# MASSIVE STARS IN QUASI-STATIC EQUILIBRIUM*

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## ABSTRACT

The binding energy properties of quasi-static solutions to the equations of general relativity for starlike condensations of $10^2$ to $10^9$ solar masses are presented. With increasing central temperature and decreasing radius, the binding energy first increases through positive values, passes through a maximum, and then decreases rapidly to large negative values. The central temperature at which maximum binding occurs follows the rough law $T_c \sim 3.4 \times 10^{13}(M_\odot/M)^\circ$ K and the binding energy at this point is roughly $3.6 \times 10^{54}$ erg, nearly independent of mass.

## INTRODUCTION

Starlike condensations of $10^6$–$10^8$ solar masses have been suggested recently (Hoyle and Fowler 1963*a*, *b*) as the origin of the large energies ($10^{58}$ to $10^{61}$ erg) required to account for strong radio sources on the usual hypothesis of electron synchrotron radiation. Hoyle and Fowler make quantitative estimates on the supposition that a particular solution of the classical hydrostatic equations—the polytrope of index 3—provides a suitable approximation to the structure of the massive star during all phases prior to a presumed eventual implosion of central regions and explosion of outer regions.

It is the purpose of this note to point out that an understanding of the behavior of a very massive body during gravitational contraction through starlike phases requires the solution of the full dynamic equations of general relativity. Whereas the equations of classical hydrodynamics permit quasi-static solutions which are consistent with a collapse rate which decreases with time, the quasi-static solutions to the relativistic equations suggest that, if the star does not fragment, the rate of collapse must increase with time prior to nuclear burning. For stellar masses above $10^8$ solar masses, it does not appear likely that nuclear fuel can be burned rapidly enough to halt the collapse.

## EQUATIONS AND SOLUTION

In the case of spherical symmetry, the time-dependent metric may be written as (see, e.g., Tolman 1934)

$$(ds)^2 = e^{\nu(r,t)}(dt)^2 - \frac{1}{c^2}\{e^{\lambda(r,t)}(dr^2) + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]\}. \quad (1)$$

When time variations are neglected,

$$e^{-\lambda} = (1 - 2GM(r)/rc^2), \quad (2)$$

where

$$\frac{\partial M(r)}{\partial r} = 4\pi r^2 \rho, \qquad M(0) = 0 \quad (3)$$

and

$$\frac{\partial P}{\partial r} = -(\rho + P)\frac{e^\lambda}{c^2}\left[4\pi G P r + \frac{GM(r)}{r^2}\right]. \quad (4)$$

Here $Pc^2$ and $\rho c^2$ are, respectively, the pressure and energy density in a local frame of reference, $c$ is the velocity of light, and $G$ is the gravitational constant.

If $R$ is defined as the position at which the matter density vanishes, then $M_s = M(R)$

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represents the entire energy content of the star in mass units. Since the star radiates energy, $M_s$ must change with time. If it is assumed that the total number of nucleons in the star remains constant, then

$$N=\int_0^{\text{surface}} q(r)\,4\pi r^2 e^{\lambda/2} dr = \text{constant}, \tag{5}$$

where $q(r)$ is the local nucleon density in number/cm$^3$ and $4\pi r^2 e^{\lambda/2}dr$ is the volume of a spherical shell in a local frame of reference. The total rest mass of the system is obtained by integrating

$$\frac{\partial M_m(r)}{\partial r}=\rho_m 4\pi r^2 e^{\lambda/2}, \tag{6}$$

where $\rho_m$ is the local rest mass density of nuclei and associated electrons. Since they vanish when all components of the system are at rest at infinity, electron-positron pairs are not included in $M_m(r)$. The total binding energy of the system is simply $(M_0 - M_s)c^2$, where $M_0 = M_m(R)$.

Hoyle and Fowler (1963*a*) give arguments for assuming that the temperature gradient in a massive star is close to the adiabatic gradient. If flow is adiabatic, then

$$\frac{\partial \rho}{\partial r}=\frac{\rho+P}{\rho_m}\frac{\partial \rho_m}{\partial r}. \tag{7}$$

Table 1 gives the value of $(M_0 - M_s)/M_0$ as a function of rest mass and central temperature for pure hydrogen configurations satisfying equations (3) through (7). Electron-positron pairs are included in the equation of state, as described in the Appendix. Entries are omitted when leptons become partially degenerate near the center.

