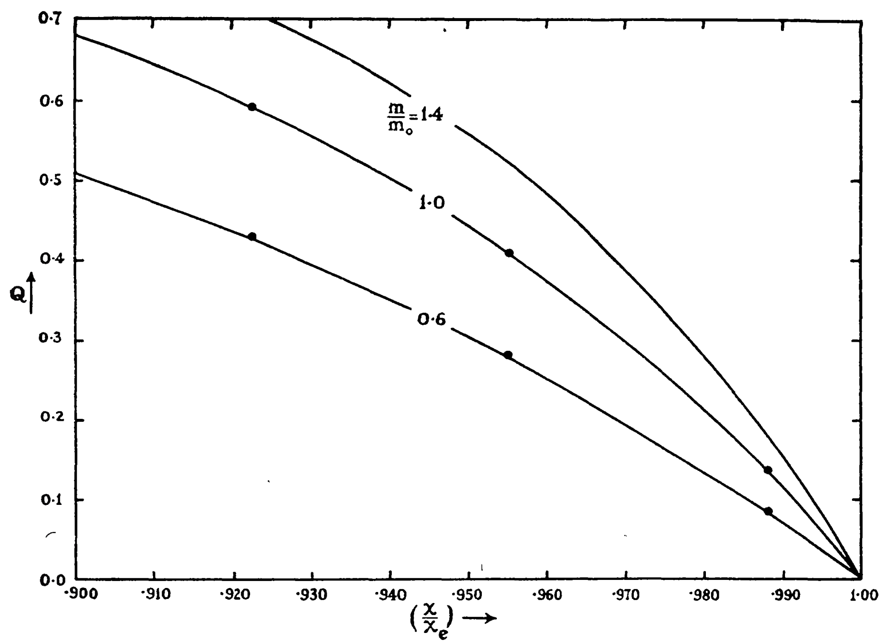

FIG. 1.—*The cut-off function for a spherical cluster with constant gravitational potential. The full drawn curves are plots of equation* (3.7) *for different values of the stellar mass ratio. Points labelled with* ● *are from the solution of the Boltzmann equation, obtained by Spitzer and Härm* (1958).

Since by the Virial Theorem $x_e^2 = 6$, keeping the dominant terms in equation (3.3) results in

$$\frac{2\epsilon x^4}{n_0 j^3 \Gamma A} e^{m/m_0 x^2} = x \frac{d^2 Q}{dx^2} - \left(1 + 2\frac{m}{m_0} x^2\right) \frac{dQ}{dx}. \tag{3.6}$$

For $\epsilon = 0$, the above equation is easily integrated to

$$Q(x) = \frac{1 - e^{-m/m_0 (x_e^2 - x^2)}}{1 - e^{-m/m_0 x_e^2}}, \tag{3.7}$$

and this clearly satisfies the boundary values at $x = 0$ and $x = x_e$. It should be noted that in the numerator there occurs an energy difference, while in the denominator, the term $\exp(-m/m_0 x_e^2)$. Thus the result for $Q(x)$ is rather insensitive to the exact place where the boundary values are applied. Figure 1 shows the comparison between this $Q(x)$ and the exact cut-off obtained by solving the Boltzmann equation for all x. We may draw attention to the fact that the results are virtually identical with the solution of the Boltzmann equation for $m/m_0 \geqslant 0{\cdot}5$, and also the wide range over which agreement is obtained. The exact calculations are slightly above the full curves for x less than $0{\cdot}95x_e$ but even for $m/m_0 = 0{\cdot}6$ there is only about 1 per cent deviation at $0{\cdot}90x_e$. The agreement becomes worse as the mass ratio decreases, for the reasons just discussed. Also, we notice a more severe depopulation of high energy stars with the smaller mass. This analysis of the square-well potential seems to indicate that a similar calculation for the distribution of high energy stars may be applied to an actual system with a varying potential, provided we restrict ourselves to stellar masses at least greater than one-half the average mass.

