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Secular Perturbations of Asteroids with High Inclination and Eccentricity

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Secular perturbations of asteroids with high inclination and eccentricity moving under the attraction of the sun and Jupiter are studied on the assumption that Jupiter's orbit is circular. After short-periodic terms in the Hamiltonian are eliminated, the degree of freedom for the canonical equations of motion can be reduced to 1.

Since there is an energy integral, the equations can be solved by quadrature. When the ratio of the semimajor axes of the asteroid and Jupiter takes a very small value, the solutions are expressed by elliptic functions.

When the z component of the angular momentum (that is, Delaunay's H) of the asteroid is smaller than a certain limiting value, there are both a stationary solution and solutions corresponding to libration cases. The limiting value of H increases as the ratio of the semimajor axes increases, i.e., the corresponding limiting inclination drops from 39°.2 to 1°.8 as the ratio of the axes increases from 0.0 to 0.95.

I. INTRODUCTION

THE stability of the solar system has been proved in the sense that no secular change occurs in the semimajor axes of planetary orbits, and that secular changes of the eccentricities and inclinations are limited within certain small domains. However, the classical theory of secular perturbations for the eccentricity and inclination is based on the assumption that the squares of the eccentricity and inclination are negligible. Although this assumption may be reasonable for major planets, it may not be for some asteroids.

The assumption in the classical theory means that a term such as $Be^2 \sin^2 i \cos 2g$ is negligibly small as compared with the principal term $A(e^2-\sin^2 i)$ in the secular part of the disturbing function. However, as the value of B increases much more rapidly than does that of A with the ratio of the semimajor axes of the asteroid and the disturbing planet, the B term cannot be neglected when the eccentricity and inclination assume large values. For example, the rate of change of the argument of perihelion, which is proportional to $A+B\sin^2 i \cos 2g$, may vanish at a certain point when the inclination of the asteroid takes a reasonably large value.

In the case of a close artificial satellite moving around the oblate earth, B vanishes in the first-order disturbing function. Therefore, we could rather easily solve the equations of motion for arbitrary values of the eccentricity and inclination. However, as the semimajor axis of the satellite becomes larger, the gravitational effects of the moon become more important and perturbations due to the moon become large and complicated, as has been discussed by Musen (1962), who used high-speed computing machines. A lunar problem with high inclination was similarly studied by Lidov (1962). Secular perturbations of meteors and comets have been investigated by Hamid (1962).

The present paper treats an analytical theory on secular perturbations of asteroids with high inclination and eccentricity by assuming that only Jupiter, moving in a circular orbit, is the disturbing body. This theory may, of course, be applied also to comets or satellites disturbed by the sun.

The conventional technique for developing the disturbing function cannot be adopted here, since neither the eccentricity nor the inclination is considered small. Nor can numerical harmonic analysis be adopted, since variations of orbital elements may not be regarded as small quantities. Therefore, the disturbing function has to be developed into a power series of α, the ratio of the semimajor axes of the asteroid and Jupiter, although convergence of the series may be slow.

Short-periodic terms depending on the two mean anomalies can be eliminated from the disturbing function by Delaunay's transformations. The longitudes of the ascending nodes of Jupiter and the asteroid disappear by the theorem on elimination of nodes. Therefore, the equations of motion for the asteroid are reduced to canonical equations of one degree of freedom with a time-independent Hamiltonian. Therefore, the equations can be solved by a quadrature.

In fact, the solutions can be expressed by elliptic functions approximately when α takes a very small value. For this case there are both one stationary and some libration solutions when $(1-e^2)\cos^2 i$, which is constant, is smaller than 0.6.

As α increases, the upper limit of $(1-e^2)\cos^2 i$ for the existence of a stationary solution increases. When α is 0.85, the limit is as large as 0.90.

Without the aid of a high-speed computer, it is rather difficult to estimate the effects of Jupiter's eccentricity and of other disturbing planets for general cases. However, the results of the present analysis may serve as a guide for future research in numerical integration.

II. EQUATIONS OF MOTION

Consider an asteroid moving under the attraction of the sun and Jupiter. The mass of the asteroid m is negligibly small compared with Jupiter's mass and the solar mass, which is taken as the unit.

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Differential equations of motion for the asteroid are written in canonical form with Delaunay's variables:

$$L=ka^{\frac{1}{2}},\qquad l=\text{mean anomaly},$$
$$G=L(1-e^2)^{\frac{1}{2}},\qquad g=\text{argument of perihelion},\qquad (1)$$
$$H=G\cos i,\qquad h=\text{longitude of ascending node},$$

where k^2 is the gravity constant of Gauss.

Jupiter's canonical elements are expressed by primes and k' such as

$$k'^2=k^2m^{-2}m'^2/(1+m')=k^2m^{-2}\mu'^2(1+m'),\qquad (2)$$

where μ' is the reduced mass of Jupiter.