For 100 solar masses the result is as might be expected from the classical virial theorem—the binding energy is directly proportional to the thermal energy content of the system and therefore increases linearly with central temperature.

Inspection of other rows in Table 1 shows that the binding energy passes through a maximum at $T_c \sim 3.4 \times 10^{13}\ (M_\odot/M_0)\ ^\circ$ K, goes to zero at $T_c \sim 6.8 \times 10^{13}\ (M_\odot/M_0)\ ^\circ$ K, and then becomes increasingly negative with increasing central temperature. The maximum binding energy is roughly $3.6 \times 10^{54}$ erg, nearly independent of rest mass. For $M_0 = (10^8 - 10^9)M_\odot$, the relative maximum in binding energy which occurs in the neighborhood of $T_c \sim 10^9\ ^\circ$ K is correlated with the creation of pairs near the center.

It might be argued that the occurrence of negative binding energies is due to a faulty choice of density gradient. However, by constructing an approximate analogue to the classical virial theorem, it becomes evident that the negative binding energy phenomenon is actually implicit in the relativistic equations.

At sufficiently low densities and temperatures, both electrons and nucleons may be treated as non-relativistic and non-degenerate so that, for a pure hydrogen configuration,

$$\begin{aligned}\rho &\cong \rho_m(1+3\theta)+\rho_r\,,\\ P &\cong 2\theta\rho_m+\tfrac{1}{3}\rho_r\,,\end{aligned} \tag{8}$$

where $\theta = kT/M_H c^2$, $\rho_r = aT^4/c^2$, $k$ = Boltzmann's constant, $M_H$ = hydrogen rest mass, and $a$ = radiation density constant.

Denoting the average of any variable $\gamma$ by $\langle\gamma\rangle = \int \gamma dM_m / \int dM_m$, one finds from equations (3), (6), and (8) that

$$\begin{aligned}\frac{M_0-M_s}{M_0} &= \langle\epsilon\rangle = \langle 1-(1-2\phi)^{1/2}(1+3\theta+\sigma)\rangle\\ &= \langle -(3\theta+\sigma)+\phi(1+\delta)(1+3\theta+\sigma)\rangle\,,\end{aligned} \tag{9}$$

| Log ($M_0/M_\odot$) | Log $T_c$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6.0 | 6.5 | 7.0 | 7.5 | 8.0 | 8.5 | 9.0 | 9.5 |
| 2.0 | $+1.67\cdot10^{-7}$ | $+5.28\cdot10^{-7}$ | $+1.67\cdot10^{-6}$ | $+5.27\cdot10^{-6}$ | $+1.68\cdot10^{-5}$ | . . | . . | . . . |
| 3.0 | $+1.65\cdot10^{-7}$ | $+5.21\cdot10^{-7}$ | $+1.65\cdot10^{-6}$ | $+5.20\cdot10^{-6}$ | $+1.64\cdot10^{-5}$ | $+5\ 08\cdot10^{-5}$ | . . | . . . . . . . . |
| 4.0 | $+1.63\cdot10^{-7}$ | $+5.14\cdot10^{-7}$ | $+1.62\cdot10^{-6}$ | $+5.11\cdot10^{-6}$ | $+1.58\cdot10^{-5}$ | $+4.68\cdot10^{-5}$ | $+9\ 43\cdot10^{-5}$ | . . . |
| 5.0 | $+1.61\cdot10^{-7}$ | $+5.07\cdot10^{-7}$ | $+1.58\cdot10^{-6}$ | $+4.76\cdot10^{-6}$ | $+1.27\cdot10^{-5}$ | $+1.65\cdot10^{-5}$ | $-3\ 49\cdot10^{-4}$ | $-1.26\cdot10^{-2}$ |
| 6.0 | $+1.58\cdot10^{-7}$ | $+4.77\cdot10^{-7}$ | $+1.28\cdot10^{-6}$ | $-1.80\cdot10^{-6}$ | $-1.71\cdot10^{-5}$ | $-2.88\cdot10^{-4}$ | $-4.12\cdot10^{-3}$ | $-6.76\cdot10^{-2}$ |
| 7.0 | $+1.29\cdot10^{-7}$ | $+1.70\cdot10^{-7}$ | $-1.64\cdot10^{-6}$ | $-2.79\cdot10^{-5}$ | $-3.20\cdot10^{-4}$ | $-3.53\cdot10^{-3}$ | $-4.40\cdot10^{-2}$ | $-5\ 65\cdot10^{-2}$ |
| 8.0 | $-1.66\cdot10^{-7}$ | $-2.76\cdot10^{-6}$ | $-3.16\cdot10^{-5}$ | $-3.32\cdot10^{-4}$ | $-3.56\cdot10^{-3}$ | $-4.27\cdot10^{-2}$ | $-8.64\cdot10^{-2}$ | $-8.34\cdot10^{-2}$ |
| 9.0 | $-3.13\cdot10^{-6}$ | $-3.25\cdot10^{-5}$ | $-3.38\cdot10^{-4}$ | $-3.56\cdot10^{-3}$ | $-4.50\cdot10^{-2}$ | $-8.81\cdot10^{-2}$ | $-7\ 60\cdot10^{-2}$ | $-7.64\cdot10^{-2}$ |

