THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 136 SEPTEMBER 1962 NUMBER 2

THE CHANGE OF REDSHIFT AND APPARENT LUMINOSITY OF GALAXIES DUE TO THE DECELERATION OF SELECTED EXPANDING UNIVERSES

ALLAN SANDAGE

Mount Wilson and Palomar Observatories

Carnegie Institution of Washington, California Institute of Technology

(With an Appendix by G. C. McVITTIE, University of Illinois Observatory, Urbana)

Received February 2, 1962; revised April 13, 1962

ABSTRACT

The redshift and apparent luminosity of any given galaxy are not constant with time for most models of the expanding universe. Redshifts decrease with time because of the braking action of the gravitational field in all exploding models, except for the one where the matter density is zero. Apparent luminosities decrease with time, except for the oscillating model in the contracting phase and for galaxies with very large $\Delta\lambda/\lambda_0$ values, because the distances between galaxies are increasing Redshifts increase with time for every galaxy in the steady-state model.

The theory and numerical results of the deceleration are presented for four selected world models. For a galaxy with redshift $z = \Delta\lambda/\lambda_0 = 0.4$ at the present epoch, the change of redshift with time is found to be $dcz/dt = -11 \times 10^{-6}$ km/sec year for the oscillating model in the expanding phase at $q_0 = +1$; $dcz/dt = -5.9 \times 10^{-6}$ km/sec year for the Euclidean model; $dcz/dt = -4\ 3 \times 10^{-6}$ km/sec year for the hyperbolic model at $q_0 = 0\ 3516$; and $dcz/dt = +9\ 2 \times 10^{-6}$ km/sec year for the steady-state model. These all assume that $H^{-1} = 13 \times 10^9$ years at the present epoch With present optical techniques there is apparently no hope of detecting such small changes in redshift for time intervals smaller than 10^7 years. If radio techniques are used with observation of the 21-cm H I line, the detection of a frequency shift of 3×10^{-2} cycles/sec year is required for the quoted deceleration of the oscillating case, which again appears to be impossible with present methods

It should be possible to choose between various models of the expanding universe if the deceleration of a given galaxy could be measured. Precise predictions of the expected effect can be made for each of the presently available family of models, but, unfortunately, the expected change in $z \equiv \Delta\lambda/\lambda_0$ for reasonable observing times (say 100 years) is exceedingly small. Nevertheless, the predictions are interesting, since they form part of the available theory for the evolution of the universe. Section I gives the theory of the change of z with time, Section II gives the theory of the change of apparent luminosity with time, and Section III gives the results of numerical calculation.

I. THEORY OF DECELERATION

Any model which accepts homogeneity and isotropy admits of the well-known Robertson-Walker line element

$$ds^2 = c^2dt^2 - R^2(t)du^2 , \tag{1}$$

where du^2 represents an auxiliary three-space of constant Riemannian curvature, is a function of the dimensionless co-moving space co-ordinates, and is independent of time. In detail, du^2 can be expressed as

$$du^2 = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{2}$$

where k is the curvature index taking values of $+1$, 0, or -1 for elliptic, Euclidean, or hyperbolic space. For any light-track it is assumed that $ds^2 = 0$, which, from equation (1), leads to

$$u = c \int_{t_1}^{t_0} \frac{dt}{R(t)} = \text{Constant}, \tag{3}$$

where t_0 is the time of light-reception by an observer on earth, and t_1 is the time of light-emission by the galaxy in question. Because the co-ordinates r, θ, and ϕ are co-moving, u is a constant (independent of time) for any particular galaxy and can be computed when $R(t)$ is known. For radial tracks of photons, $d\theta = d\phi = 0$, and u is obtained in terms of the co-ordinate r from equation (2) as

$$u = \int_0^r \frac{dr}{\sqrt{1 - kr^2}} = \sin^{-1} r, \; r, \; \sinh^{-1} r \qquad \text{for } k = +1, \, 0, \, \text{or} -1 \tag{4}$$

where the observer is at $r = 0$.

The redshift, z, is given explicitly by the equation

$$1 + z = \frac{R_0}{R_1}, \tag{5}$$

which is obtained from equation (3) by considering the ratio of Δt_0 to Δt_1 of light-pulses sent out between time t_1 and $t_1 + \Delta t_1$ and received between time t_0 and $t_0 + \Delta t_0$. In equation (5), R_0 is the value of the spatial dilatation factor at the time of light-reception, t_0, and R_1 is its value at light-emission, t_1.

All the foregoing is, of course, well known and is given only for orientation in what follows. It is well to point out again that all these relations are very general. They do not depend on any specific dynamics or field equations but only on the geometrical assumptions of homogeneity and isotropy and on the properties of light-tracks.

We now address the problem of finding the change of z with time for various models when the dynamics are specified (cf. Friedman 1922, eqs. [4] and [5], or Robertson 1933, eq. [3.2]) by the Einstein field equations. The general solution of the dynamical equations is given in parametric form (cf. Sandage 1961*b*, hereafter called "Paper II") by

$$R = a(1 - \cos\theta), \qquad t = \frac{a}{c}(\theta - \sin\theta), \tag{6}$$

for $k = +1$, where the constant $a = 4\pi G \rho R^3/3c^2$ and where θ is the development angle. For the case $k = -1$,

$$R = a(\cosh\theta - 1), \qquad t = \frac{a}{c}(\sinh\theta - \theta). \tag{7}$$

For $k = 0$

$$R = (6\pi G \rho R^3)^{1/3} t^{2/3}. \tag{8}$$

a) Models with $k = \pm 1$

Consider, first, the change of z with time for the models with $k = \pm 1$. Let the development angle at the time of observation be θ_0, which can be found from equation (12) or

(13) of Paper II, once the deceleration parameter q_0 is known. The angle θ_1 (the development angle when light left a particular galaxy of redshift z) can be found from

$$1+z=\frac{1-\cos\theta_0}{1-\cos\theta_1}, \tag{9}$$

which follows from equations (5) and (6). Therefore, both θ_0 and θ_1 are known, once q_0 and z_0 are given. Consequently, the invariant u is known for the galaxy in question because equations (3), (6), and (7) require that

$$u=\theta_0-\theta_1=\text{Constant for all time}. \tag{10}$$

Consider, now, some new epoch of observation $t_{0,\,i}$ in the future (or past) when q_0 is different. Denote the development angle at this new epoch by $\theta_{0,\,i}$ and the angle when

