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The Structure of Star Clusters. I. An Empirical Density Law

IVAN KING
University of Illinois Observatory, Urbana, Illinois
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New observations make it possible to re-examine the question of the density distribution in globular clusters. Jeans' r^{-4} law is based on insufficient observations and on an incorrect theory. On Palomar Schmidt plates the densities can be followed far enough out to define a limit to a globular cluster. The observed location of this limit agrees with the limit to be expected as a result of galactic tidal forces. The central regions of all globular clusters are similar, except for the effect of the tidal cutoff. An empirical formula has been found that represents the density from center to edge in globular clusters of all degrees of central concentration. The formula has three parameters, which is the minimum number permitted by the physical circumstances. Globular clusters are therefore as similar in structure as they could possibly be. Galactic clusters and Sculptor-type dwarf galaxies also appear to follow the same density law. From dynamical considerations it would appear that all these stellar systems are subject to two types of relaxation, which produce nearly identical effects. The first relaxation is produced by the initial mixing of the system. Thereupon stellar encounters slowly change the density parameters without affecting the basic law. Relative to globular clusters, giant elliptical galaxies have an excess of brightness near the center. This difference can be explained as a result of relaxation, equipartition, and an excess of dwarf stars.

I. INTRODUCTION

BECAUSE of their richness and symmetry, globular clusters have always presented an intriguing field for dynamical study. Yet the theory of their structure is still in a far from satisfactory state. A physically realistic model presents mathematical complexities too great to unravel, while models simple enough to handle lead to results that reflect mathematical simplifications rather than physical realities.

The present situation seems to be as follows: The theory of stellar encounters shows [Chandrasekhar 1942, Eq. (5.227)] that the mean free path in a star cluster is many times the radius of the cluster. Spatial mixing is therefore much more effective than relaxation through encounters; one may expect the structure of a star cluster to be closely represented by a solution of the encounterless Liouville equation, with encounters producing a slow evolution from one such solution to another. Unfortunately, a very wide range of solutions is possible; the problem is to decide which of them will give a realistic representation of a star cluster. Here the theory has not yet produced an answer. The irreversible nature of relaxation processes suggests that a cluster should reach a quasi-equilibrium in which it changes "as slowly as possible," but this condition has never been precisely formulated, and the general problem remains unsolved.

In its present impasse the theory can derive considerable benefit from an observational study of the distribution of stars in clusters. Even though velocity distributions are not directly observable, a determination of the spatial density law will so severely restrict the range of possible models that it may then be possible to set fairly narrow guide lines for the theory. Various studies of star distributions have been made in the past, but the availability today of photoelectric photometers and large Schmidt cameras suggests that a new attack on the problem should be profitable. The writer has therefore undertaken an extensive study of the distribution of stars in globular clusters, based chiefly on new observations made at the Mount Wilson and Palomar Observatories. Partial results are now available in about fifteen clusters. The results will be published in a series of papers, as the study of each cluster is completed. A general law is already evident, however, and the purpose of the present paper is to present this law, illustrate it with a selection of the data, and discuss some of its dynamical consequences.

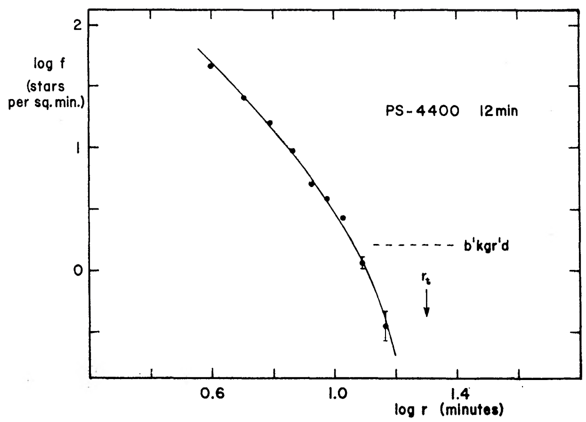

FIG. 1. Star counts in M15. Maximum-exposure 48-inch Schmidt plate. Mean errors are indicated above and below last two points. Arrow indicates value of r_t; dotted line indicates backgrount count.

II. A RE-EXAMINATION OF JEANS' LAW

A frequently cited density law for globular clusters was derived by Jeans (1916). He showed theoretically that the outer parts of a globular cluster should have a star density that varies as r^{-4}, where r is the distance from the cluster center. In projection this becomes an r^{-3} law. To support his conclusion Jeans cited the star counts of Bailey (1915). Unfortunately Jeans' proof is incorrect, and Bailey's observations support the law only weakly.

Jeans' theoretical result depends on two unjustified assumptions. First, he assumes that the velocity distribution is isotropic at every point in the cluster. This is an arbitrary and unjustified restriction. Second—and more serious—he assumes that at large r the potential U can be represented by the power series

$$-U=(GM/r)(1+c_1/r+c_2/r^2+\cdots), \tag{1}$$

with $c_1 \neq 0$. Jeans' r^{-4} law follows from the nonvanishing of c_1, for which no proof is given.

As for the observational test, an examination of Bailey's counts shows that in the outer parts of a cluster so few stars are included that the densities are quite uncertain. Even so, the counts, when plotted on logarithmic scales, suggest that the exponent of r increases with increasing r. Furthermore, Jeans concedes that two of Bailey's 10 clusters do not fit his law.

III. LIMITS OF A GLOBULAR CLUSTER

When counts are extended to the more numerous faint stars, the densities become more reliable and can also be followed farther from the cluster center. As an example, Fig. 1 shows the result of a count made on a photograph of M15 taken with the 48-inch Schmidt camera of the Palomar Observatory. The densities have been corrected for the background (or more properly, foreground) density, which is easily determined on the wide-angle Schmidt plates. The vertical lines through the points have half-lengths corresponding to the square root of the counted number of stars, in accordance with the Poisson distribution. The smooth curve is computed from a density formula that is discussed in Sec. IV.

Both scales in Fig. 1 are logarithmic. On such a plot any simple power law would be a straight line. Far from straightening out, however, these points plunge more and more steeply with increasing r, suggesting that the surface density f actually drops to zero at some finite value of r. As for the nature of the drop to zero, a little experimentation with this and similar sets of data in other clusters led to the form

$$f=f_1(1/r-1/r_t)^2, \tag{2}$$

where f_1 is a constant and r_t is the value of r at which f reaches zero. This formula is easily tested by plotting $f^{\frac{1}{2}}$ against $1/r$, as in Fig. 2. The points should lie on a straight line, whose intercept on the $1/r$ axis is $1/r_t$. Figure 3 is a similar plot for three other clusters. The data for ω Centauri and 47 Tucanae are photoelectric surface brightnesses measured by Gascoigne and Burr (1956), while the M13 data come from counts on another photograph made with the 48-inch Schmidt. The fit is good in the outer parts of all four clusters.

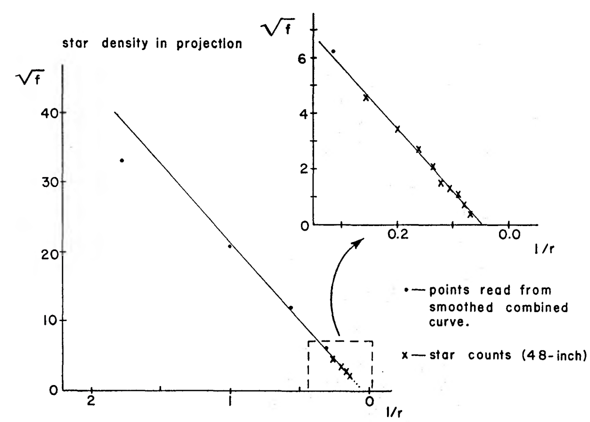

FIG. 2. Test of Eq. (2) in M15. Dots are mean of several plates; crosses come from Fig. 1. Upper right section is an enlargement of dashed rectangle.

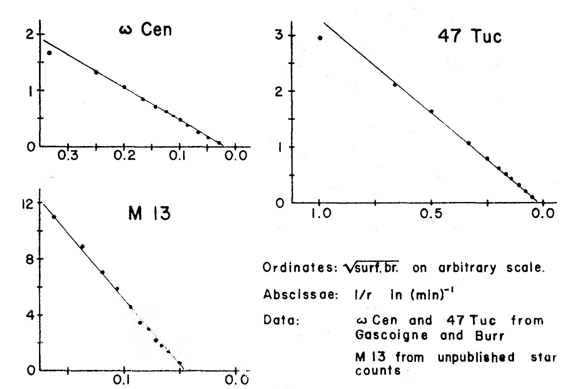

FIG. 3. Test of Eq. (2) in three more clusters.

Naturally the central parts cannot be expected to fit Eq. (2), since it predicts an infinite density at the center.