Coordinates of Jupiter are referred to the center of the sun, and those of the asteroids to the center of gravity of the sun and Jupiter. The Hamiltonian for this system is written as follows:

$$F=\frac{k^4}{2L^2}+m\mu'^{-1}\frac{k'^4}{2L'^2}+k^2\mu'\left\{\left[r^2-2rr'\frac{s}{1+m'}+\left(\frac{r'}{1+m'}\right)^2\right]^{-\frac{1}{2}}-\frac{r}{r'^2}\frac{s}{1+m'}\right\},\qquad (3)$$

where

$$s=(xx'+yy'+zz')/rr'.\qquad (4)$$

When we adopt the invariable plane as (x,y) plane, the following relations hold:

$$h=h',\qquad (5)$$

$$m^2(G^2-H^2)=\mu'^2(G'^2-H'^2).\qquad (6)$$

TABLE I. Limiting value of $(H_0/L^*)^2$ and i_0.

α	$(H_0/L^*)^2$	i_0	i_0 approx
0.00	0.60 000	39°231	39°231
0.05	0.60 116	39.164	39.164
0.10	0.60 464	38.960	38.960
0.15	0.61 043	38.620	38.620
0.20	0.61 849	38.146	38.146
0.25	0.61 880	37.536	37.535
0.30	0.64 133	36.791	36.790
0.35	0.65 599	35.911	35.905
0.40	0.67 274	34.894	34.875
0.45	0.69 154	33.738	33.694
0.50	0.71 230	32.437	32.355
0.55	0.73 495	30.986	30.860
0.60	0.75 940	29.374	29.239
0.65	0.78 556	27.586	27.566
0.70	0.81 330	25.600	25.925
0.75	0.84 252	23.380	24.410
0.80	0.87 305	20.874	23.078
0.85	0.90 488	17.964	21.926
0.90	0.94 581	13.460	20.963
0.95	0.99 900	1.811	...

Relation (5) is known as an integral of elimination of nodes in the three-body problem. Since the inclination of Jupiter's orbit is of the order of m/μ' by relation (6), the invariable plane coincides almost with Jupiter's orbital plane.

As the Hamiltonian F depends on h and h' as a combination $h-h'$, the variables h and h' can be eliminated from F by the relation (5). Therefore, H and H' are constant.

As the inclination of Jupiter can be regarded as zero, the expression of s takes the following form:

$$s=\cos(f+g)\cos(f'+g')+\cos i\sin(f+g)\sin(f'+g'),\qquad (7)$$

where f and f' are true anomalies.

The variables l and l' can be eliminated from the Hamiltonian by either Delaunay's or von Zeipel's method, that is, by a canonical transformation

$$(L,G,L',G',l,g,l',g')\rightarrow(L^*,G^*,L'^*,G'^*,l^*,g^*,l'^*,g'^*).$$

Since the new Hamiltonian F^* does not depend on l^* and l'^*, L^* and L'^* are constant. The Hamiltonian takes the following form when terms with m'^2 as a factor are neglected:

$$F^*=(k^4/2L^{*2})+m'W^*,\qquad (8)$$

with

$$W=\frac{k^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{(r'^2-2rr's+r^2)^{\frac{1}{2}}}dl\,dl'.\qquad (9)$$

The last term in (3) has been dropped, since it does not produce any secular term.

Although variations of G'^* are negligibly small, those of g'^* and relative variation of G'^*-H' appearing in the equation of g'^* as this combination are of the order of m', since Jupiter's inclination $(G'^*-H')^{\frac{1}{2}}$ is of the order of m/m'. Therefore, the canonical equations of two degrees of freedom with an energy integral should be solved simultaneously.

When we assume that Jupiter's eccentricity is negligibly small, g'^* and G'^* disappear in F^*. Then the degree of freedom is reduced to one, and the equations of motion for the asteroid are written as

$$\frac{dG^*}{dt}=m'\frac{\partial W^*}{\partial g^*},\quad \frac{dg^*}{dt}=-m'\frac{\partial W^*}{\partial G^*},\qquad (10)$$

with an integral

$$W^*=\text{const.}\qquad (11)$$

Equations (10) can be solved by a quadrature.

III. STATIONARY POINT

When Jupiter's eccentricity is assumed to be zero, W^* takes the following form:

$$W^*=\sum_{j=0}A_j(\alpha,G^*,H)\cos 2jg^*,\qquad (12)$$

where

$$\alpha=(k'L^*/kL'^*)^2. \tag{13}$$

Therefore, dg^*/dt vanishes when $\sin 2g^*$ is zero. System (10) has a stationary solution, when one of the following equations has a solution G^* that satisfies an inequality (16):

$$\sum_{j=0} \frac{\partial A_j}{\partial G^*}=0 \quad (\cos 2g^*=1), \tag{14}$$

$$\sum_{j=0} (-1)^j \frac{\partial A_j}{\partial G^*}=0 \quad (\cos 2g^*=-1), \tag{15}$$

$$H \leq G^* \leq L^*. \tag{16}$$

It has been proved numerically that Eq. (14) does not have such a solution except for $H=G^*=0$ and the equation $dg^*/dt=0$ has no meaningful solution other than $\sin 2g^*=0$, at least when α is less than 0.8.