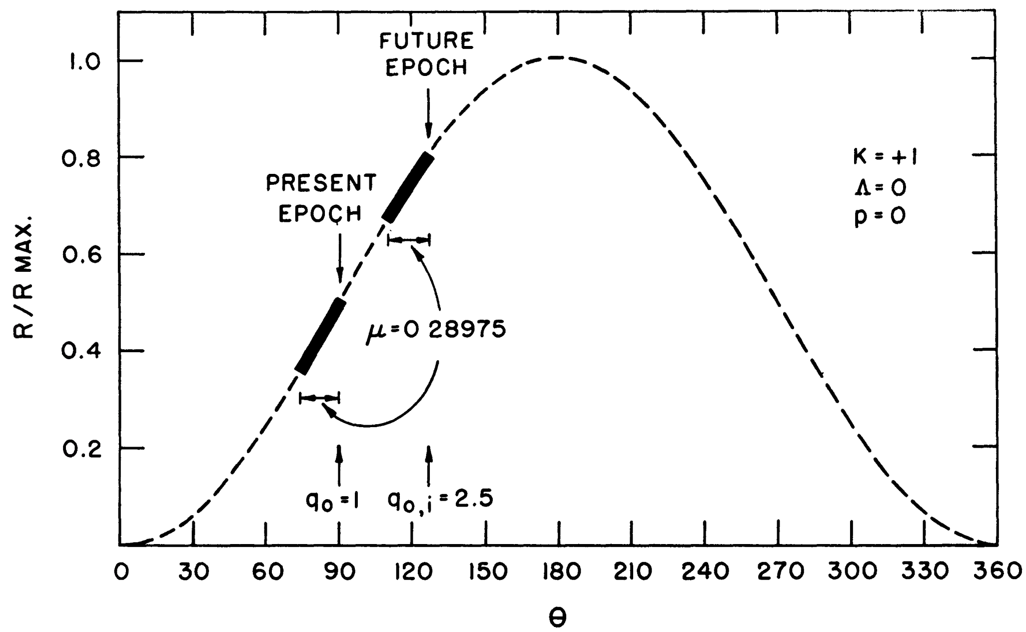

FIG. 1.—Illustrates the procedure for finding Δz in an oscillating universe. The dotted curve is the $[R, \theta]$ relation of eq. (6). The present epoch is taken, for illustration, at $q_0=+1$. For a galaxy whose present redshift is $z=0$ 4, eq (10) requires that $\theta_0-\theta_1=0.28975$. The redshift of this galaxy at the future epoch when $q_0=2.5$ is found by marking the same difference angle back along the $[R, \theta]$ track and using eq. (9) at this new epoch.

light left the galaxy in question by $\theta_{1,\,i}$. If $t_{0,\,i}$ is specified, $\theta_{0,\,i}$ is found from equation (12) or (13) of Paper II because $q_{0,\,i}$ is known. Then $\theta_{1,\,i}$ is found from equation (10), which requires $\theta_0-\theta_1=\theta_{0,\,i}-\theta_{1,\,i}$, and the new redshift is computed from

$$1+z_i=\frac{1-\cos\theta_{0,\,i}}{1-\cos\theta_{1,\,i}}, \tag{9$'$}$$

which solves the problem. For $k=-1$, replace "cos" by "cosh" in equations (9) and (9′).

The procedure is graphically illustrated in Figure 1, which gives R versus θ obtained from equation (6), for the oscillating model with $k=+1$. The situation shown is for an assumed present epoch at $q_0=+1$ $(\theta_0=\pi/2)$ and for a galaxy whose present redshift is $z=0.4$. We look out in space to an angle θ_1, such that $\theta_0-\theta_1=0.28975$, which follows from equation (9) with $1+z=1.4$. If this same galaxy is observed at some arbitrary time $t_{0,\,i}$ in the future, say when $q_{0,\,i} = 2.5$, we still look out in space through the same difference angle $\theta_0 - \theta_1 = 0.28975$, but at this time the ratio of the two values of R—shown as the ends of the solid bar on the $R(t)$-curve—is only 1.18834, which shows that $z_i = 0.18834$.

To compute the deceleration in c.g.s. units, it is necessary to know the time interval between the corresponding values of q_0. The time scale in the $k = +1$ model is obtained from equation (6), which gives

$$t = \frac{T}{2\pi}(\theta - \sin\theta), \tag{11}$$

where T is the oscillation time of the model ($\theta = 2\pi$). In the calculations it is convenient to work in the dimensionless ratio t/T, where t can later be calibrated in seconds, once T is known as follows. Substitution of equation (21) of Paper II into equation (11) gives

$$T = \frac{2\pi q_0}{H_0(2q_0 - 1)^{3/2}}, \qquad q_0 > 0.5\,. \tag{12}$$

For example, if $H_0^{-1} = 13 \times 10^9$ years when $q_0 = 1$ in the expanding phase, then $T = 8.2 \times 10^{10}$ years. In half this time, the expansion will stop and contraction begin.

The hyperbolic model ($k = -1$) has no natural time unit such as T because the expansion never stops. But again it is convenient to work in dimensionless units such as $t/t(\pi)$, where $t(\pi)$ is the time to reach $\theta = \pi$. Equation (7) shows that

$$\frac{t(\theta)}{t(\pi)} = \frac{\sinh\theta - \theta}{8.40715} \tag{13}$$

for any arbitrary θ. The time $t(\pi)$ can be expressed in seconds from knowledge of the Hubble parameter at $\theta = \pi$ and from the value of q at $\theta = \pi$. From equation (11) of Paper II, $q(\pi) = 0.079413$, and equation (22) of the same paper gives

$$t(\pi) = \frac{0.86539}{H(\pi)}. \tag{14}$$

The Hubble parameter is not known a priori at $q(\pi)$, but it is presumably known at the present epoch t_0 for which q_0 is known, and $H(\pi)$ can be found from the observed H_0 value, using the known value of q_0 and equation (8) of Paper II. The method is illustrated in Section III, where the decelerations are given in c.g.s. units.

b) The Euclidean Model ($k = 0$)

As usual, let t_0 be the present epoch and t_1 the time at which light left a particular galaxy in question. Follow this galaxy into the future to a new time of observation t_i and let the times of reception and emission of light at this future epoch be t_i and $t_{x,\,i}$, respectively. These definitions are illustrated in Figure 2, which plots $R(t) = t^{2/3}$.

After integration and divison by t_0, the invariant of equation (3) requires that

$$\left(\frac{t_x}{t_0}\right)^{1/3} = \left(\frac{t_i}{t_0}\right)^{1/3} + \left(\frac{t_1}{t_0}\right)^{1/3} - 1\,. \tag{15}$$

The ratio t_i/t_0 for any galaxy whose redshift at time t_0 is z_0 is found from equation (5), which shows that

$$1 + z_0 = \left(\frac{t_0}{t_1}\right)^{2/3}. \tag{16}$$

The ratio t_i/t_0 is the independent variable which can be chosen arbitrarily to make predictions for any desired time t_i in the future. Therefore, t_x/t_0 can be found from equation

(15) for the galaxy in question. The redshift of this galaxy observed at time t_i is then given by equation (5) as

$$1+z_i=\left(\frac{t_i}{t_x}\right)^{2/3}=\left[\left(\frac{t_i}{t_0}\right)\left(\frac{t_0}{t_x}\right)\right]^{2/3}, \tag{17}$$

which solves the problem.