It thus appears that globular clusters are limited in size—exactly as one should expect as a result of galactic tidal forces. The existence of a tidal cutoff has been pointed out theoretically by von Hoerner (1957), who gives for the limiting radius of a cluster the formula

$$r_t = R(M/2M_g)^{\frac{1}{3}}, \tag{3}$$

where R is the distance of the cluster from the galactic center and M and M_g are the masses of cluster and galaxy, respectively. [Through a typographical error the 2 was printed in the numerator instead of the denominator of von Hoerner's Eq. (33).] This formula actually gives the instantaneous limiting radius of a cluster moving in a straight-line orbit toward or away from the galactic center. For the effective limiting radius, von Hoerner points out, however, one should choose not the instantaneous value at the cluster's present location but rather the value of the limiting radius at the perigalactic point of the cluster's orbit. The reason is that the cluster is cut back most severely at that time, and internal relaxation is too slow to increase the size of the cluster between successive perigalactic passages.

The limiting radius is not hard to estimate for the perigalacticon of an elliptic orbit around the galactic center. Let R and θ be polar coordinates of the cluster with respect to the galactic center, and let a system of rotating coordinates (x,y) be defined with origin at the cluster center and the x axis always pointing away from the galactic center.

If a cluster follows a noncircular orbit around the galactic center, its limiting radius cannot be rigorously defined, since the Jacobi integral of the restricted three-body problem does not then apply. However, a tidal limit can be estimated as follows. As the cluster passes its perigalacticon, a star at a large distance from the cluster center will be detached by galactic tidal forces whereas a star at a small distance will not. We can then define the limit as that point, on the line connecting the center of the cluster with the galactic center, at which a star can remain on the line of centers with an acceleration along that line that is zero with respect to the cluster center. That is, at the moment of perigalactic passage that star is pulled neither toward nor away from the cluster. The acceleration of the cluster with respect to the galactic center at that time is

$$d^2R/dt^2 = R\omega^2 - dV/dR, \tag{4}$$

where ω is its angular velocity and $V(R)$ is the potential of the galaxy. The acceleration of the star at the same moment is

$$\frac{d^2R_s}{dt^2} = R_s\omega^2 - \left(\frac{dV}{dR}\right)_{R_s} - \frac{GM(R_s-R)}{|R_s-R|^3}, \tag{5}$$

and the relative acceleration is

$$\frac{d^2}{dt^2}(R_s-R) = (R_s-R)\omega^2 - \left(\frac{dV}{dR}\right)_{R_s} + \frac{dV}{dR} - \frac{GM(R_s-R)}{|R_s-R|^3}$$
$$\cong \left(\omega^2 - \frac{d^2V}{dR^2} - \frac{GM}{|R_s-R|^3}\right)(R_s-R). \tag{6}$$

This will be zero when $R_s - R$ has the magnitude r_{lim}, given by

$$r_{\text{lim}}^3 = \frac{GM}{\omega^2 - d^2V/dR^2}. \tag{7}$$

If we represent the force field of the galaxy by an inverse-square force due to a mass M_g, then

$$d^2V/dR^2 = -2GM_g/R^3. \tag{8}$$

The cluster's orbit about the galactic center will be an ellipse, with the angular velocity at any point given by

$$\omega^2 = GM_g a(1-e^2)/R^4. \tag{9}$$

At the perigalactic point R takes the value

$$R_p = a(1-e), \tag{10}$$

and Eq. (7) then simplifies to

$$r_{\text{lim}} = R_p[M/M_g(3+e)]^{\frac{1}{3}}. \tag{11}$$

Since it appears that most globular clusters move in rather elongated orbits (Kinman 1959), a good compromise formula is

$$r_{\text{lim}} = R_p(M/3.5M_g)^{\frac{1}{3}}. \tag{12}$$

Because of the presence of the exponent $\frac{1}{3}$, none of the quantities in the parentheses needs to be known very accurately.

The weakness of this dependence on the force field seems especially fortunate when one considers the inaccuracy of the inverse-square-force approximation for the galactic field. Because of the wide distribution of the attracting mass, the tidal force will generally be weaker than that given by Eq. (8), and r_{lim} will be somewhat larger than the value given by Eq. (12). This is especially true for clusters whose orbits dip close to the galactic center. However, in the one-third power the approximation used is probably adequate for the present purpose, which is simply to verify that the observed limiting radii are of the order of magnitude to be expected as a result of galactic tidal forces. A more detailed discussion of tidal forces can be postponed until enough limiting radii have been determined to make possible a general discussion of the shapes of cluster orbits.

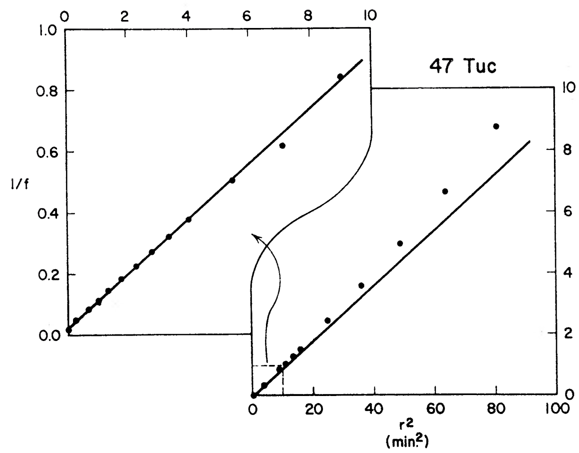

FIG. 4. Test of Eq. (13). Upper left is an enlargement of dashed square.

Before comparing r_{lim} with observed values of r_t, two more complicating factors deserve mention. First, at a given perigalactic passage a cluster will lose only a fraction of the stars that are capable of reaching the distance r_{lim}, since most such stars will not be in position to be pulled away at that time. Thus the effective limit will be somewhat larger than Eq. (12) suggests. The second effect, however, works in the opposite direction. If under the influence of tidal forces a star can reach a certain maximum distance from the cluster center, then without tidal forces it will not travel so far from the center. Consequently a cluster will shrink somewhat as it recedes from perigalacticon. These two effects tend to cancel each other, so that Eq. (12) probably gives a realistic value for r_{lim}.

For a given globular cluster R_p is of course unknown, but an upper limit is the cluster's present distance from the galactic center. M15, for instance, is now about 10 kpc from the galactic center. Its absolute magnitude (Hogg 1959) is $0^{\mathrm{m}}.7$ brighter than that of M92, whose mass has been estimated by Schwarzschild and Bernstein (1955) as $1.4\times10^5\odot$; thus the mass of M15 must be about $2.7\times10^5\odot$. If the mass of the galaxy is $10^{11}\odot$, then Eq. (12) gives $r_{\mathrm{lim}}=92$ pc. The distance to M15 is 10 to 12 kpc, depending on the absolute magnitude assigned to the RR Lyrae stars; at 11 kpc r_t subtends 29 minutes of arc. The value of r_t found from Fig. 3 is 21 minutes. For M13 the corresponding figures are $r_{\mathrm{lim}}=38'$ and $r_t=22'$. Since the true values of r_{lim}, calculated from the true values of R_p rather than from the present values of R, will be somewhat smaller, it is quite clear that in the outer parts of globular clusters we are observing the cutoff imposed by galactic tidal forces.

IV. OVER-ALL DENSITY LAW

In the central regions of a globular cluster Eq. (2) is completely inadequate, and another formula must be found. Here a brief examination of the available data showed that the surface density in the inner parts of a concentrated cluster can be represented by the

TABLE Ia. Values of $\log f(r)/f(0)$ for standard curves. Subtract 10 from all entries.

	$\log r_t/r_c$								
$\log r/r_c$	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.50	∞
−1.25	9.998	9.998	9.998	9.998	9.998	9.998	9.998	9.998	9.998
−1.00	9.993	9.994	9.995	9.996	9.996	9.996	9.996	9.996	9.996
−0.75	9.979	9.983	9.984	9.985	9.986	9.986	9.986	9.986	9.986
−0.50	9.940	9.949	9.954	9.956	9.957	9.958	9.958	9.958	9.958
−0.25	9.828	9.857	9.870	9.876	9.879	9.882	9.883	9.884	9.884
0.00	9.527	9.619	9.659	9.678	9.687	9.693	9.696	9.698	9.699
0.25	8.861	9.163	9.274	9.325	9.350	9.364	9.371	9.378	9.380
0.50	∞	8.371	8.702	8.864	8.890	8.922	8.938	8.952	8.958
0.75		∞	7.847	8.200	8.341	8.410	8.444	8.473	8.486
1.00			∞	7.324	7.692	7.840	7.913	7.970	7.996
1.25				∞	6.808	7.184	7.338	7.452	7.499
1.50					∞	6.297	6.678	6.911	6.999
1.75						∞	5.791	6.333	6.500
2.00							∞	5.673	6.000
2.25								4.785	5.500
2.50								∞	5.000

TABLE Ib. Values of $\log(r_t^2 f/r_c^2 k)$ for standard curves. Subtract 10 from all entries.