It is an easy matter to estimate the effect of ϵ. With a reference time T_R defined by Spitzer and Härm (1958),

$$(T_R)^{-1} = 2\pi G^2 m_0^2 j^3 n_0 \log_e\left(\frac{D\langle v^2\rangle}{2m_0 G}\right), \tag{3.8}$$

the left hand side of equation (3.6) now becomes

$$\epsilon \, \frac{T_R}{A} \, x^4 \, e^{m/m_0 x^2}. \tag{3.9}$$

We may approximate A by writing,

$$A \cong \frac{N_m j^3}{\pi^{3/2}}\left(\frac{m}{m_0}\right)^{3/2}, \tag{3.10}$$

where N_m is the total number of stars of mass m. Now for ϵ, let us take k times the mean divergence over all velocity space, or

$$|\epsilon| = k\,\frac{|\dot{N}_m|}{\frac{4}{3}\pi v_e^3} \cong k\,\frac{10^{-2} N_m}{T_R \frac{4}{3}\pi v_e^3}. \tag{3.11}$$

$\dot{N}_m$ is the rate of escape of stars of mass m from the system. We now find, for $m/m_0 \approx 1$,

$$|\epsilon|\,\frac{T_R}{A}\,x^4\,e^{x^2} \approx 10k, \tag{3.12}$$

and we see that for k equal to one per-cent of the mean divergence, the terms on the right hand side of equation (3.6) are about three orders of magnitude larger than the divergence term on the left. A small divergence of the flux at and near the phase space boundary affects the slope, but the amount is not important for this study. Even including the possibility of a small divergence of $\mathbf{F}$ at this region of phase space, which might occur in the later stages of evolution when the over-all escape rate is relatively large, it would seem that only a very slight error is introduced by using $\nabla_v . \mathbf{F} = 0$ at and near x_e.

4. *The distribution of high energy stars in a spherical system.*—A comparison of the calculations in the previous section with the solution of the Boltzmann equation indicated a very small difference between the two over the whole range of energy, provided the mass ratio was (say) greater than one-half. Furthermore, the results were insensitive to the exact place where the boundary values were applied; the "exact" value of f at v_e; and finally, a small divergence of flux of representative points at and near v_e. This analysis was done, however, for an isotropic velocity distribution and constant potential, and we now wish to relax these restrictions to study a more realistic system. But an inadequacy is immediately evident if we use a Maxwellian distribution of field star velocities;

the "interaction functions" h and g in the Fokker-Planck equation are now independent of μ, and hence derivatives such as

$$\frac{\partial h}{\partial \mu}, \frac{\partial^2 g}{\partial v \partial \mu}, \frac{\partial^2 f}{\partial v \partial \mu},$$

which normally would be present in the encounter term are now absent. A more exact discussion would have the stars interacting with each other, but if we remember that encounters are only important in the inner region, then it would seem that little error will be committed by making this field star approximation. In a previous study (Michie, 1961) the velocity space anisotropy was found to become important only beyond the sphere containing approximately half the mass of the system. For a globular cluster such as 47 Tuc, this corresponds to a distance of about 14 parsecs from the centre. We feel then some justification for using a Maxwellian field star distribution of velocities within a rather large region of the system. So although this approximation can be justified on these grounds, it is important to realize that by neglecting the small velocity anisotropy in the inner regions, we pay the price of not obtaining conditions which in part determine the dependence of f upon μ.

The more modest problem now is this: what is the distribution of energy consistent with an assumed velocity space anisotropy? If we pose this simpler question, then we can obtain information concerning the high energy stars in a set of clusters or galaxies, but must forego any attempt to obtain an accurate velocity anisotropy at very large distances from the centre. However, we will be able to get the information which, for example, is important for discussing the loss of stars of different mass from systems with a varying potential, systems which are self-supporting with different central concentration of stars, and built on different models.

We start this analysis by writing for f,

$$f(r, v, \mu; t) = A\, e^{-m/m_0(\alpha E + \beta J^2)} Q(r, v, \mu; t), \tag{4.0}$$

where

$$E = \tfrac{1}{2} v^2 + \Phi(r), \tag{4.1}$$

$$J^2 = r^2 v^2 (1 - \mu^2), \tag{4.2}$$

and

$$\Phi(0) = 0.0. \tag{4.3}$$

A, α and β are functions of time. By requiring

$$Q|_{E=0} = 1.0, \tag{4.4}$$

it is clear that f approaches the equilibrium distribution $\exp(-m/m_0 \alpha E)$ as E tends toward zero. Also, the assumed form of the anisotropy (the J^2 term) represents a depopulation of stars in the more circular orbits at large distances from the centre; the J^2 term produces an increasingly radial distribution of velocities at increasing distances from the centre. If we substitute (4.0) into equation (2.8) we obtain,