Figure 2 illustrates the case for a galaxy with $z = 0.4$ at time t_0. We require the redshift at a time $t_i/t_0 = 2$ in the future. From equation (16) $t_0/t_1 = 1.6565$. Equation (15) then gives $t_x/t_0 = 1.3495$, which, when substituted in equation (17), requires that $z_i = 0.2999$. The time interval $t_i - t_0$ can be expressed in seconds, once the Hubble time H_0^{-1} is known at time t_0 and when the ratio t_i/t_0 is specified. Section III gives the decelerations in c.g.s. units.

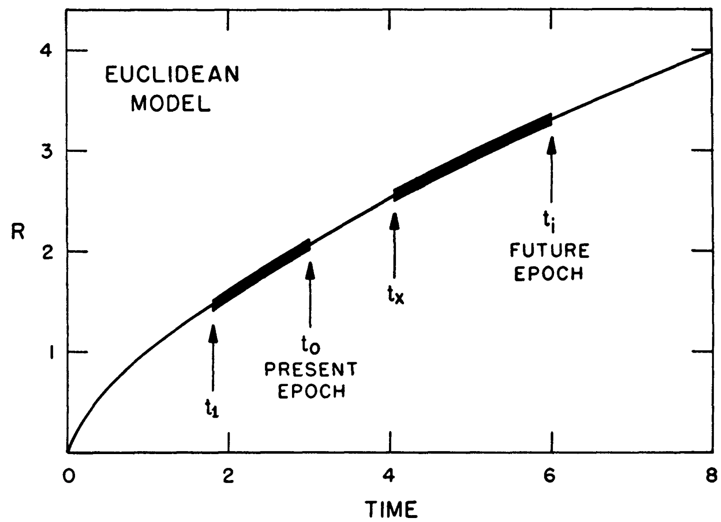

FIG. 2.—Same as Fig. 1, but for the Euclidean model. The definitions of the various times used in the text are shown The case illustrated is for a galaxy with $z = 0.4$ at the present epoch, which requires, from eq. (16), that $t_0/t_1 = 1\ 6565$. The future epoch is chosen such that $t_i/t_0 = 2$, which requires, from eq. (15), that $t_i/t_x = 1.3495$. Equation (17) then gives $z_i = 0\ 2999$.

c) *The Steady-State Model*

With $R = Be^{Ht}$ and

$$1+z_0=\frac{R_0}{R_1}=e^{Ht_0-Ht_1}, \tag{18}$$

equation (3) requires that

$$u=\frac{c\,z_0}{BH}\,e^{-Ht_0}=\text{Constant}, \tag{19}$$

where again t_0 is the present epoch and t_1 is the time when light was emitted. Consider, now, a new time of observation $t_{0,\,i}$ in the future, such that

$$t_{0,\,i}=t_0+nH^{-1}, \tag{20}$$

where n is an independent variable and is equal to the number of Hubble times we project into the future. Multiply equation (20) by H and use equation (19) at the two epochs of observation t_0 and t_i to obtain

$$z_i=e^n z_0 \tag{21}$$

for the answer. This is the only case of the four which requires an acceleration rather than deceleration. The physical reason for deceleration in the exploding models is clearly the braking action of the gravitational field. However, each galaxy in the steady-state model is accelerating, which can be accomplished only by the constant application of a repulsive force. Many workers have considered this to be a weakness in the usual formulation of the steady-state model because no physical theory for the origin of this repulsive force is given, except in the case of the "electric universe" of Lyttleton and Bondi (1959), which has apparently run into difficulty (Swann 1961).

II. THE CHANGE OF APPARENT LUMINOSITY WITH TIME

Because the distances to galaxies are presently increasing, the apparent luminosities must become fainter with time. The decrease is independent of, and superposed on, any secular change due to the evolution of the stellar content. Only the change due to increasing distance is considered in this section.

Tolman (1930) and later Robertson (1938) showed that the apparent bolometric luminosity l of a galaxy observed at time t_0 is given by

$$l=\frac{L}{4\pi R_0^2 r^2 (1+z)^2}, \tag{22}$$

where r is related to u by equation (4). For illustration, consider the oscillating model with $k = +1$ and $r = \sin u$. Converting equation (22) to astronomical magnitude, substituting equation (6) for R_0, equation (10) for u, equation (9) for $1 + z$, and considering L to be constant with time, gives

$$m_{\text{bol}} = 5 \log \left[\frac{a(1-\cos\theta_0)^2}{1-\cos\theta_1}\sin(\theta_0-\theta_1)\right]+C. \tag{23}$$

This is the general equation, of which Mattig's result (1958; Sandage 1961*a*, eq. [24]) is an important special case, obtained as follows. Substitution of

$$H_0=\frac{c\sin\theta_0}{a(1-\cos\theta_0)^2} \tag{24}$$

(Paper II, eq. [7]) in equation (23) gives

$$m_{\text{bol}} = 5 \log \left[\frac{c\sin\theta_0\sin(\theta_0-\theta_1)}{H_0(1-\cos\theta_1)}\right]+C \tag{25}$$

as an equivalent statement of equation (23). We assume that the world model is a priori unspecified and must be determined from an observation of q_0 using the magnitude-redshift relation by the method of Paper I (Sandage 1961*a*). We therefore look at all models satisfying equation (25) which have the same numerical value of H_0, found from observation of the real world. We now show that equation (25) reduces to Mattig's equation (13) (1958) for the $[m, z, q_0]$ relation if c and H_0 are absorbed in the constant term. The following relations have been, or can be, derived from equations (12) and (16) of Paper II:

$$\cos\theta_0=\frac{1-q_0}{q_0}, \tag{26}$$

$$\sin\theta_0=\frac{(2q_0-1)^{1/2}}{q_0}, \tag{27}$$

$$\cos\theta_1=\frac{z+\cos\theta_0}{1+z}=\frac{q_0(z-1)+1}{q_0(1+z)}, \tag{28}$$

and

$$\sin\theta_1 = \frac{(2q_0-1)^{1/2}(2q_0 z+1)^{1/2}}{q_0(1+z)}. \tag{29}$$

These last four equations require that

$$\sin(\theta_0-\theta_1) \equiv r = \frac{(2q_0-1)^{1\ 2}}{q_0^2(1+z)}\{q_0 z+(q_0-1)[(2q_0 z+1)^{1/2}-1]\}, \tag{30}$$

which is identical with Mattig's (1958) equation (12) when the relation between H_0, R_0, and q_0 is substituted from equation (10′) of Sandage (1961*a*). Finally, substitution of equations (27), (28), and (30) in equation (25), and considering cH_0^{-1} to be constant, gives

$$m_{\text{bol}} = 5\log\frac{1}{q_0^2}\{q_0 z+(q_0-1)[(2q_0 z+1)^{1/2}-1]\}+C, \tag{31}$$

which was to be proved.