TABLE 1

$(M_0-M_s)/M_0$ for Pure Hydrogen Stars. Adiabatic Approximation where $\phi = GM(r)/rc^2$, $\sigma = \rho_r/\rho_m$, and $\phi(1+\delta) = 1-(1-2\phi)^{1/2}$. Multiplying both sides of equation (4) by $4\pi r^3 dr$ and integrating the left-hand side by parts yields

$$\langle (1-2\phi)^{1/2}(6\theta+\sigma)\rangle = \langle \phi(1-2\phi)^{-1/2}(1+5\theta+\tfrac{4}{3}\sigma)(1+\alpha)\rangle , \quad (10)$$

where $\alpha = 4\pi P r^3/M(r)$. Elimination between equations (9) and (10) gives, after some rearrangement,

$$\langle\epsilon\rangle = \langle 3\theta - \phi(1-2\phi)^{-1/2}[\alpha+(5\theta+\tfrac{4}{3}\sigma)(1+\alpha)+\delta+3\theta(1+\delta)(1-2\phi)^{1/2}]\rangle . \quad (11)$$

From equation (4) it is evident that when gas pressure is small relative to radiation pressure, as it certainly is in the massive stars under consideration, $\langle\sigma\rangle$, $\langle\phi\rangle$, $\langle\alpha\rangle$, and $\langle\delta\rangle$ are all nearly proportional to $M_0^{1/2}T_c$. The negative term in equation (11) therefore increases quadratically with increasing central temperature and must eventually dominate the positive contribution which depends only linearly on central temperature. It is further clear that the temperature at which negative binding energies first appear decreases nearly linearly with increasing mass.

TABLE 2

$(M_0 - M_s)/M_0$ AS A FUNCTION OF POLYTROPIC INDEX $n$, $T_c = 10^8$ ° K

| Log $(M_0/M_\odot)$ | $n$ | | | | | |
|---|---|---|---|---|---|---|
| | 0.5 | 1.0 | 2 0 | 3 0 | 4 0 | 4.5 |
| 8.0.. | $-5\ 35\cdot 10^{-3}$ | $-4\ 99\cdot 10^{-3}$ | $-4.27\cdot 10^{-3}$ | $-3.56\cdot 10^{-3}$ | $-2\ 85\cdot 10^{-3}$ | $-2\ 50\cdot 10^{-3}$ |
| 7.0.. . | $-4.62\cdot 10^{-4}$ | $-4.31\cdot 10^{-4}$ | $-3.75\cdot 10^{-4}$ | $-3\ 20\cdot 10^{-4}$ | $-2\ 65\cdot 10^{-4}$ | $-2\ 37\cdot 10^{-4}$ |
| 6 0 .. | $-2\ 80\cdot 10^{-5}$ | $-2\ 57\cdot 10^{-5}$ | $-2\ 12\cdot 10^{-5}$ | $-1.71\cdot 10^{-5}$ | $-1\ 38\cdot 10^{-5}$ | $-1\ 20\cdot 10^{-5}$ |
| 5.0... | $+1\ 46\cdot 10^{-5}$ | $+1.42\cdot 10^{-5}$ | $+1.34\cdot 10^{-5}$ | $+1.27\cdot 10^{-5}$ | $+1.19\cdot 10^{-5}$ | $+1\ 15\cdot 10^{-5}$ |
| 4.0.. | $+1\ 88\cdot 10^{-5}$ | $+1.81\cdot 10^{-5}$ | $+1\ 69\cdot 10^{-5}$ | $+1\ 57\cdot 10^{-5}$ | $+1\ 45\cdot 10^{-5}$ | $+1\ 39\cdot 10^{-5}$ |
| 3.0..... | $+1\ 89\cdot 10^{-5}$ | $+1.83\cdot 10^{-5}$ | $+1\ 71\cdot 10^{-5}$ | $+1\ 59\cdot 10^{-5}$ | $+1\ 48\cdot 10^{-5}$ | $+1\ 42\cdot 10^{-5}$ |