$$\begin{aligned} \frac{\epsilon\sqrt{\pi}\, v^2}{n_0 \Gamma A} e^{m/m_0(\alpha E + \beta J^2)} &= \frac{\partial^2 Q}{\partial v^2}\left[\frac{F_1(jv)}{(jv)^2} - \frac{1}{jv} e^{-(jv)^2} - \frac{2}{3} jv\, e^{-(jve)^2}\right] v \\ &+ \frac{\partial Q}{\partial v}\left\{2\frac{m}{m_0}F_1(jv) - \frac{F_1(jv)}{(jv)^2} + e^{-(jv)^2}\left[2jv\left(1 - \frac{m}{m_0}\right) + \frac{1}{jv}\right]\right. \\ &\left. - \frac{4}{3} jv e^{-(jve)^2} - 2\alpha v^2 K\left[\frac{F_1(jv)}{(jv)^2} - \frac{1}{jv} e^{-(jv)^2} - \frac{2}{3} jv\, e^{-(jve)^2}\right]\right\}, \end{aligned} \tag{4.5}$$

where

$$\kappa = \frac{m}{m_0}\left[1 + 2\frac{\beta}{\alpha}r^2(1-\mu^2)\right],$$

and, of course, the above equation is written for $v \gtrsim v_e$. (In the complete equation (4.5) the coefficient of Q is extremely small at and near the phase space boundary at v_e. This, combined with the fact that Q itself is very small and is zero at v_e, allows the terms in Q and $\partial Q/\partial\mu$ to be safely neglected.) Because spatially there is a central concentration of stars, $(jv_e)^2$ at the centre is greater than $\langle j^2 v_e^2 \rangle$ which, by the Virial Theorem, approximately equals 6·0. Now in the filed star velocity distribution, it is necessary for the field stars to have the same "kinetic temperature" as the test stars. Hence j^2 is an increasing function of r, for the velocity dispersion decreases monotonically from the centre. Since v_e decreases from the centre, $(jv_e)^2 = x_e^2$ is a rather slowly varying function. The result is that x_e does not decrease very rapidly and remains large at large distances from the centre of the system. The dominant terms in equation (4.5) are therefore those containing $F_1(jv)$, and we may make the approximation of keeping only F_1 over an extensive region. Equation (4.5) now reduces with $(x = jv)$ to,

$$\frac{2\epsilon x^4}{n_0 j^3 \Gamma A} \mathrm{e}^{m/m_0(\alpha E + \beta J^2)} = x\frac{\partial^2 Q}{\partial x^2} - \frac{\partial Q}{\partial x}(1 + 2x^2 k), \tag{4.6}$$

and k is given by

$$k = \frac{\alpha}{j^2}\kappa - \frac{m}{m_0} = \frac{m}{m_0}\left\{\frac{\alpha}{j^2}\left[1 + 2\frac{\beta}{\alpha}r^2(1-\mu^2)\right] - 1\right\}. \tag{4.7}$$

Not surprisingly, equation (4.6) is similar in form to equation (3.6) derived for Φ = constant. But the boundary conditions now must be with respect to E and not v, for f must approach the equilibrium distribution as E (and therefore r) tend toward zero. Since $2j^2 E_e = x_e^2$ at $r = 0$, it is an easy matter to verify that the following expression for $Q(r, v, \mu; t)$,

$$Q = \frac{1 - \mathrm{e}^{-kx_e^2(0)(1 - E/E_e)}}{1 - \mathrm{e}^{-kx_e^2(0)}}, \tag{4.8}$$

satisfies the differential equation (4.6) with $\epsilon = 0$ as well as the boundary conditions,

$$Q|_{E=0} = 1\cdot 0. \tag{4.9}$$

$$Q|_{E=E_e} = 0\cdot 0. \tag{4.10}$$

The differential equation for Q contained derivatives with respect to v only. We may therefore consider $A = A(r, \mu; t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$. However, by a direct substitution into the Boltzmann equation it is an easy matter to see that the dependence of A upon r and μ is extremely slight. This calculation (which is quite lengthy) indicates that by putting $A = A(t)$ only, the error is very small—of the order 0·01 per cent over an extremely wide range in z (defined in equation 5.5). Beyond $z = 500$, the error is of the order 0·1 per cent.