	$\log r_t/r_c$							
$\log r/r_t$	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.50
−0.25	9.549	9.704	9.756	9.774	9.780	9.781	9.782	9.782
−0.15	9.053	9.171	9.213	9.223	9.229	9.230	9.230	9.231
−0.10	8.670	8.775	8.810	8.819	8.825	8.826	8.827	8.818
−0.05	8.034	8.128	8.158	8.164	8.173	8.173	8.173	8.178

formula

$$f=\frac{f_0}{1+(r/r_c)^2}, \tag{13}$$

where f_0 is the central surface density and r_c is a scale factor that may be called the core radius. The adequacy of this formula is illustrated in Fig. 4, which makes use of the data of Gascoigne and Burr (1956) in 47

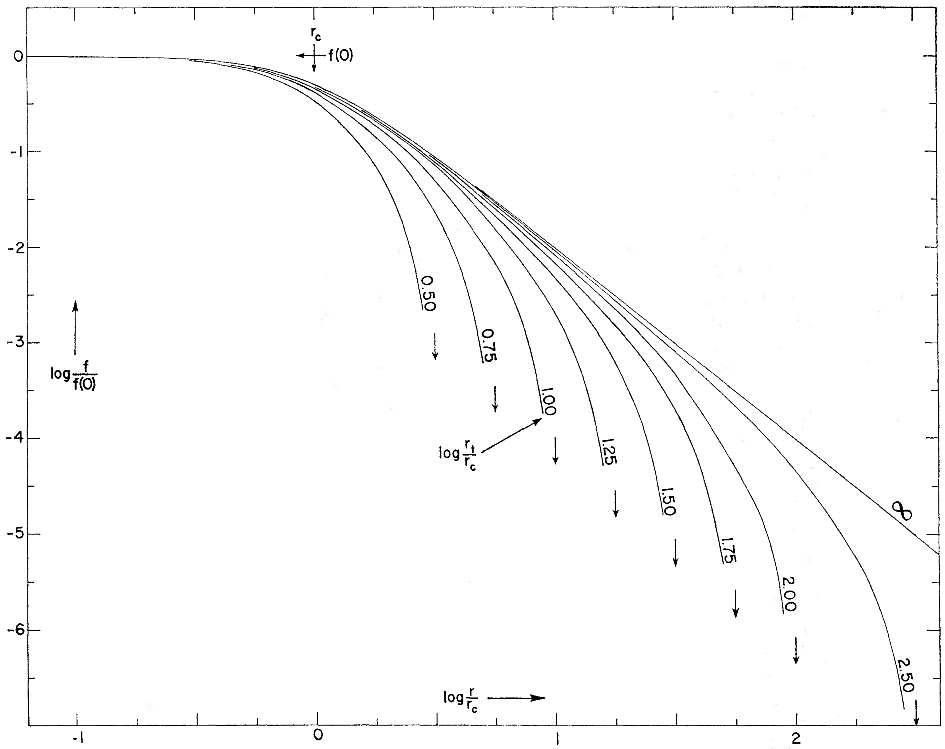

FIG. 5. Standard curves calculated from Eq. (14), plotted so as to osculate at left. These may be used directly with inch graph paper.

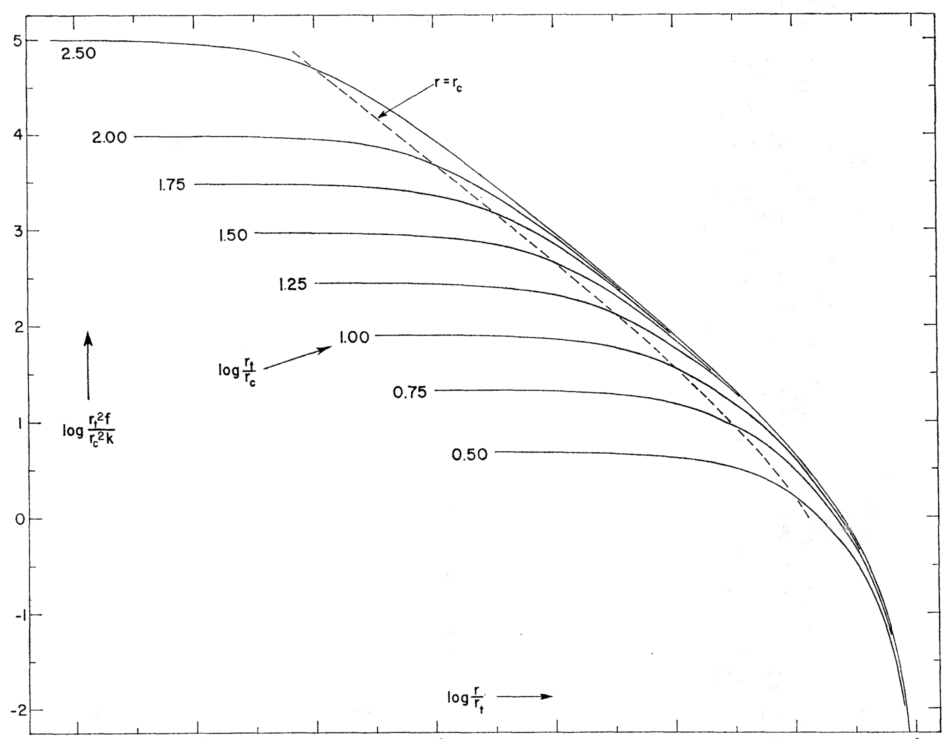

FIG. 6. Same curves as in Fig. 5 but shifted so as to osculate at right.

TABLE II. Values of $2.5\log n/\pi r_c^2 f(0)$ for standard curves. Subtract 10 from all entries.

	$\log r_t/r_c$								
$\log r/r_c$	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.50	∞
−1.00	4.992	4.993	4.993	4.994	4.994	4.994	4.994	4.995	4.995
−0.80	5.981	5.984	5.984	5.985	5.986	5.986	5.986	5.986	5.986
−0.60	6.952	6.960	6.962	6.964	6.966	6.966	6.966	6.967	6.967
−0.40	7.884	7.902	7.910	7.915	7.917	7.917	7.918	7.918	7.919
−0.20	8.728	8.772	8.791	8.802	8.807	8.809	8.811	8.812	8.813
0.00	9.414	9.511	9.555	9.577	9.588	9.594	9.598	9.601	9.602
0.20	9.864	10.063	10.153	10.197	10.210	10.233	10.240	10.246	10.248
0.40	10.044	10.406	10.572	10.654	10.696	10.719	10.731	10.742	10.747
0.50	10.055	...	...	...	...	...	...	...	...
0.60		10.556	10.832	10.970	11.042	11.080	11.101	11.119	11.127
0.75		10.574	...	...	...	...	...	...	...
0.80			10.957	11.173	11.287	11.348	11.381	11.410	11.423
1.00			10.982	11.281	11.452	11.545	11.596	11.640	11.660
1.25				11.311	11.561	11.711	11.795	11.868	11.901
1.50					11.584	11.796	11.927	12.044	12.099
1.75						11.814	11.997	12.179	12.266
2.00							12.012	12.272	12.411
2.50								12.334	12.653

Tucanae. The coordinates of Fig. 4 are $1/f$ and r^2, so that Eq. (13) is represented by a straight line. The fit is good out to 5 minutes from the center. Since the edge formula (2) was shown in Fig. 3 to fit 47 Tucanae inward to 2 minutes from the center, it is evident that the two formulas between them fit the entire cluster.

It remains to find a single formula that embodies the characteristics of both Eq. (2) and Eq. (13). Such a formula is

$$f=k\left\{\frac{1}{[1+(r/r_c)^2]^{\frac{1}{2}}}-\frac{1}{[1+(r_t/r_c)^2]^{\frac{1}{2}}}\right\}^2. \tag{14}$$

In a typical globular cluster r_t/r_c is of the order of 30, so that for small to moderate values of r/r_c Eq. (14) differs only slightly from Eq. (13) with

$$f_0=k\left\{1-\frac{1}{[1+(r_t/r_c)^2]^{\frac{1}{2}}}\right\}^2. \tag{15}$$

For $r\gg r_c$, on the other hand, Eq. (14) comes very close to Eq. (2) with

$$f_1=kr_c^2. \tag{16}$$

Note that the second term in brackets in Eq. (14) could be replaced by a single constant; it is written in this more complicated form in order to show the role of r_t.