Although equation (31) is correct for the analysis of the $[m, z]$ data obtained at the present epoch, it cannot be used to follow the evolution of a model universe with time because H_0 is not a constant. To follow the change of m for a given galaxy as the universe unfolds, we must use equation (23), which implicitly contains the variation of H_0. All parameters are known at all times in equation (23) for galaxies with redshift z by using the analysis of Section I and equations (26) and (28). The calculations based on equation (23) are given in the next section. Similar results for the $k = -1$ model follow in a straightforward way by replacing the circular functions with hyperbolic functions.

III. NUMERICAL RESULTS AND INTERPRETATION

a) The Oscillating Case

Table 1 gives the numerical results for the $[z, t]$, and $[m, t]$ relations for the oscillating model ($k = +1$, $q_0 \geq 0.5$). In making the computation, it was convenient to consider q_0 as the independent variable rather than time, but, for any given q_0 value, the dimensionless time intervals t/T were found from equations (11) and (27). The calculations were begun by assuming various z values at $q_0 = +1$ and following galaxies with these z values backward and forward in time. Each row of the table refers to a given galaxy; therefore, the variation of z and m is found by reading across the table in any given row. The final row shows the t/T value for the particular q_0 value. Figure 3 gives the resulting $[z, t]$ relation.

The magnitudes were computed from equation (23) with $C = 20.266$ to conform with the normalization of Paper I and with a absorbed into the constant. It will be recalled that this normalization is based on observational data for the first brightest galaxy in the great clusters and refers to an $m_r - k_r$ zero point.

The oscillating model is the most interesting of the four because it predicts that the expansion will eventually stop (at $\theta = \pi$, $q_0 = \infty$), and contraction will begin. The redshift of each galaxy will eventually drop to zero, and a blueshift will then begin. However, because of the finite speed of light, the *observed* reversal from redshift to blueshift takes place at different times for different galaxies. There will exist a time when the nearby galaxies appear to be approaching while more distant galaxies are still receding. This remarkable situation can be computed by the methods of Sections I and II. An illustration of the expected $[m, z]$ relation is given in Table 2 for the time when $q_0 = 5$ in the contracting phase (see Fig. 1 of Paper II). The magnitudes are computed from equation (23) with $C = 20.266$. The situation can also be seen in Figure 3 of this paper in the neighborhood of $q_0 \simeq 13$ to $q_0 \simeq 4$ in the contracting phase, where, at a given time of observation, both negative and positive redshifts exist.

TABLE 1

REDSHIFTS AND MAGNITUDES FOR GALAXIES IN THE OSCILLATING MODEL*

Expanding Phase

	q = 0.6		q = 0.8		q = 1.0		q = 1.6		q = 2.5		q = 5.0		q = 13.0		q = ∞	
z(q = +1)	z	m	z	m	z	m	z	m	z	m	z	m	z	m	z	m
0.02	0.0454	9.439	0 0259	11.159	0.0200	11.770	0.0134	12.449	0.0100	12.770	0.0067	13.019	0.0040	13.157	0.0001	13.235
0.05	0.1163	11.537	0.0651	13 167	0.0500	13.760	0.0335	14.418	0.0248	14.729	0.0166	14.967	0.0101	15.097	0.0006	15.155
0.10	0.2422	13.144	0.1312	14.702	0.1000	15.265	0.0666	15.890	0.0493	16.184	0.0332	16.406	0.0206	16.523	0.0021	16.565
0.20	0.5260	14.908	0.2666	16.264	0.2000	16.770	0.1314	17.335	0.0972	17.597	0.0658	17.790	0.0417	17.884	0.0070	17.892
0.40	1.2497	16.921	0.5503	17.873	0.4000	18.275	0.2563	18.733	0.1883	18.941	0.1284	19.084	0.0836	19.140	0.0213	19.092
0.60	2.2528	18.312	0.8512	18.849	0.6000	19.155	0.3750	19.519	0.2737	19.682	0.1868	19.784	0.1236	19.809	0.0379	19.717
0.80	3.6603	19.462	1.1694	19.562	0.8000	19.780	0.4882	20.060	0.3537	20.183	0.2414	20.246	0.1613	20.250	0.0550	20.122
1.00	5.6705	20.036	1.5052	20.130	1.0000	20.265	0.5965	20.468	0.4292	20.557	0.2923	20.594	0.1966	20.571	0.0718	20.412
1.50	16.135	22.941	2.4238	21.204	1.5000	21.145	0.8486	21.182	0.6000	21.198	0.4062	21.173	0.2755	21.105	0.1111	20.887
2.00	52.833	25.655	3.4600	22.008	2.0000	21.770	1.0786	21.666	0.7508	21.622	0.5046	21.549	0.3432	21.446	0.1459	21.182
4.00	Beyond		8.9466	24.145	4.0000	23.277	1.8443	22.743	1.2222	22.536	0.8000	22.334	0.5431	22.143	0.2500	21.771
10.00	the		46.991	27.840	10.000	25.267	3.3863	23.961	2.0611	23.509	1.2847	23.130	0.8581	22.824	0.4118	22.313
30.00	Horizon		37.5×10^4	47.440	30.000	27.652	5.9671	25.102	3.2443	24.354	1.8989	23.782	1.2351	23.361	0.5975	22.717
t/T	0.01515		0.05568		0.09084		0.16364		0.22509		0.30209		0.37595		0.50000	

