ON THE OCCURRENCE OF MULTIPLE FREQUENCIES AND BEATS IN THE β CANIS MAJORIS STARS

S. CHANDRASEKHAR AND NORMAN R. LEBOVITZ
University of Chicago and Massachusetts Institute of Technology

Received July 7, 1962

ABSTRACT

An explanation is suggested for the occurrence of two nearly equal frequencies and associated beats in the light- and in the velocity-variations of the β Canis Majoris stars. It is shown that if the ratio of the specific heats γ is 1 6 and the star is rotating, any disturbance will excite two normal modes with nearly equal frequencies.

I. INTRODUCTION

In the two preceding papers (Chandrasekhar and Lebovitz 1962*c*, *d;* these papers will be referred to hereafter as "Paper I" and "Paper II," respectively) we have determined the effect of rotation on the different modes of oscillation of a homogeneous compressible mass and of a distorted polytrope. The detailed results of these two papers confirm, what had in fact been disclosed by the general theory (Chandrasekhar and Lebovitz 1962*a*), that a rotation couples two modes of oscillation which are, in the absence of rotation, purely radial and purely non-radial and, further, that a ratio of the specific heats $\gamma = 1.6$ plays a critical role in this phenomenon. The explanation which we wish to suggest for the occurrence of multiple periods and beats in the β Canis Majoris stars is on this theoretical basis; it clarifies and extends an earlier suggestion of ours (Chandrasekhar and Lebovitz 1962*b*). But first we shall briefly describe the nature of the coupling which is predicted and the origin of the critical role of $\gamma = 1.6$.

II. THE CASE WHEN ROTATION IS ABSENT

For the sake of simplicity we shall describe the phenomena in the framework of the approximate theory in which the Lagrangian displacement $\boldsymbol{\xi}$ is assumed to be a linear function of the co-ordinates. The arguments can be made more exact for the homogeneous model considered in Paper I, and they can be made more general for the others; but these refinements are not necessary for our present purposes.

The two modes of oscillation which are coupled by rotation are, in the absence of rotation, characterized by the displacement

$$\xi_1 = X_R x_1 \,, \qquad \xi_2 = X_R x_2 \,, \qquad \xi_3 = X_R x_3 \qquad (R\text{-mode}) \,, \tag{1}$$

and

$$\xi_1 = X_S x_1 \,, \qquad \xi_2 = X_S x_2 \,, \qquad \xi_3 = -2 X_S x_3 \qquad (S\text{-mode}) \,, \tag{2}$$

where X_R and X_S are two arbitrary constants. The corresponding characteristic frequencies are (in the same approximation)

$$\sigma_R^2 = (3\gamma - 4)\frac{|\mathfrak{W}|}{I} \qquad \text{and} \qquad \sigma_S^2 = \frac{4}{5}\frac{|\mathfrak{W}|}{I} \,, \tag{3}$$

where $\mathfrak{W}$ is the gravitational potential energy and I is the moment of inertia of the configuration.

The "radial" character of the displacement belonging to the R-mode and the solenoidal character of the displacement belonging to the S-mode are apparent from equations (1) and (2).

We notice that

$$\sigma_R^2 = \sigma_S^2 \qquad \text{if} \qquad \gamma = 1.6 \,. \tag{4}$$

Therefore, when $\gamma = 1.6$, the two characteristic frequencies coincide; and we have a case of degeneracy. Accordingly, in this case any linear combination of the proper solutions represented by equations (1) and (2) can be considered as belonging to the common characteristic frequency. This is the origin of the critical role of $\gamma = 1.6$.

III. THE CASE WHEN ROTATION IS PRESENT

Now when the configuration is set in rotation, the R- and S-modes (as we have designated them) get coupled, and the corresponding Lagrangian displacements are (cf. Paper II, eq. [98])

$$\xi_1 = X_R x_1 + Y_R x_2 , \qquad \xi_2 = X_R x_2 - Y_R x_1 , \qquad \xi_3 = Z_R x_3 , \tag{5}$$

and

$$\xi_1 = X_S x_1 + Y_S x_2 , \qquad \xi_2 = X_S x_2 - Y_S x_1 , \qquad \xi_3 = Z_S x_3 , \tag{6}$$

where

$$\frac{X_R}{Z_R} \neq 1 , \qquad \frac{X_S}{Z_S} \neq -\tfrac{1}{2} \qquad (\Omega \neq 0) , \tag{7}$$

$$Y_R = -2i \frac{\Omega}{\sigma_R} X_R , \qquad Y_S = -2i \frac{\Omega}{\sigma_S} X_S , \tag{8}$$ [1]

and σ_R and σ_S are the corresponding characteristic frequencies. A general property of solutions (5) and (6) is (cf. Paper I, eq. [26])

$$\left(\frac{X_R}{Z_R}\right)\left(\frac{X_S}{Z_S}\right) = -\tfrac{1}{2} + O(\Omega^2) \qquad (\Omega \to 0) . \tag{9}$$