The crucial test, of course, is to compare Eq. (14) directly with observations. This can be done most easily by computing standard curves of $\log f$ against $\log r$, so that scale factors can be removed by sliding the curves horizontally and vertically. The curves are tabulated in Table I and plotted in Figs. 5 and 6. The individual curves are labeled by values of the logarithm of the parameter

$$c=r_t/r_c, \tag{17}$$

which may be referred to simply as the concentration ratio. Interpolation between the plotted curves is greatly simplified by using Fig. 5 for the central region of a cluster and Fig. 6 for the outer parts. If an observed curve is fitted to the two figures alternately, the process of determining c converges rapidly.

Another useful sort of curve gives the total number of stars in projection within a distance r of the center. This function can be found by integrating f with respect to $2\pi r dr$; the result is

$$n(x)=\pi r_c^2 k\left[\ln(1+x)-4\frac{(1+x)^{\frac{1}{2}}-1}{(1+x_t)^{\frac{1}{2}}}+\frac{x}{1+x_t}\right], \tag{18}$$

where

$$x=(r/r_c)^2, \tag{19}$$

$$x_t=(r_t/r_c)^2. \tag{20}$$

Since these curves are most useful for discussing results of surface photometry, the values of n are given in Table II in magnitudes. Figures 7 and 8 show the curves, which for convenience are again plotted with osculation first at one end and then at the other. The total number of stars in each cluster model is given by

$$n(x_t)=\pi r_c^2 k\left\{\ln(1+x_t)-\frac{[3(1+x_t)^{\frac{1}{2}}-1][(1+x_t)^{\frac{1}{2}}-1]}{1+x_t}\right\}. \tag{21}$$

When $x_t\gg 1$, this is approximately

$$n(x_t)\doteq\pi r_c^2 k\ln(r_t^2/20r_c^2). \tag{22}$$

Through the kind cooperation of the Editor, Figs. 5–8 have been printed on such a scale that they can be used directly with inch-scale graph paper. The horizontal and vertical units are 1.25 inches and 0.5 inch, respectively.

The empirical density law expressed in Eqs. (14) and (18) can now be compared with observation. Figures 9 and 10 show the surface brightnesses measured by Gascoigne and Burr (1956) in 47 Tucanae and ω Centauri. Each figure is divided into two parts, because of the reduction procedure used by Gascoigne and

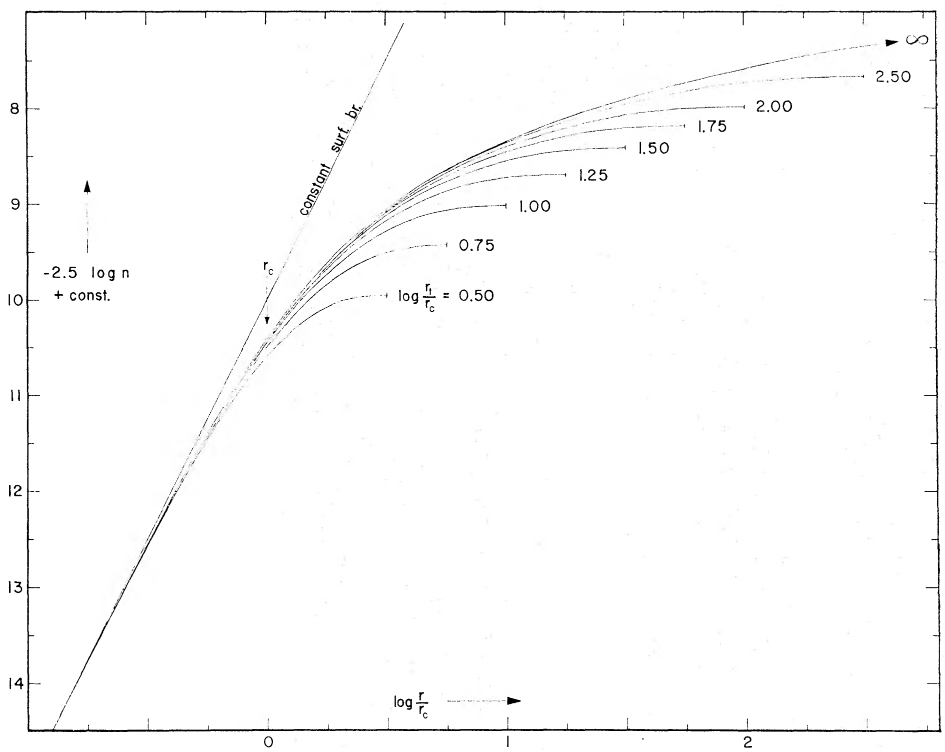

FIG. 7. Integral-magnitude curves calculated from Eq. (14), plotted so as to osculate at left. May be used directly with inch graph paper.

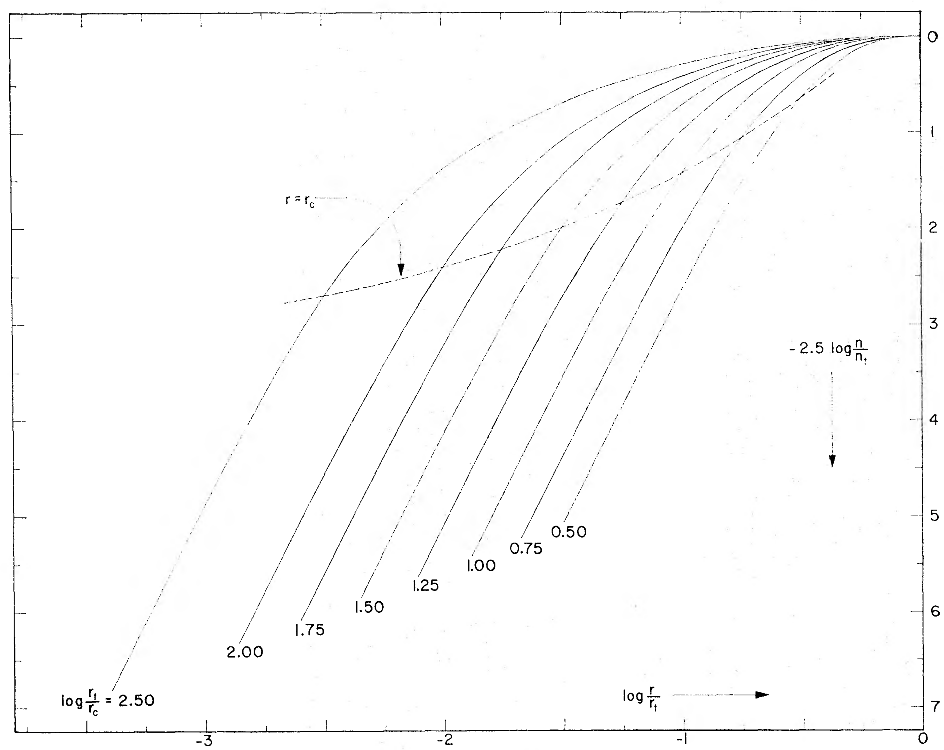

FIG. 8. Same curves as in Fig. 7 but shifted so as to coincide at $r = r_t$.

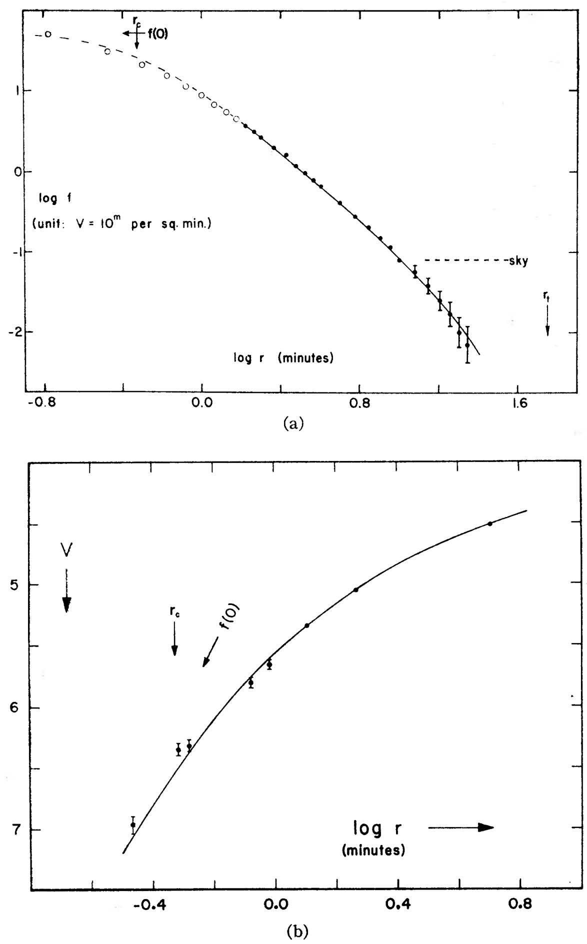

FIG. 9. Test of Eq. (14) in 47 Tucanae (Gascoigne and Burr): (a) outer parts, (b) central region.