Contracting Phase

	q = 13.0		q = 5.0		q = 2.5		q = 1.0		q = 0.8		q = 0.6	
z(q = +1)	z	m	z	m	z	m	z	m	z	m	z	m
0.02	−0.0038	13.140	−0.0064	12.991	−0.0096	12.728	−0.0192	11.686	−0.0247	11.049	−0.0435	9.246
0.05	−0.0089	15.056	−0.0151	14.898	−0.0228	14.625	−0.0454	13.554	−0.0582	12.900	−0.0991	10.752
0.10	−0.0159	16.444	−0.0277	16.274	−0.0421	15.986	−0.0833	14.870	−0.1061	14.191	−0.1755	12.254
0.20	−0.0259	17.739	−0.0470	17.547	−0.0724	17.233	−0.1429	16.041	−0.1802	15.319	−0.2868	13.256
0.40	−0.0358	18.886	−0.0712	18.662	−0.1129	18.306	−0.2222	17.000	−0.2768	16.167	−0.4199	13.978
0.60	−0.0384	19.471	−0.0847	19.220	−0.1423	18.824	−0.2727	17.445	−0.3371	16.619	−0.4965	14.261
0.80	−0.0374	19.842	−0.0924	19.571	−0.1548	19.160	−0.3077	17.707	−0.3783	16.848	−0.5462	14.404
1.00	−0.0345	20.105	−0.0997	19.816	−0.1665	19.386	−0.3333	17.880	−0.4083	16.997	−0.5812	14.486
1.50	−0.0234	20.525	−0.1000	20.204	−0.1837	19.737	−0.3750	18.136	−0.4569	17.207	−0.6355	14.580
2.00	−0.0110	20.781	−0.0983	20.437	−0.1921	19.943	−0.4000	18.276	−0.4860	17.316	−0.6668	14.613
4.00	+0.0331	21.272	−0.0816	20.873	−0.2000	20.318	−0.4444	18.505	−0.5383	17.478	−0.7215	14.620
10.00	+0.1089	21.703	−0.0419	21.242	−0.1989	20.598	−0.4762	18.655	−0.5777	17.562	−0.7620	14.557
30.00	+0.1983	22.008	+0.0100	21.493	−0.1690	20.813	−0.4918	18.725	−0.5998	17.582	−0.7855	14.467
t/T	0.62405		0.69791		0.77491		0.90916		0.94432		0.98485	

*The values for a given galaxy are found by reading a given row.

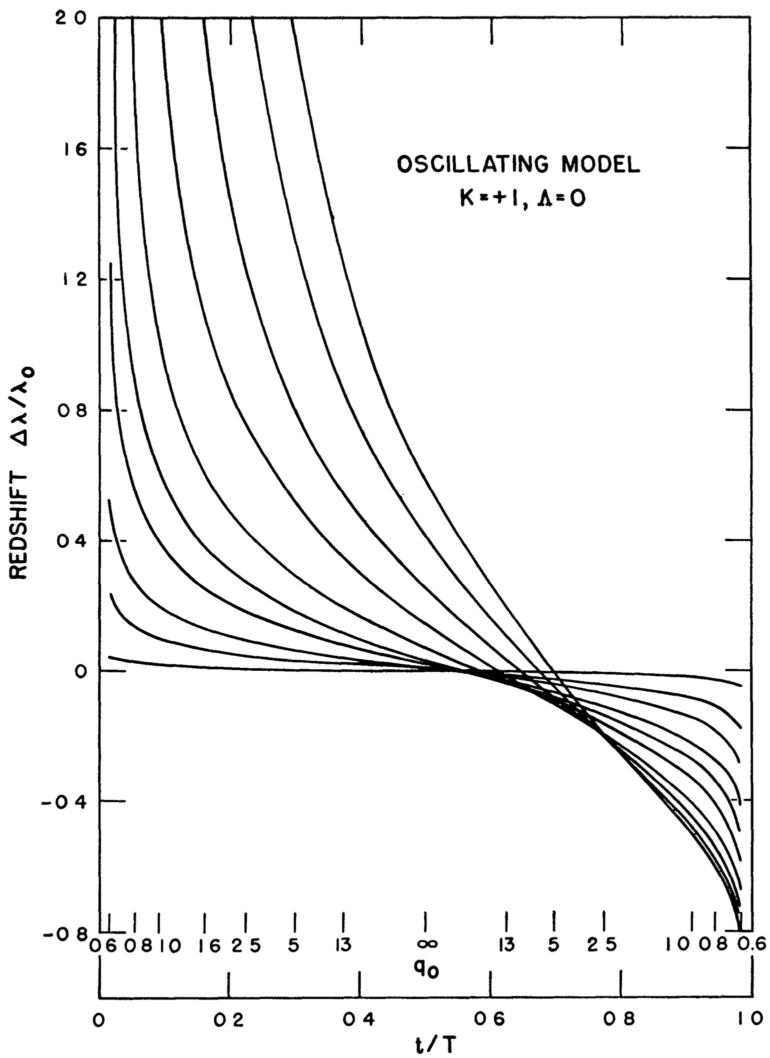

FIG. 3.—The change of redshift with time for the oscillating model. The abscissa gives t/T, where T is the total oscillation time of the model T can be expressed in seconds from eq. (12) when H_0 and q_0 are known. Values of q_0 for various phases t/T are shown above the abscissa.

TABLE 2

THE PREDICTED REDSHIFT-MAGNITUDE RELATION FOR $q_0 = 5$ IN CONTRACTING PHASE

z	$m_r—k_r$	z	$m_r—k_r$	z	$m_r—k_r$
−0 02	16 294	−0 10	19 649	0 00	21 454
− 05 .	17 702	− .05	21 188	+ 20	21 923
−0 07 ..	18 609	−0 02	21 364	+0 50	22 203

Another interesting special case exists for this model at $q_0 = \infty$ when the expansion of the universe has ceased. Even though $\dot{R} = 0$, all observed galaxies still exhibit a redshift because light left each galaxy at a time when $R_1 < R_0 = R_{\max}$. Furthermore, the magnitude-redshift relation at this time has a slope of 2.5 rather than 5, which is usually expected. This follows from equation (23) which requires, when $\theta_0 = \pi$, that

$$m_r - k_r = 5 \log \left(\frac{4 \sin \theta_1}{1 - \cos \theta_1}\right) + 20.266 . \tag{32}$$

Equation (28) then gives

$$1 - \cos \theta_1 = \frac{2}{1 + z},$$

and therefore

$$\sin \theta_1 = \frac{2 \sqrt{z}}{1 + z}.$$

Equation (32) then reduces to

$$m_r - k_r = 2.5 \log z + 23.276 . \tag{33}$$

Finally, it is of interest to compute the expected deceleration in c.g.s. units. Figure 3 shows that the deceleration is greatest in those galaxies with the largest redshifts. The practical limit of the 200-inch telescope for the observation of redshifts is about $z = 0.4$. Then consider a galaxy with $z = 0.4$, observed at a time when $q_0 = 1$ ($t/T = 0.09084$), in a model where $H_0^{-1} = 13 \times 10^9$ years. The change in redshift in the interval from $q_0 = 1$ to $q_0 = 1.6$ is 0.144 (see Table 1). The corresponding time interval is $0.07280T = 5.95 \times 10^9$ years by equations (11) and (12). These numbers give an average deceleration over the time interval of

$$\frac{\Delta c z}{\Delta t} = -0.73 \text{ cm/sec year} = -7.3 \times 10^{-6} \text{ km/sec year}.$$