Equation (15) has a solution when H is equal to or smaller than a limiting value H_0. When H is equal to H_0, the stationary solution appears at $G^*=1$. As H decreases, Eq. (14) has a smaller value of G^* as the root, and when H is zero, $G^*=0$ corresponds to the stationary value. When H is equal to H_0, the corresponding inclination is derived by

$$H_0=L^* \cos i_0. \tag{17}$$

Both H_0 and i_0 depend on α and are derived by numerical harmonic analysis of $\partial W^*/\partial G^*$. The results are given in Table I and as a solid line in Fig. 1. Besides the numerical harmonic analysis, values of i_0 are derived analytically by developing the disturbing function into power series of α up to the eighth degree, shown in the last column of Table I and as a broken line in Fig. 1. Comparison of the two lines in Fig. 1 shows that the analytical method can provide rather good values for i_0 up to $\alpha=0.7$.

In the first approximation, i_0 and H_0 do not depend on Jupiter's mass m'. The value of i_0 drops from 39°.2 to 1°.8 as α increases from 0.0 to 0.95. However, there are few asteroids that have H smaller than H_0. When α is larger than 0.95, there may be a stationary solution for any value of H.

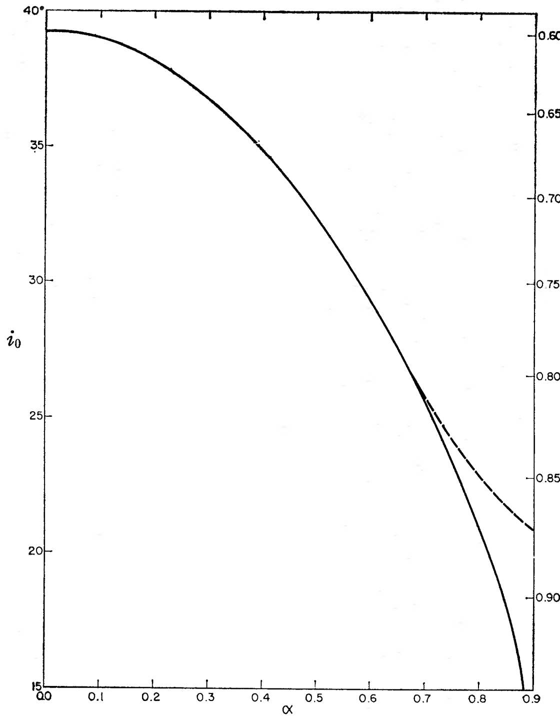

FIG. 1. Limiting value of Θ (or i). Solid line is computed by numerical harmonic analysis and broken line by power series, where $\Theta=(H/L)^2$.

IV. DISTURBING FUNCTION

When α takes a small value, the principal part of the disturbing function is developed into a power series of r/r' by means of Legendre's polynomials as

$$R=k^2m'\frac{1}{(r^2-2rr's+r'^2)^{\frac{1}{2}}}=\frac{k^2m'}{r'}\sum_{j=0}^{\infty} P_j(s)\left(\frac{r}{r'}\right)^j. \tag{18}$$

When l' is eliminated, the P_1 term vanishes, and after Jupiter's eccentricity is neglected, all other odd-order terms are dropped from the disturbing function, that is,

$$R_1=\frac{1}{2\pi}\int_0^{2\pi}(R)_{e'=0}dl'=\frac{k^2m'}{a'}\sum_{j=0}^{\infty} P_{2j}(s_1)\left(\frac{r}{a'}\right)^{2j}. \tag{19}$$

Expressions of s_1^{2j} and $P_{2j}(s_1)$ are given below:

$$\begin{aligned}
s_1^2=&\tfrac{1}{4}[1+\cos^2 i+\sin^2 i \cos 2(f+g)],\\
s_1^4=&(3/64)[3+2\cos^2 i+3\cos^4 i+4\sin^2 i(1+\cos^2 i)\\
&\times\cos 2(f+g)+\sin^4 i \cos 4(f+g)],\\
s_1^6=&(5/512)[2(5+3\cos^2 i+3\cos^4 i+5\cos^6 i)\\
&+3(5+6\cos^2 i+5\cos^4 i)\sin^2 i \cos 2(f+g)\\
&+6(1+\cos^2 i)\sin^4 i\cos 4(f+g)\\
&+\sin^6 i\cos 6(f+g)],
\end{aligned} \tag{20}$$

$$\begin{aligned}
s_1^8=&(35/16\,384)[35+20\cos^2 i+18\cos^4 i+20\cos^6 i\\
&+35\cos^8 i+8(7+2\cos^2 i+7\cos^4 i)(1+\cos^2 i)\\
&\times\sin^2 i\cos 2(f+g)+4(7+10\cos^2 i+7\cos^4 i)\\
&\times\sin^4 i\cos 4(f+g)+8(1+\cos^2 i)\sin^6 i\cos 6(f+g)\\
&+\sin^8 i\cos 8(f+g)],
\end{aligned}$$