Burr. Outside $r=1'$ they used direct measures of surface brightness, but inside a one-minute radius their surface brightnesses come from numerical differentiation of the total brightness observed through concentric apertures of various sizes. In the central regions it is therefore more meaningful to compare directly with their concentric-aperture measures, using the curves of Figs. 7 and 8. The solid curves in Figs. 9 and 10 are all traced by interpolation from the curves in Figs. 5–8, and in each case the curves of parts a and b have identical parameters. The vertical lines through the observed points are sampling mean errors, due to the fact that the observed brightness is due to a finite number of stars. The calculation of these mean errors, which takes the luminosity function into account, will be described in a later paper of this series, dealing with surface photometry.

The values of the parameters derived in Figs. 9 and 10 are as follows:

	k	r_c	r_t
47 Tuc	51.2	0′.47	56′.2
ω Cen	6.68	2′.45	43′.7

The unit of k is $V=10$ mag. per square minute. The values of r_t are not well determined; star counts can be expected to give a much stronger result in the outer parts.

In Fig. 1, which gives star counts in M15, the smooth curve was traced from Fig. 6. Similarly, Fig. 11 shows the M13 counts referred to in Sec. III, with a curve taken from Fig. 6. In both these cases the fit is almost independent of the choice of r_c, since the counts do not reach into the dense central regions. Separate measurements of surface brightnesses confirm that the ratio r_t/r_c is large in both clusters.

It is unfortunate that most clusters cannot be covered from center to edge in a single star count. Short-exposure photographs show too few stars in the outer parts, while on a long-exposure plate the star images are hopelessly crowded in the center. Only in clusters of very low central concentration can counts of faint stars be extended from the center of the cluster to the edge. An example is shown in Fig. 12, which illustrates counts made on a 48-inch Schmidt photograph of NGC 5053.

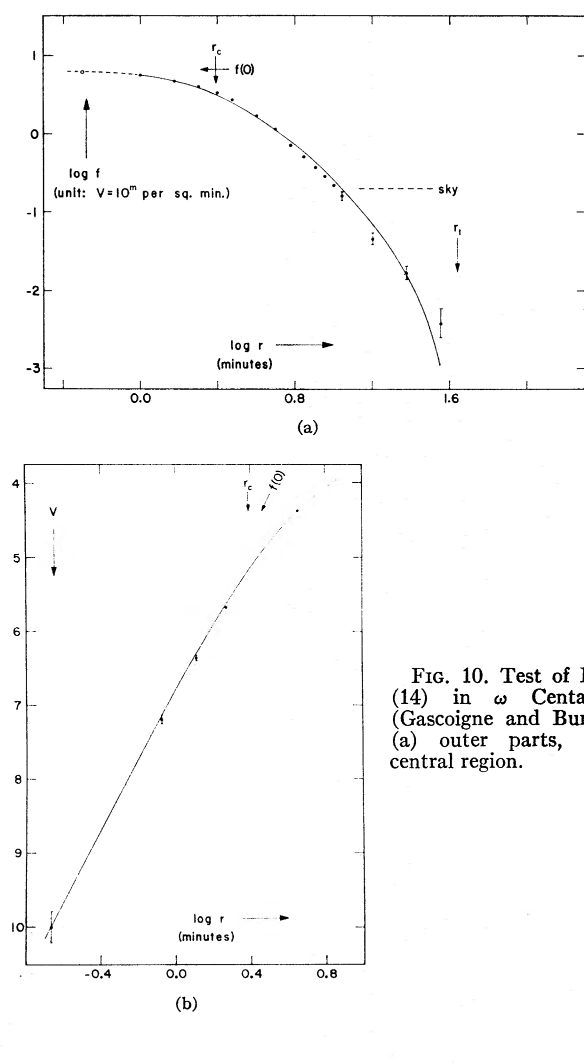

FIG. 10. Test of Eq. (14) in ω Centauri (Gascoigne and Burr): (a) outer parts, (b) central region.

In all five of these clusters the empirical law expressed by Eq. (14) agrees quite satisfactorily with the observational data. Additional data have now been accumulated in a dozen more clusters, and no significant disagreements with this law have been found. The observational data presented in the present paper are chosen for their breadth and quality, not for their agreement with the chosen law.

In the globular clusters examined so far, the extreme values of the concentration $c=r_t/r_c$ are $3\frac{1}{2}$ and 125. The values of r_c are close to one-half the 0.5 core diameters given by Mowbray (1946), but the values of r_t are considerably larger than the limiting radii previously quoted for globular clusters. The concentrations correlate fairly well with the three concentration classes given by Kron and Mayall (1960), but they correlate much less well with the concentration classes estimated by Shapley (1930) and by Mowbray. The latter two sets of eye estimates on photographs seem to be influenced by the absolute values of surface

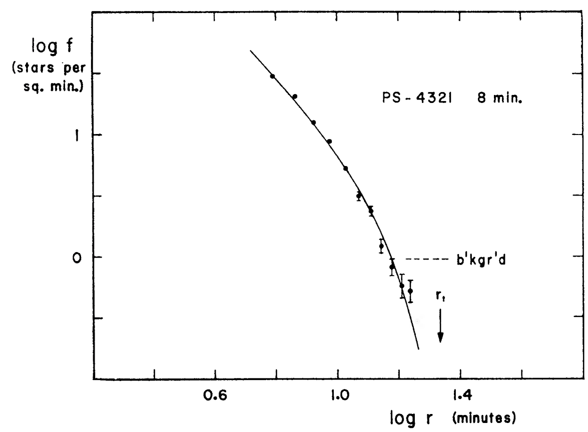

FIG. 11. Star counts in M13; maximum-exposure 48-inch Schmidt plate.

brightness and core radius and therefore do not express a scale-invariant property of the clusters. At present there is no evidence that the ellipticities observed in some clusters have any influence on the radial density law.

One further complication is worthy of mention: in a given cluster the bright and faint stars do not have the same distribution. Any tendency toward equipartition of energy will give higher velocities to the less-massive stars; and these fainter stars will therefore remain, on the average, farther from the cluster center. Near the edge of the cluster the density distribution is dominated by the tidal cutoff, which operates equally on all stars regardless of mass; hence near the edge the distributions of bright and faint stars should be closely similar. Near the center, however, where the density distributions depend strongly on velocity dispersions, bright and faint stars should have different distributions. Observationally it is quite unfortunate

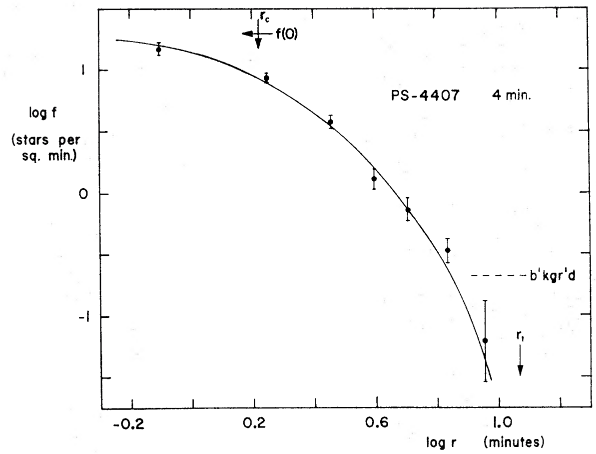

FIG. 12. Star counts in NGC 5053. Medium-exposure 48-inch Schmidt plate.

that these segregation effects are best studied near the cluster centers, for again we are restricted to the low-concentration clusters in which faint stars can be resolved close to the center. Again NGC 5053 provides a good example, for its center is resolvable even at the limiting magnitude of the 200-inch reflector. Figure 13 shows counts made on such a limiting-magnitude plate, which was taken by Dr. Allan R. Sandage.

One difficulty immediately arises, which is characteristic of cluster star counts made on reflector plates. The coma-free field of the 200-inch reflector, used with the $f/3.7$ corrector, has a radius of only 8 minutes of arc. Even on this slightly off-center plate the usable field did not extend beyond the limits of the cluster; as a result the background count cannot be directly determined. The outermost points in Fig. 13 are therefore plotted twice. The open circles give the actual counts, while the filled circles have been corrected for an assumed background density, which was chosen so as to make the outermost points satisfy the empirical law of Eq. (14), with the same value of r_t as in Fig. 12.