Taking the limit for infinitesimal time intervals gives $c dz/dt = -11 \times 10^{-6}$ km/sec year by the considerations in the appendix, and this appears to be far below the level of any technique of measurement. If we take an optimistic view of the present accuracy of optical determinations of redshift, a change of 10 km/sec might possibly be detected for a galaxy with $z = 0.4$ if emission lines are present. If deceleration were the only cause of velocity changes, it would take 0.9×10^6 years before a change of 10 km/sec would occur. Another possible technique of observation would be to use the 21-cm radio emission line of neutral hydrogen. If this line were observed in a galaxy whose redshift was $z = 0.4$, the frequency of the light should increase at the rate of 3×10^{-2} cycle/sec per year, which again is far outside the limit of stable detection with today's conventional equipment. The accuracy of the measurement would have to be about one part in 10^{11} for a 1-year measuring interval.

b) The Hyperbolic Model

Calculations for $k = -1$ are given in Table 3 and are illustrated in Figure 4, where the times along the abscissa are $t/t(\pi)$ and where values of q_0 are shown above the $t/t(\pi)$ axis. The magnitudes are computed from the analogue of equation (23) valid for this case:

$$m_{\text{bol}} = 5 \log \left[\frac{(\cosh \theta_0 - 1)^2 \sinh(\theta_0 - \theta_1)}{(\cosh \theta_1 - 1)}\right] + C, \tag{34}$$

where C was taken to be 21.938. This follows from an arbitrary normalization that requires $m_r - k_r$ for a galaxy with $z = 0.02$ at the time when $q_0 = 0.3516$ [$t/t(\pi) = 0.03895$] to be the same as a galaxy at $z = 0.02$ when $q_0 = +1$ for the $k = +1$ model.

TABLE 3

REDSHIFTS AND MAGNITUDES FOR GALAXIES IN THE HYPERBOLIC MODEL*

z(70°)	θ = 50°		θ = 60°		θ = 70°		θ = 80°		θ = 100°		θ = 120°		θ = 130°	
	z	m	z	m	z	m	z	m	z	m	z	m	z	m
0.02	0.0267	10.192	0.0227	11.036	0.0200	11.770	0.0180	12.426	0.0154	13.580	0.0139	14.597	0.0133	15.072
0.05	0.0672	12.224	0.0570	13.054	0.0500	13.780	0.0450	14.430	0.0384	15.576	0.0344	16.588	0.0330	17.061
0.1	0.1362	13.796	0.1146	14.606	0.1000	15.317	0.0896	15.957	0.0761	17.090	0.0680	18.094	0.0652	18.564
0.2	0.2796	15.432	0.2317	16.200	0.2000	16.883	0.1778	17.503	0.1495	18.610	0.1328	19.599	0.1270	20.063
0.4	0.5888	17.179	0.4730	17.866	0.4000	18.495	0.3504	19.077	0.2890	20.136	0.2538	21.096	0.2417	21.551
0.6	0.9293	18.282	0.7237	18.889	0.6000	19.466	0.5184	20.013	0.4198	21.027	0.3648	21.962	0.3461	22.407
0.8	1.3042	19.116	0.9842	19.643	0.8000	20.171	0.6821	20.684	0.5434	21.657	0.4676	22.568	0.4422	23.005
1.0	1.7150	19.797	1.2540	20.245	1.0000	20.725	0.8419	21.206	0.6603	22.140	0.5632	23.030	0.5309	23.460
1.5	2.9196	21.133	1.9694	21.382	1.5000	21.748	1.2261	22.156	0.9285	23.004	0.7764	23.846	0.7270	24.260
2	4.4210	22.177	2.7439	22.225	2.0000	22.484	1.5914	22.826	1.1683	23.599	0.9608	24.401	0.8945	24.802
4	15.090	25.221	6.4577	24.403	4.0000	24.274	2.9070	24.399	1.9379	24.940	1.5169	25.624	1.3897	25.987
10	501.60	33.331	24.941	27.747	10.0000	26.624	6.0096	26.305	3.3711	26.440	2.4395	26.940	2.1809	27.246
30	....	......	352.53	33.891	30.0000	29.339	12.598	28.216	5.5046	27.774	3.6192	28.052	3.1460	28.293
q	0.41570		0.38457		0.35163		0.31809		0.25309		0.19524		0.16989	
t/t(π)	0.01369		0.02405		0.03895		0.05947		0.12267		0.22651		0.29902	

z(70°)	θ = 140°		θ = 150°		θ = 160°		θ = 170°		θ = 180°		θ = 190°	
	z	m	z	m	z	m	z	m	z	m	z	m
0.02	0.0129	15.529	0.0125	15.973	0.0122	16.406	0.0120	16.829	0.0117	17.246	0.0116	17.655
0.05	0.0319	17.517	0.0310	17.959	0.0303	18.391	0.0297	18.814	0.0291	19.230	0.0288	19.639
0.1	0.0630	19.018	0.0611	19.459	0.0596	19.889	0.0584	20.311	0.0573	20.725	0.0565	21.134
0.2	0.1224	20.513	0.1186	20.950	0.1156	21.378	0.1131	21.797	0.1109	22.210	0.1093	22.616
0.4	0.2321	21.992	0.2243	22.424	0.2181	22.846	0.2130	23.261	0.2087	23.670	0.2053	24.074
0.6	0.3314	22.842	0.3195	23.267	0.3100	23.685	0.3022	24.096	0.2958	24.502	0.2907	24.904
0.8	0.4222	23.434	0.4062	23.854	0.3934	24.267	0.3830	24.675	0.3743	25.078	0.3675	25.478
1.0	0.5056	23.883	0.4855	24.298	0.4694	24.708	0.4564	25.113	0.4456	25.513	0.4371	25.910
1.5	0.6886	24.670	0.6584	25.076	0.6344	25.477	0.6150	25.875	0.5991	26.271	0.5864	26.664
2	0.8436	25.201	0.8037	25.598	0.7720	25.993	0.7466	26.386	0.7260	26.777	0.7095	27.166
4	1.2940	26.357	1.2204	26.731	1.1629	27.107	1.1174	27.485	1.0807	27.864	1.0515	28.244
10	1.9930	27.572	1.8524	27.912	1.7447	28.262	1.6608	28.618	1.5943	28.980	1.5416	29.346
30	2.8155	28.571	2.5752	28.874	2.3954	29.195	2.2578	29.530	2.1502	29.874	2.0658	30.225
q	0.14706		0.12673		0.10879		0.09308		0.07941		0.06759	
t/t(π)	0.38890		0.49953		0.63493		0.79986		1.00000		1.24209	

*The values for a given galaxy are found by reading a given row.