A comparison of equations (9) and (11) shows that when binding energy becomes negative, $(\langle\sigma\rangle) \sim (\langle\phi\rangle) \gg (\langle 3\theta\rangle)$. It may be inferred that the appearance of negative binding energies is to be attributed as much to the difference between the pressure relative to energy density properties of particles with rest mass and of particles with negligible rest mass as to an effect on the equilibrium conditions due to space curvature in a strong gravitational field.

The precise relationship between total energy content and central temperature of course depends on structural details. To illustrate that negative binding energies can nevertheless not be avoided in the quasi-static approximation by altering structural details, a set of models characterized by different degrees of central condensation has been constructed. For each model it has been assumed that

$$\frac{1}{\rho_m}\frac{\partial \rho_m}{\partial r} = \frac{n}{1+n}\frac{1}{P}\frac{\partial P}{\partial r}, \quad (12)$$

$n$ being the so-called polytropic index of the star.

Table 2 gives the binding energy relative to rest mass energy as a function of mass and polytropic index when $T_c = 10^8$ ° K. For $M_0 \geq 10^6 M_\odot$, binding energy is negative for all distributions having a finite radius ($n < 5$) but becomes slightly less negative with increasing $n$. For $M_0 \leq 10^5 M_\odot$, binding energy is positive but increases slightly with decreasing $n$.

This behavior follows from the fact that, for a given central temperature, low values of $n$ correspond to more compact (smaller radius, higher mean density) distributions of matter and consequently to larger values of $\langle\phi\rangle$. As may be seen from equation (11), second-order terms proportional to $\phi$ dominate the first-order term when binding energy is negative. A more compact distribution (smaller $n$) thus leads to more negative binding energies. This is the case when $M_0 \geq 10^6\, M_\odot$.

When binding energy is positive, second-order terms in $\langle\epsilon\rangle$ may be neglected. From equations (9) and (11) one obtains

$$\langle\epsilon\rangle = \langle 3\theta\rangle = \langle\phi - (\sigma + 3\theta)\rangle = \tfrac{1}{2}\langle\phi - \sigma\rangle . \tag{13}$$

Denoting the ratio of gas pressure to total pressure by $\beta$, then $\sigma = (1 - \beta)6\theta/\beta$. With the help of equation (13), $\langle\phi\rangle = \langle 6\theta/\beta\rangle = \langle 6\theta\rangle/\bar{\beta}$, giving finally the classical expression for binding energy

$$\langle\epsilon\rangle = \tfrac{1}{2}\bar{\beta}\langle\phi\rangle . \tag{14}$$

For given central temperature, a more diffuse distribution (larger $n$) now corresponds to weaker binding. This is the situation when $M_0 \leq 10^5\, M_\odot$.

## CONCLUDING REMARKS

It is clear that, contrary to the classical picture, a very massive star cannot continue to condense indefinitely through a sequence of quasi-static configurations. Upon approaching the maximum in the binding energy curve for such configurations, the star must either continue to condense dynamically or break up into smaller parts with a net increase in binding energy.

Assuming that collapse is more probable than fragmentation, it is probable that the rate of collapse increases with increasing condensation. When dynamic terms are included, but macroscopic velocities are small compared to $c$, then to first order in dynamic terms one must have that

$$\frac{M_0 - M_s}{M_0} \sim \langle\epsilon\rangle - \left\langle r\,\frac{d^2 r}{dt^2} + \tfrac{1}{2}\left(\frac{dr}{dt}\right)^2\right\rangle \frac{1}{c^2}, \tag{15}$$

where $dr/dt$ and $d^2r/dt^2$ are macroscopic velocity and acceleration, respectively.