$$\begin{aligned}
P_2(s_1)=&\tfrac{1}{8}[-(1-3\cos^2 i)+3\sin^2 i\cos 2(f+g)],\\
P_4(s_1)=&(3/512)[3(3-30\cos^2 i+35\cos^4 i)\\
&-20\sin^2 i(1-7\cos^2 i)\cos 2(f+g)\\
&+35\sin^4 i\cos 4(f+g)],\\
P_6(s_1)=&(5/8192)\\
&\times[-10(5-105\cos^2 i+315\cos^4 i-231\cos^6 i)\\
&+105\sin^2 i(1-18\cos^2 i+33\cos^4 i)\cos 2(f+g)\\
&-126\sin^4 i(1-11\cos^2 i)\cos 4(f+g)\\
&+231\sin^6 i\cos 6(f+g)],
\end{aligned} \tag{21}$$

$$P_8(,_1)=(35/2\,097\,152)\times[35(35-1260\cos^2 i+6930\cos^4 i-12\,012\cos^6 i+6435\cos^8 i)-2520\sin^2 i\times(1-33\cos^2 i+143\cos^4 i-143\cos^6 i)\times\cos 2(f+g)+2772\sin^4 i(1-26\cos^2 i+65\cos^4 i)\times\cos 4(f+g)-3432\sin^6 i(1-15\cos^2 i)\times\cos 6(f+g)+6435\sin^8 i\cos 8(f+g)].$$

Finally, by using the following formula (Tisserand 1889),

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^q\cos pf\,df=(-1)^p\frac{(q+2)(q+3)\cdots(p+q+1)}{1\cdot 2\cdots p}\left(\frac{e}{2}\right)^p\times F\left(\frac{p-q-1}{2},\frac{p-q}{2},p+1,e^2\right),\quad(22)$$

(F being a hypergeometric function), W^* is derived as follows:

$$W^*=(k^2/a')\alpha^2\{\tfrac{1}{16}[-(1-3\theta^2)(5-3\eta^2)+15(1-\theta^2)(1-\eta^2)\cos 2g^*]+(9/2^{12})\alpha^2\times[(3-30\theta^2+35\theta^4)(63-70\eta^2+15\eta^4)-140(1-\theta^2)(1-7\theta^2)(1-\eta^2)(3-\eta^2)\cos 2g^*+735(1-\theta^2)^2(1-\eta^2)^2\cos 4g^*]+(5/2^{17})\alpha^4\times[-10(5-105\theta^2+315\theta^4-231\theta^6)\times(429-693\eta^2+315\eta^4-35\eta^6)+315(1-\theta^2)\times(1-18\theta^2+33\theta^4)(1-\eta^2)(143-110\eta^2+15\eta^4)\times\cos 2g^*-4158(1-\theta^2)^2(1-11\theta^2)(1-\eta^2)^2\times(13-3\eta^2)\cos 4g^*+99\,099(1-\theta^2)^3(1-\eta^2)^3\cos 6g^*]+(175/2^{28})\alpha^6[7(35-1260\theta^2+6930\theta^4-12\,012\theta^6+6435\theta^8)(12\,155-25\,740\eta^2+18\,018\eta^4-4620\eta^6+315\eta^8)-27720(1-\theta^2)(1-33\theta^2+143\theta^4-143\theta^6)\times(1-\eta^2)(221-273\eta^2+91\eta^4-7\eta^6)\cos 2g^*+396\,396(1-\theta^2)^2(1-26\theta^2+65\theta^4)(1-\eta^2)^2\times(17-10\eta^2+\eta^4)\cos 4g^*-490\,776(1-\theta^2)^3\times(1-15\theta^2)(1-\eta^2)^3(17-3\eta^2)\cos 6g^*+15\,643\,485(1-\theta^2)^4(1-\eta^2)^4\cos 8g^*]\},\quad(23)$$

where

$$\theta=H/G^*,\quad \eta=G^*/L^*.\quad(24)$$

The limiting value of H is derived from the equation

$$(\partial W^*/\partial G^*)_{\cos 2g^*=-1,\eta=1}=0,$$

that is,

$$-5\Theta+3+(15/32)(-49\Theta^2+46\Theta-5)\alpha^2+(175/512)(-297\Theta^3+417\Theta^2-143\Theta+7)\alpha^4+(18\,375/65\,536)(-1573\Theta^4+2974\Theta^3-1738\Theta^2+320\Theta-9)\alpha^6=0,\quad(25)$$

where

$$\Theta=(H/L^*)^2.\quad(26)$$

Equation (25) gives the limiting value Θ_0 corresponding to H_0 as a function of α. When α is zero, Θ_0 is equal to 0.6.