Can we justify putting $\epsilon = 0$? To answer whether or not this would be consistent is not difficult, for we notice that $f(r, v, \mu; t)$ can now be expressed in terms of $(r, E, J^2; t)$, and at large values of E the dominant term in ϵ is

$$\epsilon = \simeq \frac{\partial Q}{\partial E}\frac{\partial \Phi}{\partial t} A\,\mathrm{e}^{-m/m_0(\alpha E + \beta J^2)}. \tag{4.11}$$

The left hand side of equation (4.6) now becomes

$$j^2 x^3 T_R \frac{\partial \Phi}{\partial t}\frac{\partial Q}{\partial x}, \tag{4.12}$$

and we need to compare $j^2x^3T_R\partial\Phi/\partial t$ with $1+2\chi^2k$. At and very near the centre, the situation is obviously favourable. At the distance which inside there is about one-half the total mass, an estimate of $T_R\partial\Phi/\partial t=\Delta\Phi$ indicates the coefficient of $\partial Q/\partial x$ in (4.12) is of the order 10^{-2}. It would seem fair to conclude that just as in the analysis of the square-well potential problem, the small divergence of flux of representative points near the phase space boundary E_e introduces little error in the solution for Q.

5. *Discussion.*—The distribution function just obtained is

$$f(r, v, \mu; t) = A\,e^{-m/m_0(\alpha E+\beta J^2)}\frac{1-e^{-kx_e^2(0)(1-E/E_e)}}{1-e^{-kx_e^2(0)}}, \tag{5.0}$$

where

$$k=\frac{m}{m_0}\left\{\frac{\alpha}{j^2}\left[1+2\frac{\beta}{\alpha}r^2(1-\mu^2)\right]-1\right\}, \tag{5.1}$$

and A, α, β are functions of time. In principle the time dependence and the implicit r dependence are not too difficult to obtain. For the latter, at any given time we solve

$$\nabla^2\Phi=\int_0^{ve}\int_{-1}^{1} f(r, v, \mu; t)2\pi v^2\,d\mu\,dv, \tag{5.2}$$

for the potential. For the time dependence, Michie (1961) used three moments of the Boltzmann equation and found A and α to increase and β to decrease as evolution proceeds. All this requires a great amount of computation, but certainly a good deal less than a direct solution of the Boltzmann equation would require. It must be stressed however, that the above expression for f is in no way an exact solution of the Boltzmann equation. In the first place, we have assumed a particular distribution of J^2; although the chosen form is quite plausible physically, there is no real guarantee for this distribution because our approximation for h and g eliminated any dependence on the independent variable μ. Secondly, the analysis was made only at the high energy tail of the distribution function. What we have obtained is a distribution function which accurately satisfies the Boltzmann equation for large and small E consistent with an assumed velocity space anisotropy. But for reasons already discussed, the dependence of f upon E may be fairly accurate for all values of E provided we restrict our discussion to stellar masses for which $m/m_0 \geqslant 0{\cdot}5$; and for an actual stellar system this is not a serious restriction.

Evidently the distribution of E and J^2 is no longer independent, since the expression for k may be rewritten as

$$k=\frac{m}{m_0}\left\{\frac{\alpha}{j^2}\left[1+\frac{\beta}{\alpha}\frac{J^2}{(E-\Phi)}\right]-1\right\}. \tag{5.3}$$