If the distribution of matter and temperature is assumed to be roughly similar to that given by the quasi-static solution, then, beyond a critical central temperature, $\langle\epsilon\rangle$ becomes increasingly negative with increasing condensation. Since $M_s$ must decrease with time, the quantity $\langle r(d^2r/dt^2)\rangle$ must grow large and negative to offset the decrease in $\langle\epsilon\rangle$. The rate of collapse of a major portion of the star must therefore increase with time as long as equation (15) remains approximately valid.

Schwarzschild and Härm (1959) have demonstrated that, on the basis of classical hydrodynamics, stars above 65–100 solar masses become pulsationally unstable during hydrogen burning via the CN cycle. However, if a very massive star is collapsing dynamically on reaching the hydrogen-burning phase, it is possible that nuclear fuel cannot be burned rapidly enough to stop the collapse.

If the distribution of matter during dynamic collapse is roughly the same as that given by the quasi-static solutions in the adiabatic approximation, then nuclear burning via the CN cycle does not become rapid enough to supply even the surface output until $\log T_c = 7.8$ to $7.93$ for $M_0 = (10^6 \text{ to } 10^9)M_\odot$. The lifetime against CN-cycle burning at these temperatures is on the order of $3 \times 10^6$ yrs. These estimates are based on the assumptions that surface luminosity is $L \sim 3.5 \times 10^4 L_\odot(M_0/M_\odot)$, as given by Hoyle and Fowler (1963$a$) and that $X_{\rm CN} = \sim 0.01$ and $X_{\rm H} \sim 1.0$.

From Table 1 it follows that at least $4.5 \times 10^{-2} \times 940 = 42$ MeV per nucleon is required to stop the collapse of a star of $10^9$ solar masses if it reaches a central temperature of about $10^8\,^\circ$ K. Since a total of only $\sim 6$ MeV per nucleon is available from the conversion of hydrogen to helium, hydrogen burning cannot stop the collapse, and it is likely that the star will continue to implode through all nuclear burning stages. If $M_0 =$

$10^8 M_\odot$, collapse cannot be halted if $T_c$ exceeds $1.1 \times 10^8$ ° K. But the lifetime against hydrogen burning at this temperature is on the order of $10^4$ years, whereas estimates based on equation (15) indicate that an upper limit to the time required for the central temperature to rise from $T_c \sim 7.8 \times 10^7$ ° K (onset of nuclear burning) to $T_c \sim 1.1 \times 10^8$ ° K is only $3 \times 10^5$ sec. Again, hydrogen burning cannot stop the collapse.

Similar arguments may be made for less massive stars with the result that conditions for stopping the collapse through explosive hydrogen burning become more favorable with decreasing mass. In particular, since the properties of models with $M_0 \leq 10^4 M_\odot$ and $T_c < 3 \times 10^8$ ° K are essentially identical with those obtained by means of the classical equations, the results of Schwarzschild and Härm (1959) are probably valid for $M_0 \leq 10^4 M_\odot$.

In conclusion, if it is granted that an initially bound configuration of $10^8$ solar masses or more can be formed with sufficiently low angular momentum that it reaches starlike phases without fragmenting, then it is likely that implosion will continue through nuclear burning stages. If the mass of the initial configuration is less than $10^4$ but greater than $10^2$ solar masses, then pulsational instability is likely to accompany hydrogen burning. Above $10^4$ solar masses and below a critical mass $M_c$, hydrogen burning will reverse the collapse explosively. The value of $M_c$ can be only determined by solving the full dynamic equations of general relativity.

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## APPENDIX

When temperatures are low enough that nucleons may be treated non-relativistically and yet high enough that the electron-positron gas is non-degenerate, the expressions for energy and pressure in a system of $H^1$ and $He^4$ are quite straightforward. Let

$$\rho_n = \text{nucleon rest mass density (gm/cm}^3) \,,$$

$$q_n = \rho_n \left(\frac{X}{M_p} + \frac{Y}{M_\alpha}\right) = \text{nuclei/cm}^3 \,, \tag{16}$$

$$q_e = \rho_n \left(\frac{X}{M_p} + \frac{2Y}{M_\alpha}\right) = \text{electrons/cm}^3 \text{ exclusive of pairs} \,,$$

and

$$\rho_m = \rho_n + m_e q_e = \text{rest mass/cm}^3 \text{ exclusive of pairs} \,.$$