V. CASE FOR SMALL α

When α is small enough so that we can neglect α^2 in the braces { } in W^* (23), Eqs. (10) can be integrated by using an elliptic function of Weierstrauss.

For this case the energy integral (11) is written as

$$-(1-3\Theta x^{-1})(5-3x)+15(1-\Theta x^{-1})(1-x)\cos 2g^*=\mathrm{C},\quad(27)$$

where C is a modified energy constant, Θ is the constant defined in (26), and

$$x=\eta^2.\quad(28)$$

The constant C is expressed by x_0, the value of x at $g^*=0$, as follows:

$$C=10-12x_0+6\Theta.\quad(29)$$

Since $\cos 2g^*$ can be solved as a function of x by using the integral (27), the first equation of (10), dG^*/dt, can be transformed to the equation containing x as the only dependent variable, that is,

$$dx/dt=\mp\tfrac{3}{2}n\alpha^3 m'[2(x-x_0)y]^{\frac{1}{2}},\quad(30)$$

where n is the mean motion which is constant and

$$y=3x^2-x(5+5\Theta-2x_0)+5\Theta.\quad(31)$$

In Eq. (30) the minus sign corresponds to positive values of $\sin 2g^*$, and the plus sign corresponds to negative values. And x should be between 1 and Θ, which is itself between 1 and zero.

The solutions of the equation are classified into the following four types, according to the value of x_0:

(1) When $x_0=\Theta$, the equation $y=0$ has the following two roots:

$$x=\Theta,\quad\text{and}\quad x=5/3.\quad(32)$$

Since x cannot reach 5/3, x is always equal to Θ and g^* makes a complete revolution. For this case, the inclination of the asteroid is always zero.

(2) When $x_0=1$, the roots of the equation $y=0$ are

$$x=1,\quad\text{and}\quad x=(5/3)\Theta.\quad(33)$$

Therefore, either x is always equal to 1 (circular orbit) or x changes between 1 and $5\Theta/3$ when Θ is less than $\Theta_0(=0.6)$. For $\Theta=\Theta_0$, the stationary solution exists at

$$x=1,\quad\text{and}\quad\cos 2g^*=-1.\quad(34)$$

When Θ is less than Θ_0, g^* cannot make one revolution.

(3) When x_0 is between 1 and Θ, one of the roots of $y=0$ is between 1 and Θ, and the other is larger than 1. For this case g^* makes a complete revolution, and x decreases from x_0 as $2g^*$ proceeds from 0° to 180°, where x takes a minimum value. The value of x increases again as $2g^*$ goes from 180° to 360°.

There is no reasonable solution when x_0 is less than Θ.

(4) When x_0 is larger than 1, $2g^*$ cannot reach 0°. The equation $y=0$ has two roots, both of which are between 1 and $(5/3)\Theta$. Therefore, the solution exists only if Θ is less than Θ_0. The motion of g^* is of libration for this case. The largest amplitude of x is $[1-(5/3)\Theta]$, which corresponds to $x_0=1$.

However, there is an upper limit of x_0 for the existence of the libration solution. For the limiting value of x_0, the equation $y=0$ has a double root, and Eqs. (10) have the stationary solution,

$$2g^*=180°, \quad x^2=(5/3)\Theta. \tag{35}$$

The value of C (29) takes a minimum value, since x_0 is at the maximum. Therefore, the stationary solution is stable.

For the libration case, dg^*/dt vanishes at

$$\cos 2g^*=(5\Theta-x^2)/[5(\Theta-x^2)]. \tag{36}$$

Therefore, $\cos 2g^*$ oscillates between -1 and the value given in (36). The maximum amplitude of $2g^*$ is computed from

$$2\{180°-\cos^{-1}[(5\Theta-1)/5(\Theta-1)]\}. \tag{37}$$

In each case Eq. (30) can be solved by an elliptic function of Weierstrauss $\wp$. After the variables from x and t are transformed to z and t^* by

$$z=x-(5/9)(1+\Theta)+(1/9)x_0,$$
$$t^*=-(3\times 6^{\frac{1}{2}}/4)nm'\alpha^3 t, \tag{38}$$

the equation,

$$dz/dt^*=\pm[4(z-z_0)(z-z_1)(z-z_2)]^{\frac{1}{2}}, \tag{39}$$

is solved as

$$z=\wp(t^*), \tag{40}$$

where

$$-z_0=z_1+z_2=(5/9)(1+\Theta)-(8/9)x_0,$$
$$z_1z_2=-(50/81)(1+\Theta)^2+(5/81)x_0(1+\Theta) \tag{41}$$
$$+(7/81)x_0^2+(5/3)\Theta.$$

The period of one revolution of the argument of perihelion or of the libration is evaluated from Eq. (39) as the order of m'^{-1}.