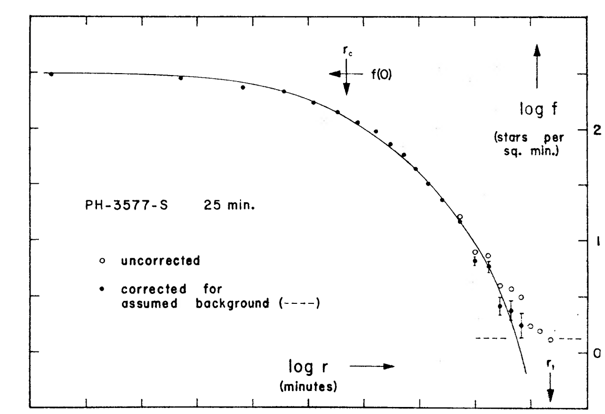

FIG. 13. Star counts in NGC 5053. Maximum-exposure 200-inch plate. Open circles are original counts, without background correction.

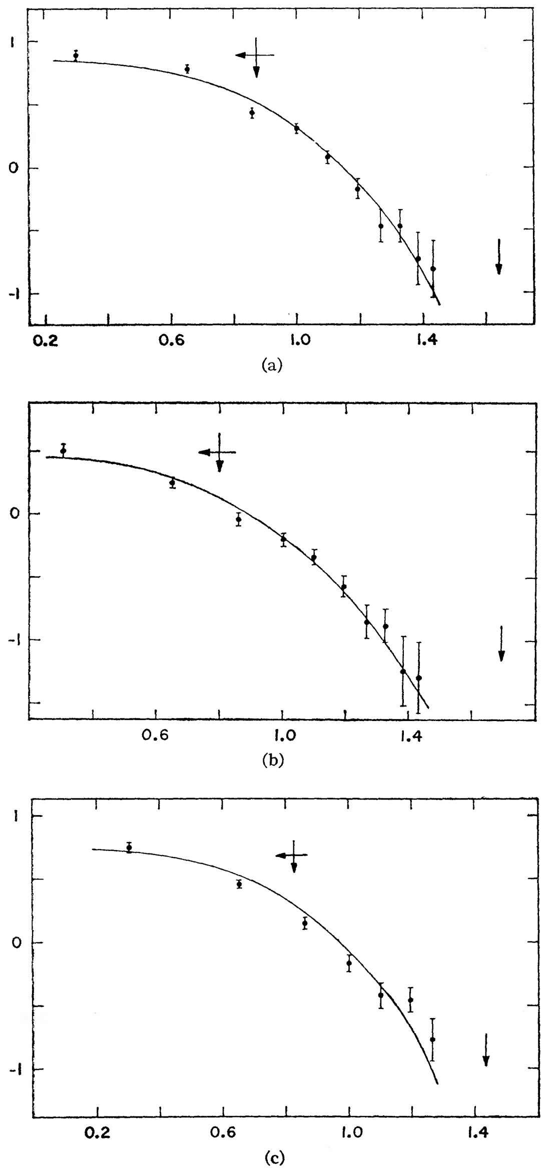

FIG. 14. Star counts in rich galactic clusters, from data of van den Bergh and Sher. Arrows indicate $f(0)$, r_c, and r_t. (a) NGC 7789. (b) M67. (c) NGC 188.

The resulting background density, 1.33 stars per square minute, is of the order of magnitude to be expected at this magnitude and latitude; but these observations are clearly of little weight in checking the validity of Eq. (14) near the edge of the cluster.

In the center of the cluster, however, the contribution of the background density is negligible. From the fit shown in Fig. 13 two conclusions can be drawn. First, these faint stars are more widely spread than the brighter stars whose distribution is shown in Fig. 12; the fainter stars have an r_c that is larger by a factor of 1.3. Second, the distribution of fainter stars can also be represented by Eq. (14). This single example therefore suggests the working hypothesis that Eq. (14) represents the distributions of both bright and faint stars, provided suitable values of k and r_c are chosen in each case.

V. OTHER SPHERICAL SYSTEMS

The law expressed by Eq. (14) can also be tested in galactic clusters. Here the test would appear to be less sensitive, because of the smaller number of cluster stars and the richer background; but two compensating factors strengthen the test. First, in a galactic cluster stars can be counted in the central regions as well as near the edge; and second, the tidal limit of a galactic cluster can be calculated with some degree of certainty. Unlike globular clusters, which move in elongated orbits with unknown perigalactic distances, galactic clusters travel about the galactic center in nearly circular orbits, so that the limiting radius can be safely calculated from the cluster's present location. In terms of the local galactic rotation field the denominator of Eq. (7) can be written

$$\omega^2 - d^2V/dR^2 = 4\omega A, \tag{23}$$

where A is the first Oort constant. Thus the limiting radius is

$$r_{\text{lim}} = (GM/4\omega A)^{\frac{1}{3}}, \tag{24}$$

and this should be the value of r_t for a galactic cluster in a constant tidal field.

As a test, data on three rich galactic clusters were taken from the luminosity-function study of van den Bergh and Sher (1960). In Fig. 14 the data are shown fitted to standard curves taken from Figs. 5 and 6. This fitting differs from the fittings in Sec. IV in that here r_t was calculated rather than determined from the observations. For this purpose A and ω were taken to be 18 and 29 km/sec kpc, respectively. The assumed masses and distances are given in Table III, along with the calculated values of r_t.

In M67 and NGC 7789 the fit is good. In NGC 188 the standard curve fits the points less well, but the fit would be greatly improved if the background density were assumed to be 5% higher—a change that lies well within the range of uncertainty. Considering this and the other uncertainties involved in making star counts in galactic clusters, Eq. (14) appears to be an adequate representation of the star densities in these three clusters.

A very different sort of stellar system is a Sculptor-type dwarf elliptical galaxy. Hodge (1961a, b, 1962) has recently published star-count data for three of these.

TABLE III. Calculated limiting radii of galactic clusters.

	mass	$(m-M)_0$	distance	r_t	dist. source
M67	800$\odot$	9.58	825 pc	49′.3	Sandage (1962)
NGC 188	900	10.95	1550	27.3	Sandage (1962)
NGC 7789	5500	11.24	1770	43.7	Rohlfs (1961)

Figure 15, based on Hodge's data, shows that Sculptor-type galaxies also fit Eq. (14). Hodge has shown that the observed limiting radii are in agreement with those to be expected as a result of the tidal force of the Milky Way.

The concentration ratios of the systems shown in Figs. 14 and 15 are low, ranging from 3 to 8.

Only in the more luminous elliptical galaxies does Eq. (14) fail. It represents their brightness distributions adequately in the outer parts, but none of the curves of Fig. 5 can represent the sharp central peak of brightness in a giant elliptical galaxy. This situation is illustrated in Fig. 16, which is based on de Vaucouleurs' (1953) study of M32. The outermost points are quite uncertain, however, since they depend on estimates of the surface brightnesses that correspond to the limiting isophotes of various observers. If the data are taken at face value, then the limiting radius of M32 is 11.2 minutes. At a distance of 600 kpc this corresponds to 2.0 kpc.

M32 should of course be limited in size by the tidal field of its large neighbor M31. If the mass of M32 is $3.6\times10^9\odot$ (Burbidge, Burbidge, and Fish 1961; King 1961) and that of M31 is $3.7\times10^{11}\odot$ (Brandt 1960), then Eq. (12) shows that the limiting radius of M32 would equal the observed 2.0 kpc if the spatial separation of the two were 14 kpc. This suggests that the closest approach of M32 to the center of M31 is 14 kpc. Since its present projected distance from the center of M31 is $23\frac{1}{2}$ minutes, or 4.1 kpc, it would appear that M32 is at least 13 kpc in front of or behind M31. The difference in distance, unfortunately, would still be too small to detect photometrically. In any case, it is well to remember that this discussion stands on rather uncertain observational data.

The tidal limit can be expected to be reasonably small whenever an elliptical galaxy has a nearby, large companion. Another example is NGC 3379, where Dennison's brightness curve (reproduced by de Vaucouleurs 1958) shows a distinct turndown that suggests a limiting radius of about 5 minutes of arc. The apparent separation of NGC 3379 from its neighbor NGC 3384 is 7 minutes.

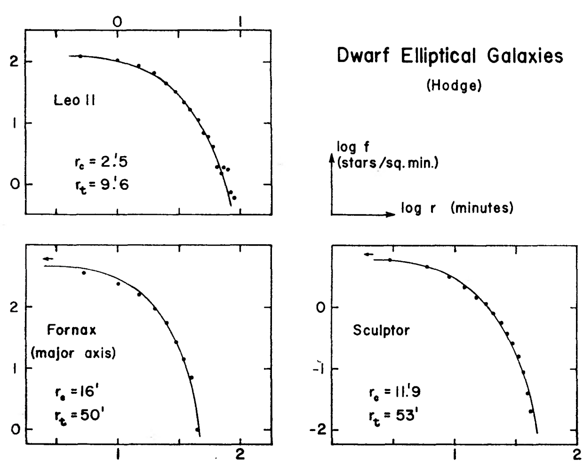

FIG. 15. Star counts in Sculptor-type galaxies. Arrows indicate Hodge's estimate of central densities.