In this form, there is a slight "correction" to the assumed distribution of J^2, for a given value of E. We also note that for a fixed value of (E, J^2), the cut-off becomes more abrupt at increasing distance from the centre, a result not surprising because of the lesser importance of encounters at greater distances from the inner regions. Let us now transform to the dimensionless variables,

$$\phi=\alpha\Phi, \tag{5.4}$$

$$z^2=r^2[A\alpha G(4\pi)^2(2/\alpha)^{3/2}m_0], \tag{5.5}$$

$$\eta=v\sqrt{\frac{\alpha}{2}}. \tag{5.6}$$

The expression for k now becomes

$$k=\frac{m}{m_0}\left\{\frac{\alpha}{j^2}\left[1+Cz^2(1-\mu^2)\right]-1\right\}, \tag{5.7}$$

while Poisson's equation transforms to

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{d\phi}{dz}\right)=e^{-\phi}\int_0^{\eta_e}\int_0^1 e^{-\eta^2}e^{-Cz^2\eta^2(1-\mu^2)}Q\eta^2\,d\mu\,d\eta, \tag{5.8}$$

and C, the model parameter, is

$$C=\frac{\beta}{A(2\alpha)^{1/2}G(4\pi)^2}. \tag{5.9}$$

The model is completely determined by a given value of C. In this form, the z dependence becomes important when $Cz^2\approx\frac{1}{2}$. For a globular cluster such as 47 Tuc, $C\approx 10^{-4}$ and the dependence on z is therefore important beyond $z=70$, which is in the very extreme outer parts—this position corresponds to about 20 parsecs. The indication is plain: if we write f in terms of E, J^2, r and t, then the r dependence is negligible throughout most of the system, and it can be considered to evolve through a series of equilibrium states as was assumed by Michie (1961). As far as evolution is concerned, the outer parts of the cluster where the r dependence cannot be neglected (for an isolated system), is dynamically unimportant. Furthermore, as evolution proceeds, C decreases, and the dependence of f upon r becomes even less important.

To summarize, the basic physical observation in this study is that if there is some method of obtaining rather accurately the distribution function at large energies, then there is reason for believing the resulting distribution over all energies will also be fairly accurate. The deviation from equilibrium occurs at large values of E, which is where the analysis was made for the determination of $f(r, v, \mu; t)$. These statements and results hold only if the mass ratio is not too small. The study began by making a comparison of the "escape time" to the dynamical time scale; and noting the very large mean-free-path for the high energy stars, it was seen that the distribution function must tend toward zero for large values of E. In phase space there is an effective absorbing wall at $E=E_e$, and $f\cong 0$ there; this produces a condition on the various derivatives of f. By writing f as a product of the equilibrium distribution and an arbitrary function (and assuming a particular distribution of J^2 for low and moderate energy) the Boltzmann equation can be approximately solved as far as the energy dependence is concerned, for the high energy tail of the distribution function. The limiting form of f for small and large E is now reasonably correct, and if the mass ratio is not too small we can expect a fairly accurate distribution of energy for all values of E, consistent with the assumed velocity anisotropy. The results of this analysis for a system with a varying potential and particular distribution of angular momentum (squared) shows a certain similarity to the square-well potential calculation. Except for the weak and relatively unimportant dependence on r (or z), the function Q is of the same form as was obtained for $\Phi=$constant. But now x_e^2 is evaluated at the bottom of the potential well. Since $x_e^2(0)$ increases as C becomes smaller (Michie 1961) the cut-off of energy in the inner regions is more abrupt the larger the mass ratio, the smaller the model parameter, and finally for the stars in more circular orbits. Conversely, we may say that a factor which produces a greater rate of energy exchange also produces a more severe depopulation of high energy stars. Spherical stellar systems obtained by use of the distribution function derived in this paper should give a close representation to actual stellar systems, since a number of approximations made in previous studies have been eliminated. First, the distribution function has been obtained from the Boltzmann equation; in particular the cut-off has been derived and not assumed. Second, the systems obtained (corresponding to different values of the model parameter C) are all self-supporting, since the velocity distribution and potential are consistent by Poisson's equation. Third, the stellar orbits are increasingly more radial at larger distances from the centre. (The importance of this has been discussed by Woolley and Robertson (1956), Oort and van Herk (1959), and Michie (1961). In particular, the last author found an increasingly radial velocity distribution with evolution of the system.) Fourth, the distribution function has been obtained for stars of different mass. In our opinion, the most important effect remaining to be included is the removal of the condition that the system be isolated. The inclusion of a realistic tidal effect, one that is not restricted to a weak-field approximation, would be a very valuable extension. Finally, we wish to point out that this distribution function now allows an accurate calculation of the loss of low mass stars from spherical systems. Also, an analysis of the type presented here might prove promising for obtaining information concerning the distribution function for rotating systems.

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