Here $M_p$, $M_\alpha$, and $m_e$ are, respectively, the mass of the proton, the mass of the $\alpha$ particle, and the mass of the electron. The mass fractions of hydrogen and helium are given by $X$ and $Y$. Energy density and pressure are additive quantities so that

$$\rho = \rho_n + q_n \tfrac{3}{2} \frac{kT}{c^2} + \rho_+ + \rho_- + \rho_r \,,$$

$$P = q_n \frac{kT}{c^2} + P_+ + P_- + \tfrac{1}{3} \rho_r \,, \tag{17}$$

where $\rho_+$ and $\rho_-$ are the energy densities and $P_+$ and $P_-$ are the pressures associated with the positrons and electrons, respectively. All other quantities are defined in the text.

At temperatures sufficiently low that electrons may be treated non-relativistically, pairs may be neglected ($\rho_+ = P_+ = 0$) and

$$\rho = \rho_m + \tfrac{3}{2} \frac{kT}{c^2} (q_n + q_e) + \rho_r$$

$$P = (q_n + q_e) \frac{kT}{c^2} + \tfrac{1}{3} \rho_r \,. \tag{18}$$

When $X = 1$, $Y = 0$, these expressions reduce to equations (8) in the text if one makes the identification $M_{\mathrm{H}} = m_e + M_p$.

The non-relativistic approximation is satisfactory as long as $\langle(\text{electron velocity}/c)^2\rangle_{\mathrm{av}} \ll 1$ or $3kT/m_e c^2 \sim T_9/2 \ll 1$. When this condition is not met, electron-positron pairs must be included and treated relativistically.

The number density of positrons and electrons is given by

$$q_{\pm} = \frac{8\pi}{h^3}\int_0^{\infty}\frac{p^2 dp}{e^{\pm\alpha+\epsilon/kT}+1}, \tag{19}$$

where $\alpha$ is the electron chemical potential, $h$ is the Planck constant, and $\epsilon^2 = (m_e c^2)^2 + (cp)^2$. In the non-degenerate approximation (Chandrasekhar 1939),

$$q_{\pm} = \frac{C}{z}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(e^{\pm\alpha})^n}{n}K_2(nz),$$

$$\rho_{\pm} = \frac{m_e C}{z}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(e^{\pm\alpha})^n}{n}\left(\frac{3K_3(nz)+K_1(nz)}{4}\right), \tag{20}$$

and

$$P_{\pm} = \frac{m_e C}{z^2}\sum(-1)^{n+1}\frac{(e^{\pm\alpha})^n}{n^2}K_2(nz),$$

where $C = 8\pi(m_e c/h)^3$, $z = (m_e c^2/kT) \cong 5.90/T_9$, and the $K$'s are defined by Chandrasekhar. The degeneracy parameter $\alpha$ may be found by solving

$$q_e = q_- - q_+ . \tag{21}$$

In actual computations, the values of the functions $K_\nu$ may be taken from the asymptotic series

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1+\frac{4\nu^2-1}{1!\,8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!\,(8z)^2}+\ldots\right]. \tag{22}$$

For $z \geq 2$, the first four terms in the expansion for $K_1$ and $K_2$ give an accuracy to better than 1 per cent.

A first approximation to the degeneracy parameter is obtained by setting

$$q_e = \tfrac{1}{2}J(e^{\alpha} - e^{-\alpha}),$$

or

$$e^{\pm\alpha} = \pm\left(\frac{q_e}{J}\right)+\left[\left(\frac{q_e}{J}\right)^2+1\right]^{1/2}, \tag{23}$$

where $J = 2\,CK_2(z)/z$. Better values for $\alpha$ may be obtained by successive iterations on equation (21); however, for all models discussed in the text, the first approximation to $\alpha$ is sufficient.

In performing stellar structure computations, the derivatives $d\rho_m/dP$ and $dT/dP$ are found from the series expansions for $(\partial\rho/\partial T)_{q_e}$, $(\partial\rho/\partial q_e)_T$, $(\partial P/\partial T)_{q_e}$, and $(\partial P/\partial q_e)_T$ in conjunction with the derivative $dq_e/d\rho_m$ and either equation (7) or equation (12) in the text.

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