The motion of the argument of perigee is derived from (27) and (29) as

$$\cos 2g^*=Q(x)/5(x-\Theta)(1-x), \tag{42}$$

where

$$Q(x)=x^2+[5(1+\Theta)-4x_0]x-5\Theta. \tag{43}$$

The variation of i is determined from

$$i=\cos^{-1}(H/G^*). \tag{44}$$

The equations for the mean anomaly and the longitude of the ascending node are given below:

$$\frac{dl^*}{dt}=n+\frac{3}{8}nm'\alpha^3\eta^{-1}\left[x-3\Theta-\frac{Q(x)}{1-x}\right], \tag{45}$$

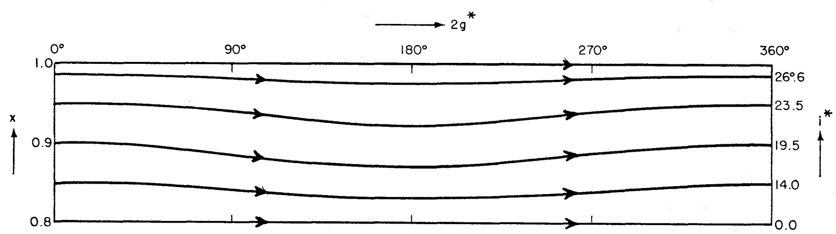

FIG. 2. Trajectories for $\alpha=0$ and $\Theta=0.8$.

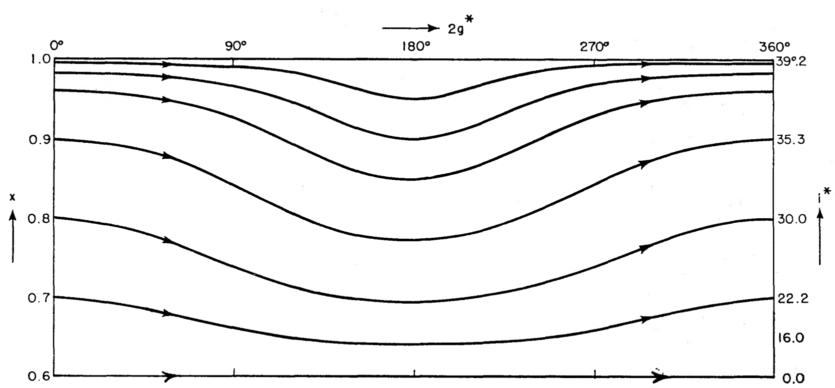

FIG. 3. Trajectories for $\alpha=0$ and $\Theta=0.6$ (limiting value).

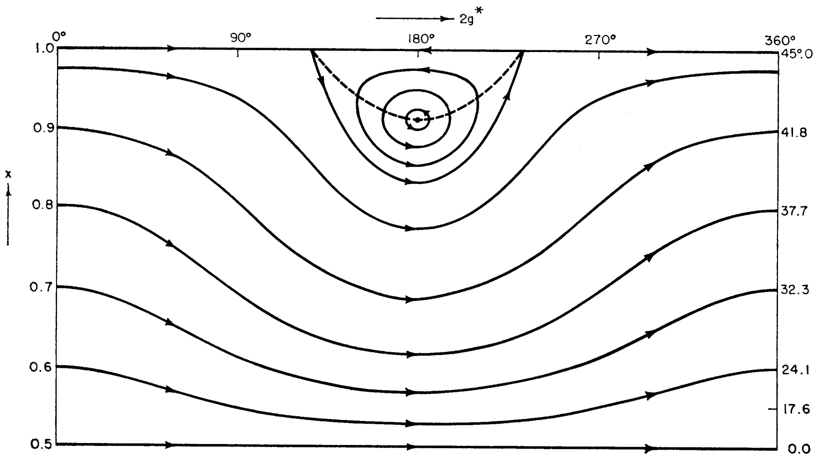

FIG. 4. Trajectories for $\alpha=0$ and $\Theta=0.5$. On the broken line dg^*/dt vanishes.

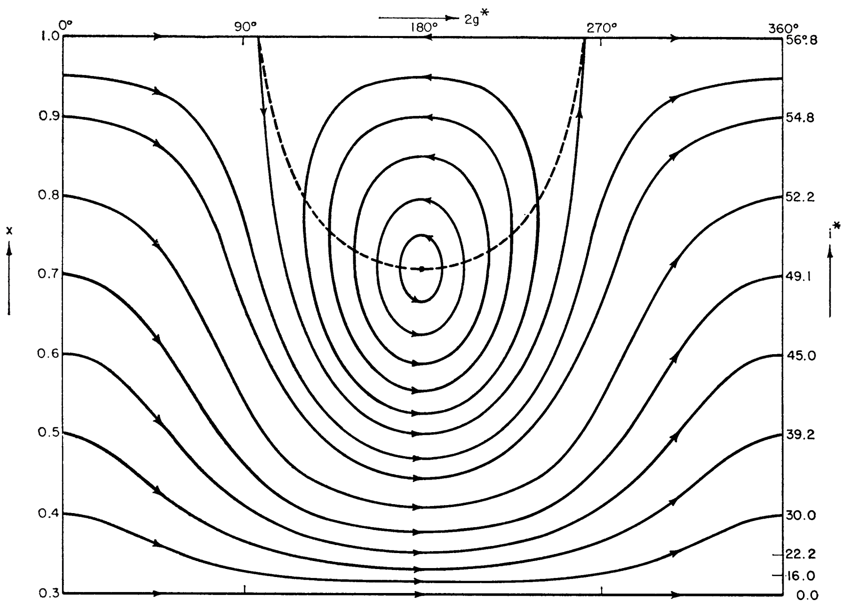

FIG. 5. Trajectories for $\alpha=0$ and $\Theta=0.3$.