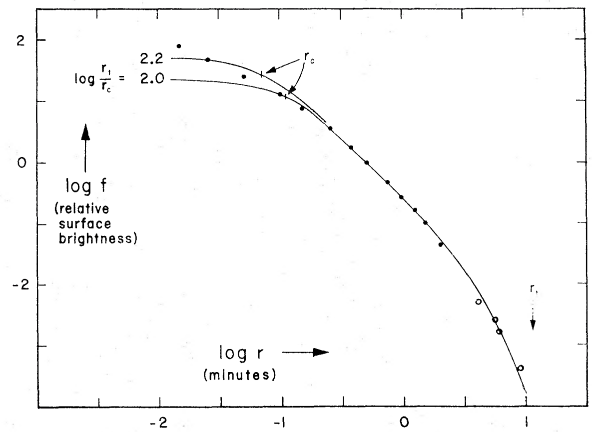

FIG. 16. Surface brightnesses in the elliptical galaxy M32 (de Vaucouleurs). Open circles are derived from estimates of limiting isophotes.

For an isolated elliptical galaxy the tidal cutoff should be very far from the center—if, indeed, a tidal cutoff exists at all. For such galaxies Hubble's (1930) observations indicate that the surface brightness at large distances from the center goes as $1/r^2$. This is exactly what Eq. (14) predicts when r_t is very large. It may therefore be suggested that the outer parts of elliptical galaxies differ from those of star clusters only in the absence, or near-absence, of a tidal limit.

VI. SPACE DENSITY AND STRIP DENSITY

From Eq. (14) formulas can be found for the spatial density

$$\varphi(r)=-\frac{1}{\pi}\int_r^{r_t}\left[\frac{d}{dx}f(x)\right]\frac{dx}{(x^2-r^2)^{\frac{1}{2}}} \tag{25}$$

and for the strip density

$$g(r)=2\int_r^{r_t}\frac{f(x)xdx}{(x^2-r^2)^{\frac{1}{2}}}. \tag{26}$$

The latter function gives the total number of stars in a strip of unit width running completely across the cluster and passing at a distance r from the center. The formulas for φ and g are

$$\varphi(r)=\frac{k}{\pi r_c[1+(r_t/r_c)^2]^{\frac{3}{2}}}\frac{1}{z^2}\left[\frac{1}{z}\cos^{-1}z-(1-z^2)^{\frac{1}{2}}\right] \tag{27}$$

$$g(r)=\frac{2kr_c}{[1+(r_t/r_c)^2]^{\frac{1}{2}}}\times\left[\frac{1}{z}\cos^{-1}z-2\operatorname{sech}^{-1}z+(1-z^2)^{\frac{1}{2}}\right], \tag{28}$$

where

$$z=\left[\frac{1+(r/r_c)^2}{1+(r_t/r_c)^2}\right]^{\frac{1}{2}}. \tag{29}$$

For comparison, Eq. (14) can be written

$$f(r)=\frac{k}{1+(r_t/r_c)^2}\left(\frac{1}{z}-1\right)^2. \tag{30}$$

VII. DISCUSSION

The observational data presented in Secs. IV and V suggest that the single law expressed by Eq. (14) represents the distributions of projected density in globular clusters of high and low central concentration, in galactic clusters, and in Sculptor-type dwarf galaxies. What inferences are to be drawn from this similarity of structure?

First, the exact mathematical form of Eq. (14) is almost certainly of no consequence. It should be regarded as merely a convenient fitting formula. Ultimately the shape of the curves can serve as a basis for a detailed theoretical discussion, but the present discussion will be confined to a much simpler problem: the number of parameters. Equation (14) describes an individual cluster by means of three parameters: a number factor k, a core radius r_c, and a limiting radius r_t. But the density distribution in a star cluster could not possibly be described by fewer than three parameters, because three separate conditions are imposed by circumstances quite independent of the cluster's internal dynamics. First, the cluster has a certain total number of stars. Second, the cluster has a certain total energy. Third, the cluster finds itself in a tidal field of a certain strength. Thus the description of a cluster will require at least three parameters, unless one of the above quantities is determined by the other two—not merely correlated, but rigidly determined. Since such a restriction is highly implausible, it therefore appears that the density distributions in star clusters are all described by a single law containing the *minimum* number of parameters. In other words, star clusters are as similar in structure as they could possibly be.

This similarity leads to some strong conclusions about the dynamical development of star clusters. First, consider the role of initial conditions. As pointed out in the preceding paragraph, the initial number of stars and the total energy must certainly leave their imprint on a cluster. On the other hand, the initial density and velocity distributions do not appear to make any difference; for if they did, then clusters would now differ in more than the minimum three parameters.

Since it is most unlikely that such widely differing systems all originated under identical initial conditions, we may conclude that all of the systems under consideration have been subjected to some regularizing, or relaxing, tendency—or perhaps to two or more relaxation processes, all tending to produce the same result. What relaxation process, then, is responsible for the similarity of star clusters? The first that springs to mind is relaxation through stellar encounters. This process has been extensively studied, and estimates have been made of the time of relaxation in star clusters. The most reliable calculation is that made by Oort and van Herk (1959) for M3. They employ a density distribution that agrees with observation, a consistent velocity distribution, and a realistic distribution function of stellar masses. For the massive stars that dominate the center of the cluster Oort and van Herk find a central time of relaxation of 1.5×10^8 yr. For less massive stars, and for regions of lower density, the relaxation time is somewhat longer; however, it is clear that during a lifetime of 10^{10} yr or more, stellar encounters have had time to exert a considerable effect on the central structure of the cluster.

For other clusters it is easily shown (Chandrasekhar 1942, p. 202) that the time of relaxation at the center is proportional to $n^{\frac{1}{2}}R^{\frac{3}{2}}$, where n is the number of stars in the cluster and R is its radius. Unpublished calculations by the writer show that in the models of Eq. (14) the effective radius R is to be closely identified with the core radius. The data of Table III and Fig. 14 then show that the times of relaxation in the three galactic clusters previously discussed are of the order of a tenth of that in M3. Here even more emphatically, stellar encounters have had ample time to operate. In the dwarf elliptical galaxies of Fig. 15, on the other hand, n is perhaps 10 times as large as in M3 (Hodge 1961a, b, 1962) while R is a hundred times as large. Therefore even at the centers of those systems the time of relaxation is over 10^{11} years, and stellar encounters cannot have had any serious effect during the commonly accepted cosmic time scale. Yet these galactic clusters and dwarf elliptical galaxies have very similar density distributions, even to the values of the concentration ratios.

Consider first the systems whose times of relaxation are very long. Stellar encounters cannot have produced any appreciable relaxation; yet, as has been argued above, relaxation has nevertheless taken place. Some other relaxation mechanism has therefore been effective. The nature of this mechanism is by no means obvious, but a likely possibility is the initial mixing at the time of formation of the system. When the stars condense out of a parent gas cloud, they have density and velocity distributions characteristic of the gas. It is most unlikely that these distributions will satisfy the virial theorem for gravitational forces alone; hence some readjustment is necessary. During the period of readjustment each star finds itself moving in a time-dependent potential field, in which the energy of an individual star does not remain constant. The total energy change that a star experiences during this settling-down period will depend not only on its position and speed but also on the direction of its motion, so that the net effect is a strong randomizing effect on the energies of individual stars—in other words, a relaxation. The characteristic time for this process is the circulation time of a star through the system—generally 10^6 to 10^8 yr.

As for the systems in which stellar encounters produce a rapid relaxation, they have also been subjected to initial mixing but have thereafter been acted upon by stellar encounters. Yet they have the same density law as systems unaffected by stellar encounters. It would therefore appear that the relaxation produced by stellar encounters has the same effect as that produced by initial mixing. Furthermore, successive relaxation times produce no further change other than changes of the parameters in Eq. (14). In short, the models described by Eq. (14) seem to constitute a unique evolutionary sequence. Initially a cluster evolves rapidly during the mixing stage, until it reaches the basic sequence. Thereafter its development is much slower and consists of a quasi-stationary evolution *along* the sequence.

An unhappy corollary is that a cluster no longer bears any indication of the point at which it joined the basic sequence. The cluster's structure therefore offers no clue to its age.