A sample deceleration calculation for a galaxy at $z = 0.4$ at the time when $q_0 = 0.3516$ [$\theta_0 = 70°$ and $t/t(\pi) = 0.03895$] gives

$$\frac{\Delta(cz)}{\Delta t} = -3 \times 10^{-6}\ \text{km/sec year}$$

for the average deceleration in the time interval from $\theta = 70°$ to $\theta = 80°$. To obtain this number we have assumed that $H_0^{-1} = 13 \times 10^9$ years at the epoch when $q_0 = 0.3516$ ($\theta_0 = 70°$) and, therefore, from equation (8) of Paper II that

$$\frac{H(70°)}{H(\pi)} = 21.1324 \qquad \text{or} \qquad H(\pi)^{-1} = 2.747 \times 10^{11}\ \text{years}.$$

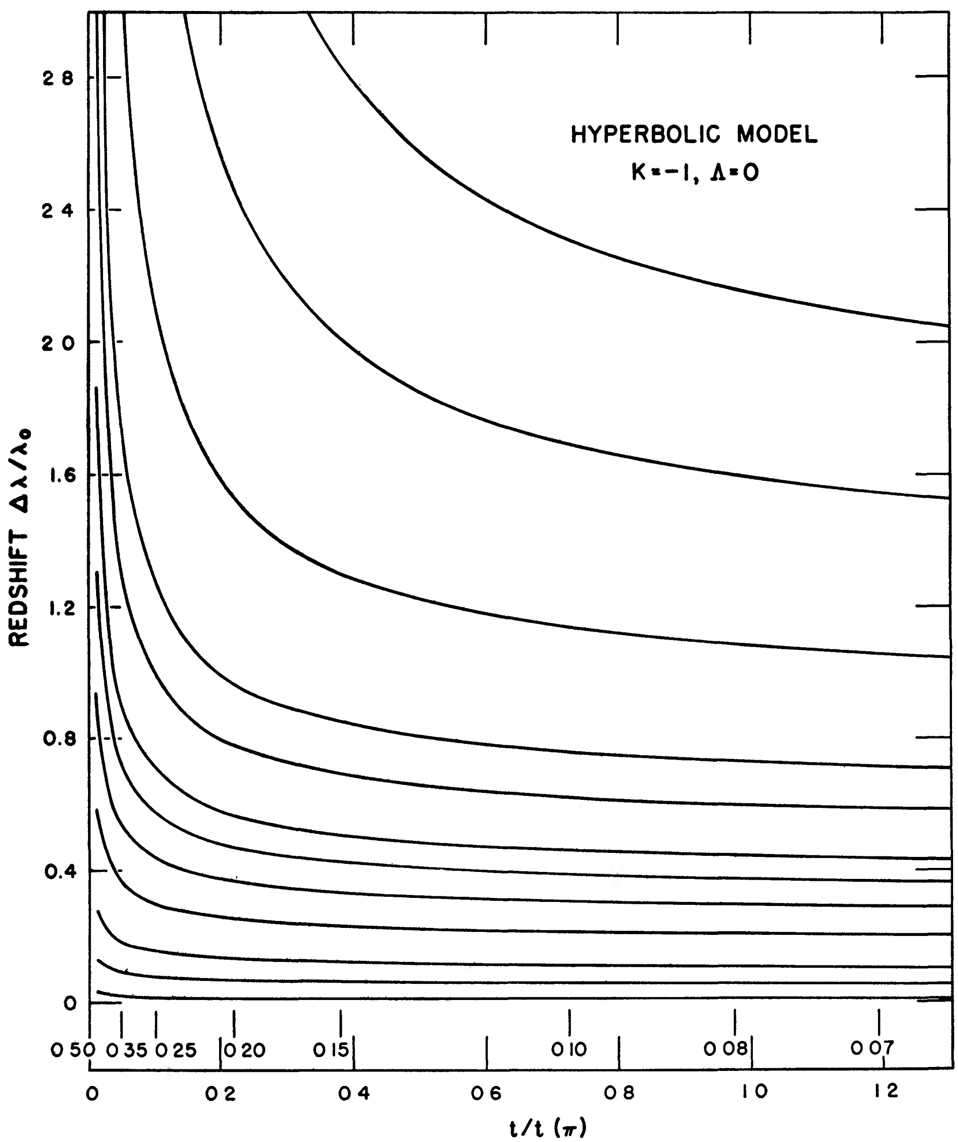

FIG. 4.—Change of redshift with time for the hyperbolic model. The time scale is expressed in terms of $t(\pi)$. Values of q_0 for various times are also shown.