$$\frac{dh^*}{dt}=-\frac{3}{8}nm'\alpha^3\theta\eta^{-1}[5-3x-5(1-x)\cos 2g^*]$$
$$=-\frac{3}{8}nm'\alpha^3\theta\eta^{-1}\left[5-3x-\frac{Q(x)}{x-\Theta}\right]. \tag{46}$$

From these equations we conclude that, with a few exceptions, dl^*/dt and dh^*/dt do not vanish. If θ is zero, that is, if i_0 is 90°, dh^*/dt vanishes. When θ is zero at any time, it remains zero unless G^* is zero at the same time. When G^* is zero, that is, the motion is in parabola, dh^*/dt vanishes. However, in this paper, it can be assumed that G^* is not zero.

VI. TRAJECTORY

Trajectories of Eqs. (10) can be plotted on the $(2g^*,x)$ plane by using the energy integral W^* (23). In Figs. 2 through 5 the trajectories are shown for $\alpha=0$.

Figure 2 corresponds to a case $\Theta=0.8$. There is no stationary point, and variations of x are limited within narrow regions. On the right-hand side the value of i corresponding to that of x is given. Near $2g^*=180°$ the velocity of g^* is slow for the upper part of the figure and fast for the lower.

In Fig. 3, Θ takes a limiting value of 0.6. There is a stationary point at $x=1$ and $\cos 2g^*=-1$. However, there is no closed trajectory.

In Fig. 4, Θ is 0.5. There are closed trajectories besides a stationary point. Two bifurcation points on a line $x=1$ are not actual singularities since they disappear when the coordinates are transformed into polar ones (e,g^*). In fact, for $x=1$, the circular orbit, g^* cannot be defined. At the bifurcation points dg^*/dt vanishes.

Figure 5 shows trajectories for $\Theta=0.3$. The position of the stationary point is in the lower part of the figure, and amplitudes of x are usually very large for this case. The speed of g^* is very fast, near $2g^*=180°$ for small values of x. In Figs. 4 and 5 broken lines indicate positions where dg^*/dt vanishes.

Usually the velocity of g^* is greater at a point where the density of trajectories is high. Therefore, the distribution of the argument of perihelion for the high-inclination asteroids cannot be evaluated easily from the present theory.

As the value of Θ decreases, the libration region becomes wide. Even when Θ is zero, there are trajectories of both libration and complete revolution. For an extreme case the orbit oscillates between a circular perpendicular one and a parabolic one of zero inclination. Although the assumption $\alpha=0$ may not be valid for this case, the results may confirm Lidov's numerical work (1962).

As the value of α increases, for a fixed value of Θ, amplitudes of x become large and the libration region expands as is expected.

In Figs. 6 and 7 trajectories are shown for the actual values of α and Θ of two asteroids. For these figures only a half of each trajectory is given.

Figure 6 corresponds to an asteroid (1036), that is, $\alpha=0.5123$ and $\Theta=0.5979$. The present values of x and $2g^*$ are, respectively, 0.7510 and 246°, and this position is marked in the figure. It is not in the libration region. The eccentricity and inclination oscillate, respectively, between 0.3 and 0.55 and between 23° and 48°, whereas the present values are 0.5 and 27°.

Figure 7 corresponds to an asteroid (1373), $\alpha=0.6569$ and $\Theta=0.5325$. The present position of the asteroid, $x=0.9184$ and $2g^*=207°$, is marked in the libration region. The present values of the eccentricity and inclination are, respectively, 0.29 and 42°. and they oscillate between 0.25 and 0.6 and between 25° and 42°. The motion of the argument of perihelion is limited between 60° and 120°.

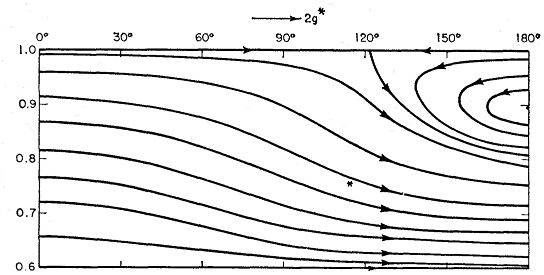

FIG. 6. Trajectories for $\alpha=0.5123$ and $\Theta=0.5979$. The asterisk shows the present position of asteroid (1036).