The data on which the above conclusions are based cover stellar systems of a wide range of characteristics, but at the same time it is well to be aware of their limitations. One serious restriction is that all these systems are old, according to the estimates provided by the theory of stellar evolution. This raises the possibility that their similarity is due neither to initial mixing nor to stellar encounters but is due instead to some other regularizing tendency that operates over the billions of years. Such an effect might arise, for instance, from the mass loss attendant upon the rapid evolution of massive stars. Von Hoerner (1958) and Oort and van Herk (1959) have shown that such a loss of mass should result in a proportional increase in the cluster radius. The clusters considered in the present paper have lost from a third to half of their original mass in this way, and the consequent re-adjustment in radius might conceivably produce a relaxation. Whether its effect is great enough, however, remains to be seen. Even so, the galactic clusters would still pose a difficulty, because their times of relaxation through stellar encounters are too short to be ignored. As before, it would be necessary to postulate that both relaxation mechanisms produce identical effects.

In any case it is desirable to study the density distributions in some young galactic clusters. Unfortunately the young clusters tend to lie at low galactic latitudes, where the cluster stars are diluted by a rich background and where patchy interstellar absorption makes the results unreliable. Among the clusters for which counts have been published by van den Bergh and Sher (1960) the best case is M37 (NGC 2099), in which the earliest spectral type is B9 (Trumpler 1930). According to Johnson *et al.* (1961) the reddening is $0^m.3$ and the corrected distance of the cluster is 1280 pc. The cluster is rich, but so is the background. In the counts given by van den Bergh and Sher the best separation of cluster and background seems to occur at intermediate magnitudes. Figure 17 is derived from their count to limiting magnitude 15.55. As in Fig. 14, r_t has been computed rather than derived from the data. The agreement with Eq. (14) seems adequate.

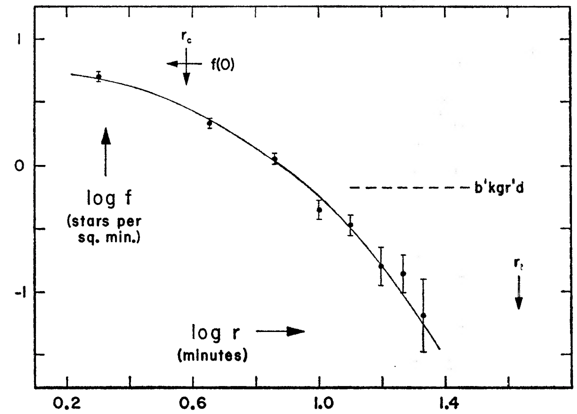

FIG. 17. Star counts in the galactic cluster M37 (van den Bergh and Sher).

Even M37 has an age greater than 10^8 yr, however, and it would be desirable to check Eq. (14) for still younger clusters. Unfortunately the extensive data collected by Wallenquist (1959) do not seem adequate for this purpose. Wallenquist appears to have underestimated the radii of the clusters and consequently chosen incorrect values for the background densities. In M37, for instance, Wallenquist chooses a limiting radius of 17′, a distance at which the data of Fig. 17 indicate that the density is 15 or 20% above that of the true background. In M11 Wallenquist chooses a limiting radius of 6′, whereas Johnson, Sandage, and Walquist (1956) state that their photometric study includes all yellow giants lying within 10 minutes of the center.

To sum up the arguments about mass loss, on the theoretical side it is by no means clear that stellar mass loss ought to produce a relaxation, while on the observational side there is some indication that relaxation takes place even before there is any major mass loss. We may therefore tentatively fall back on the interpretation already given—that relaxation is first produced by the initial mixing and is then continued by stellar encounters in a quasi-stationary fashion.

One further limitation should be mentioned. In most cases the data presented in this paper do not distinguish between bright and faint stars. Only in NGC 5053 (Figs. 12 and 13) has a difference been clearly shown. More specifically, no observational evidence has been cited to show a tendency toward equipartition of energy in a system that is younger than one time of relaxation. In NGC 5053, where faint stars are observed, the time of relaxation at the center is less than 10^9 years, while in the Sculptor-type galaxies, where the relaxation time is long, the less massive stars are too faint to observe.

This question of equipartition in young systems is an important one, for here is a point at which initial mixing and stellar encounters can be expected to produce different effects. Stellar encounters lead to equipartition of energy, whereas the interaction between a single star and a changing potential field does not depend on the mass of the star. Thus if equipartition of energy were found to exist in systems whose ages are less than their times of relaxation through stellar encounters, we should have to conclude that equipartition already existed among the prestellar blobs in the gas clouds from which the systems were formed.

Some indication of equipartition without encounters is shown at the opposite end of the size spectrum, among the giant elliptical galaxies. To see this it is necessary to examine in more detail the dynamical circumstances at the center of a stellar system. If we assume a particular form for the velocity distribution in a steady state, then integration leads to a relation between density and potential. Solution of Poisson's equation gives the potential, and thence the density, as functions of distance from the center. In this fashion Woolley and Dickens (1961) have shown that a truncated Gaussian velocity distribution leads to models whose density distributions agree with the observed densities near the centers of globular clusters. Implicit in the argument is one additional assumption, whose role is crucial to the present discussion; it is assumed that the potential field is produced by the stars under consideration—or at least by a group of stars having an identical density distribution. This assumption is satisfied if all types of stars have the same distribution, but it is also satisfied if the central density is almost completely due to stars of the type whose distribution is being studied. It is the latter condition that is satisfied in globular clusters. The massive stars that contribute most of the light account for only a small part of the total mass of the cluster, but their greater central concentration makes them predominate strongly near the cluster center. This phenomenon is shown clearly in Table 6 of Oort and van Herk (1959).

In giant elliptical galaxies, on the other hand, the density distribution is different from that in globular clusters. The extended central rise shown in Fig. 16 is characteristic not only of intermediate-luminosity systems like M32 but also of high-luminosity giants like M87 (Hubble 1930; van Houten 1961), in which the time of relaxation is far too long for stellar encounters to have had any appreciable effect. This peculiarity of giant ellipticals shows, according to the arguments just given, that either (1) the velocity distribution near the center is far from Gaussian or (2) the central density is due to stars having a different density distribution from that of the massive stars that contribute most of the light. The possibility of non-Gaussian velocity distributions can certainly not be excluded, but to the writer it seems much more likely that here too the initial mixing process will produce energy exchanges that lead to a near-Gaussian distribution of velocities. In that case it must be concluded that some considerable part of the central density comes from a differently distributed group of less luminous—and presumably less massive—stars. But the heaping up of light to the center of the system then shows that the bright stars have a lower velocity dispersion than the faint stars in whose potential field they move. This suggests equipartition of energy in giant elliptical galaxies.

It remains to explain why equipartition produces one profile in a globular cluster and another profile in a giant elliptical galaxy. The answer lies in the relative number of dwarf stars. In a globular cluster the ratio of mass to luminosity is of the order of 1 (Feast and Thackeray 1960), while in a giant elliptical galaxy it is of the order of 100 (Page 1960). This wide difference indicates that giant ellipticals have a much larger number of dwarf stars, which contribute much mass but little luminosity. The predominance of dwarfs is apparently so great that even the heaping up of giants to the center fails to outweigh them.

Turned in the other direction, this argument would say that the distribution of light in giant elliptical galaxies shows that they must contain large numbers of dwarf stars.

Two other observations lend some support to this general interpretation. First, Spinrad (1961) has argued that the spectra of giant elliptical galaxies show the presence of large numbers of dwarf stars. Second, Prendergast and Miller (1962) have found that the central regions of the elliptical galaxy NGC 3379 are slightly redder than the envelope—exactly as one would expect from a heaping up of bright red stars at the center. It must be remembered, however, that the whole dynamical interpretation presented here for giant elliptical galaxies is based on qualitative arguments and needs to be confirmed by detailed calculations.

VIII. SUMMARY OF CONCLUSIONS

(1) The density distributions in globular clusters, galactic clusters, and Sculptor-type dwarf elliptical galaxies can all be represented by the same empirical law.

(2) The density law has three parameters: a number factor, a core radius, and a limiting radius. The core radius is determined by the internal energy of the system, while the limiting radius is set by external tidal forces. Observed values of the limiting radius

agree with values calculated from the tidal force of the Milky Way.

(3) The observed density law has the minimum number of parameters; hence these stellar systems are as similar as they could possibly be. The similarity shows that all these systems have been subjected to some relaxation process. It is suggested that all have been relaxed by initial mixing and that stellar encounters thereafter produce a slow change in the density parameters without changing the basic model. It does not seem likely that cluster expansion through stellar mass loss is a major factor in the relaxation.

(4) The data considered do not include any young galactic clusters; nor do they establish whether equipartition exists in a system whose age is less than its time of relaxation.

(5) Relative to the other systems considered, giant elliptical galaxies have an excess of brightness near the center. It is suggested that this results from equipartition in a system dominated by dwarf stars.

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