So long as $\gamma \neq 1.6$, the proper solutions for the two modes given by equations (5) and (6) reduce to those given by equations (1) and (2) when $\Omega \to 0$. But this does not happen when $\gamma = 1.6$: in this case, the proper solutions belonging to the two *distinct* frequencies of oscillation (when Ω is different from zero), in the limit $\Omega = 0$, become determinate linear combinations of the solutions represented by equations (1) and (2). Thus, for the homogeneous compressible model (for which the foregoing arguments can be made exact; see Paper I, eqs. [50] and [51]),

$$\frac{X_R}{Z_R} = 2.63 , \quad \frac{X_S}{Z_S} = -0.185 , \quad \text{and} \quad Y_R = Y_S = 0 \qquad (\Omega = 0) ; \tag{10}$$

and the corresponding limits are not very different for the distorted polytropes (see Paper II, Table 4*B*). In other words, neither of the proper solutions belonging to the two distinct modes which obtain when $\Omega \neq 0$ become even approximately radial when $\Omega \to 0$: indeed, they are very far from being so.

IV. THE SUGGESTED EXPLANATION

The theoretical results summarized in Sections II and III suggest an explanation for the phenomenon of multiple periods and beats which is observed in the light- and in the velocity-variations of the β Canis Majoris stars (cf. Ledoux and Walraven 1958, Sec. 25). Earlier attempts (cf. Ledoux 1951) to explain the same phenomenon postulated the excitation of non-radial modes (*besides* the radial modes) under the influence of rotation. But one is generally reluctant to accept suggestions which appeal *directly* to the excitation of non-radial modes (besides the radial modes) on the grounds that such modes should be highly damped relative to the radial modes and, further, that their excitation would be "difficult" in view of the possible source of such excitation being in the deep interior. These arguments do not apply to the present interpretation, which is based on

[1] The occurrence of the imaginary in these relations signifies only that the terms in the Y's, in eqs. (5) and (6), oscillate 90° out of phase relative to the terms in the X's and the Z's.

the assumption that, for the stars in question, $\gamma = 1.6$; for, on this assumption, the degeneracy of the unperturbed state ($\Omega = 0$) makes indeterminate the specification of the proper solution belonging to the fundamental mode of oscillation; and the removal of this degeneracy, by the slightest amount of rotation, leads to two normal modes *neither* of which is even remotely radial. The excitation of *both* modes by any disturbance (spherically symmetric or otherwise) is not only to be expected: it is natural under the circumstances. We can "picture" what will happen as follows.

Consider a slowly rotating gaseous mass and suppose that it experiences a disturbance which has approximate spherical symmetry. The disturbance will set the mass into oscillations; and if $\gamma \neq 1.6$, only those normal modes which have the character of radial pulsations (approximately) will be excited (in agreement with one's normal expectations). However, when $\gamma = 1.6$, what will happen is quite different; for, in this case, the analysis of the original disturbance into normal modes must include the lowest modes, and neither of them, as we have seen, has a radial character. The excitation of the lowest modes with slightly different frequencies would appear to be inescapable, and, with their excitation, beats must ensue.

While we have supposed that $\gamma = 1.6$, an *exact* coincidence is not necessary for the foregoing explanation to be valid: it should clearly suffice if

$$|\gamma - 1.6| = O(e^2); \tag{11}$$

for then the difference in the unperturbed values of σ_R^2 and σ_S^2 would be "masked" by the displacements that each will undergo on account of rotation. It should be remembered in this connection that, in a star, γ (allowing for the effects of radiation pressure and ionization) is not, strictly, a constant: it will be variable, and for our present purposes it should suffice if γ takes the value 1.6 somewhere in the interior, so that condition (11) will be satisfied over a substantial part of the star.

It remains to verify that the assumption of $\gamma = 1.6$ is a reasonable one for the β Canis Majoris stars. Now it is known that, for stars of solar mass, γ is very nearly $\frac{5}{3}$ throughout the interior; and that, for stars of larger mass, γ will be less, on account of the greater importance of radiation pressure. It can be estimated[2] on the basis of our current ideas on the physical conditions in the interiors of stars that a value of $\gamma = 1.6$ will be attained among stars with masses 7–13$\odot$; this is eminently reasonable for the β Canis Majoris stars, which have spectral types and luminosity classes in the range B1 III–B2 IV.

Finally, it should perhaps be stated that the particular value $\gamma = 1.6$ has been deduced on the basis of a theory which is only approximate for configurations which are not strictly homogeneous. An exact theory may lead to a somewhat different value for γ, at which the kind of accidental degeneracy we have considered occurs. And the present interpretation will hold whenever such a degeneracy occurs.

The research reported in this paper has in part been supported by the Office of Naval Research under contract Nonr-2121(24) with the University of Chicago. The work of the second author was supported in part by the United States Air Force under contract No. AF-49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

REFERENCES

Chandrasekhar, S , and Lebovitz, N R. 1962*a*, *Ap J*, **135**, 248.
———. 1962*b*, *ibid*, p 305.
——— 1962*c*, *ibid.*, **136**, 1069.
——— 1962*d*, *ibid*, p 1082
Ledoux, P 1951, *Ap. J*, **114**, 373.
Ledoux, P , and Walraven, Th. 1958, *Hdb d Phys*, ed S. Flügge (Berlin: Springer Verlag), **51**, 398

[2] We are indebted to Professor Nelson Limber for making this estimate for us.