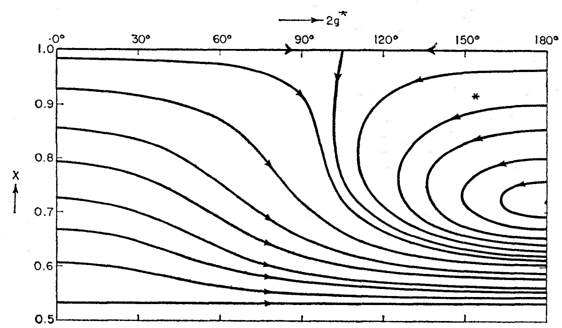

FIG. 7. Trajectories for $\alpha=0.6569$ and $\Theta=0.5325$. The asterisk shows the present position of asteroid (1373).

VII. REMARKS

A similar investigation was made by Brouwer (1947) by starting from similar equations of motion in connec-

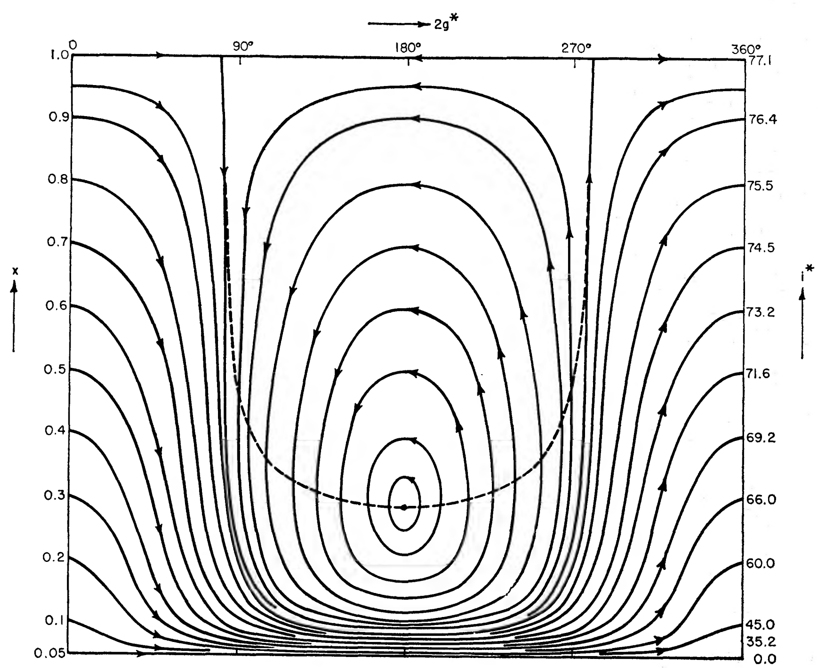

FIG. 8. Trajectories for $\alpha=0$ and $\Theta=0.05$.

tion with secular perturbations of Encke's comet. He developed the disturbing function by numerical harmonic analysis adopting a constant value for the eccentricity of the comet's orbit. This is a good approximation for this orbit since its trajectory in the $(2g^*,x)$-plane corresponds to a lower one in the figures of the present paper.

The theory discussed in the present paper can be applied to the actual asteroid motion with some restrictions, as Jupiter's eccentricity and other disturbing planets have been ignored.

The effects of Jupiter's eccentricity may be small for a case of small α and high eccentricity, since the $P_1(s_1)$ term in $R_1(19)$ vanishes and $ee' \sin(g-g')$ appears after the $P_3(s_1)$ term. When Jupiter's eccentricity is included, the canonical equations with two degrees of freedom must be solved, whereas H is still constant and there is an energy integral. However, it may be very difficult to find any meaningful stationary solution because of an apparent rapid motion of g'^* due to a very small inclination. And $l+g+h$, $g+h$, and h should be adopted instead of l, g, h. Then Jupiter's orbit can be regarded as known, although there is no integral corresponding to $H=\text{const}$ for this case.

When indirect perturbations due to other planets are considered, the integral of the elimination of nodes does not hold in the form $h=h'$. However, since Jupiter's orbital plane deviates very little from the invariable plane, H may be regarded as a stable constant, especially when the inclination of the asteroid is high. Of course there is no energy integral in the form $W^*=\text{const}$.

When the motion of a satellite around an oblate planet is considered, the perturbations due to the sun and the oblateness should be taken into consideration. When the equator of the planet coincides with the ecliptic, the present theory can be applied with little modification, since both H and W^* are constant. However, since a term of $\cos 2g$ does not appear in the first-order disturbing function due to the oblateness, the limiting value of H for the existence of a stationary solution becomes smaller or even disappears according to the ratio of the disturbing forces of the sun and the oblateness.

Sometimes, in the case of a satellite for which the period of one revolution of the sun may not be regarded as short, the solar mean anomaly l' may not be dropped. It is impossible, therefore, to make an exact study of a general case. If, however, only the principal terms are taken in the disturbing function, stationary solutions can be derived. Lunar orbits provide us with especially interesting problems.

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