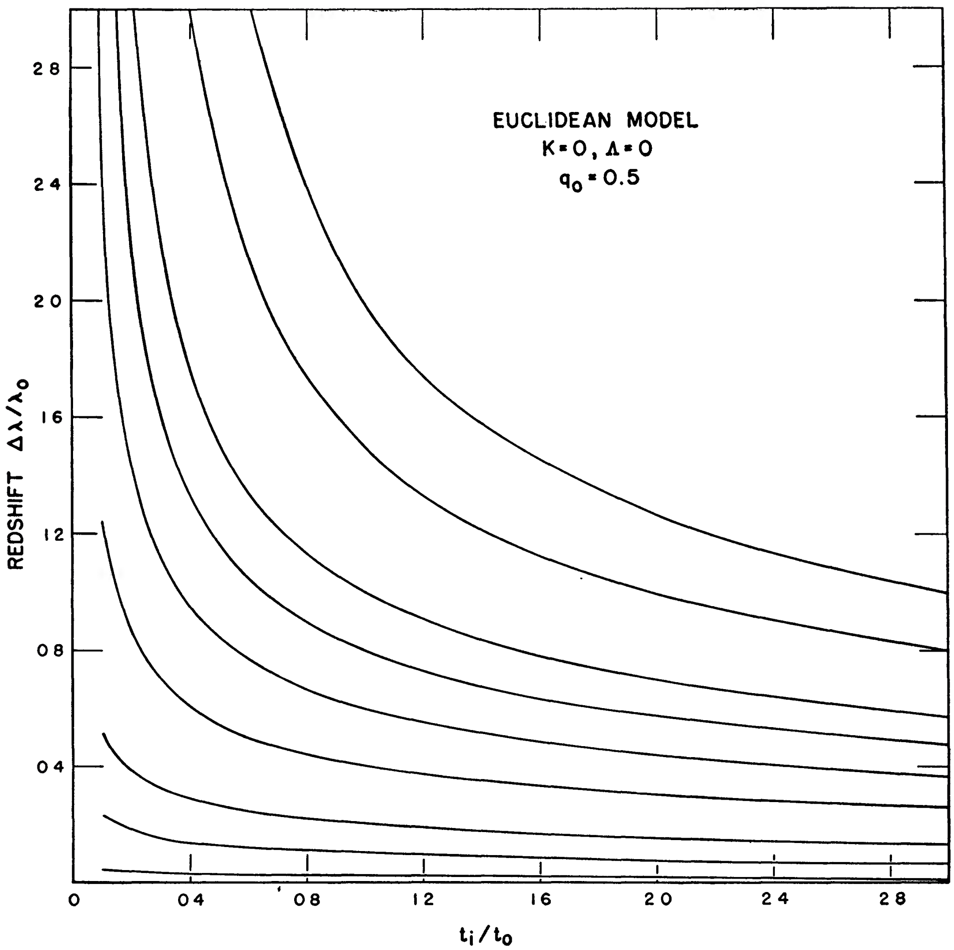

FIG. 5.—Change of redshift with time for the Euclidean model

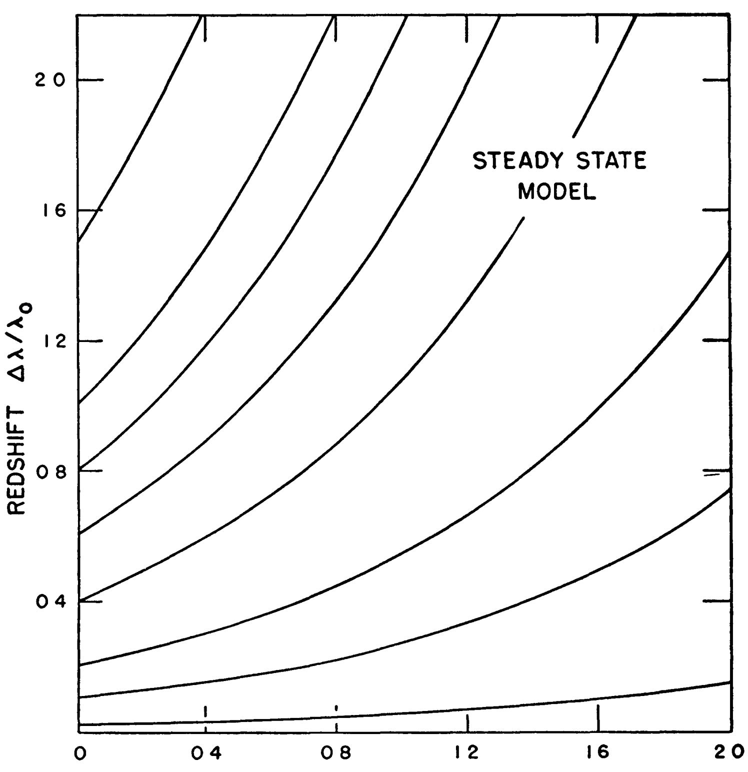

FIG 6.—Change of redshift with time for the steady-state model

TABLE 4

CHANGE OF REDSHIFT WITH TIME FOR THE EUCLIDEAN MODEL*

t_1/t_0	0.10	0.20	0.40	0.60	0.80	1.00	1.4	2.0	3.0
$z(t_0)$									
0.02	0.0438	0.0346	0.0273	0.0238	0.0216	0.0200	0.0178	0.0158	0.0138
0.05	0.1125	0.0878	0.0688	0.0597	0.0540	0.0500	0.0445	0.0394	0.0343
0.10	0.2353	0.1804	0.1394	0.1202	0.1083	0.1000	0.0887	0.0782	0.0680
0.20	0.5156	0.3808	0.2862	0.2437	0.2179	0.2000	0.1761	0.1541	0.1327
0.40	1.2518	0.8500	0.6029	0.5003	0.4405	0.4000	0.3471	0.2999	0.2550
0.60	2.3203	1.4271	0.9519	0.7698	0.6675	0.6000	0.5137	0.4385	0.3686
0.80	3.9080	2.1374	1.3352	1.0520	0.8989	0.8000	0.6763	0.5708	0.4748
1.0	6.3453	3.0136	1.7550	1.3471	1.1343	1.0000	0.8352	0.6975	0.5746
1.5	22.081	6.2454	2.9814	2.1412	1.7404	1.5000	1.2180	0.9934	0.8009
2	124.04	12.007	4.5007	3.0178	2.3702	2.0000	1.5832	1.2644	1.0009
4	....	332.63	15.0319	7.4212	5.1108	4.0000	2.9078	2.1746	1.6292
10	....	....	368.76	32.861	15.315	10.000	6.0877	4.0361	2.7602
30	....	....	....	1339.3	72.989	30.000	13.065	7.2171	4.3790

*The values for a given galaxy are found by reading a given row.

Equation (14) then gives $t(\pi) = 2.377 \times 10^{11}$ years, and the above deceleration follows from Table 3. Again taking the limit for infinitesimal time intervals gives

$$c\frac{dz}{dt} = -4.2 \times 10^{-6} \text{ km/sec years}$$

by the considerations in the appendix.

c) The Euclidean Model

Figure 5 and Table 4 give the data for this model. Magnitudes were not computed because they follow the same general pattern as for the hyperbolic model. The deceleration of a galaxy which is observed with $z = 0.4$ at the present epoch and accepting $H_0^{-1} = 13 \times 10^9$ years is

$$\frac{dcz}{dt} = -5.9 \times 10^{-6} \text{ km/sec year}.$$

d) The Steady-State Model

Figure 6 shows the acceleration of galaxies for the steady-state model. Equation (21) gives

$$\frac{\Delta cz}{\Delta t} = \frac{cz_0(e^n - 1)H}{n},$$

or

$$\lim_{n\to 0} \frac{\Delta cz}{\Delta t} \equiv \frac{d(cz)}{dt} = cz_0 H. \tag{35}$$

For $z_0 = 0.4$ and $H_0^{-1} = 13 \times 10^9$ years,

$$\frac{d(cz)}{dt} = +9.2 \times 10^{-6} \text{ km/sec year}.$$

IV. CONCLUSION

1. The foregoing considerations show that an "ideal" deceleration test exists between the exploding and the steady-state models in the sense that the *sign* of the effect is reversed. However, for the test to be useful, it would seem that a precision redshift catalogue must be stored away for the order of 10^7 years before an answer can be found because the decelerations are so small by terrestrial standards.

2. For all models, except the oscillating case, it will become more and more difficult to obtain observational information from the universe because the apparent luminosities of galaxies decrease with time. Indeed, if the oscillating case is excluded, there will be a time in the very distant future when most galaxies will recede beyond the limit of easy observation and when data for extragalactic astronomy must be collected from ancient literature.

REFERENCES

Friedman, A. 1922, *Zs. f. Phys.*, **10**, 377.
Lyttleton, R. A., and Bondi, L. H. 1959, *Proc. R. Soc. London, A*, **252**, 313.
Mattig, W. 1958, *A.N.*, **284**, 109.
Robertson, H. P. 1933, *Rev. Mod. Phys.*, **5**, 62.
———. 1938, *Zs. f. Ap.*, **15**, 69.
Sandage, A. R. 1961*a*, *Ap. J.*, **133**, 355.
———. 1961*b*, *ibid.*, **134**, 916.
Swann, W. F. G. 1961, *Ap. J.*, **133**, 738.
Tolman, R. C. 1930, *Proc. Nat. Acad. Sci.*, **16**, 511.