ELECTRON RADIATIVE TRANSITIONS IN A COULOMB FIELD

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ABSTRACT

Free-free, bound-free, and bound-bound Gaunt factors and oscillator strengths have been computed for electrons in a pure Coulomb potential. Numerical results are presented for a wide range of electron and photon energies. In addition, for the free-free case, average Gaunt factors and the rate of bremsstrahlung production have been obtained as functions of temperature for a Boltzmann distribution of electron energies.

I. INTRODUCTION

For many problems in astrophysics and plasma physics, conditions of ionization are such that it is a reasonable approximation to treat the electrons as moving in a pure Coulomb potential (with perhaps an effective screened charge for the ion). In addition, of course, hydrogen plasmas, if they are not too dense, also satisfy this condition. For many years analytic formulae have been available for computing the radiative absorption and emission properties of electrons in a Coulomb field (Sommerfeld 1953; Bethe and Salpeter 1956). These formulae involve hypergeometric functions with complex parameters. In this paper we have rewritten some of these formulae in forms more convenient for numerical computation and have computed free-free, bound-free, and bound-bound Gaunt factors and oscillator strengths as a function of electron energies and photon frequencies. In addition, for the free-free case, suitable averages over a Boltzmann distribution have been performed to give the radiative properties for an ionized plasma as a function of temperature. Some similar results have been published by other authors (e.g., Berger 1956, 1957). The present calculations considerably extend this work, to cover a range of physical conditions adequate for most astrophysical and other applications.

II. FREE-FREE TRANSITIONS

The free-free absorption coefficients in a pure Coulomb field are readily computed from the formula for bremsstrahlung, using detailed balancing. The cross-section for absorption of a photon of energy $h\nu$ by an electron of energy $E_i = \hbar^2 k_i^2/2m$, which makes a transition to energy $E_f = \hbar^2 k_f^2/2m$, is related to the cross-section for emission of the photon $h\nu$ by the electron E_f by the formula

$$\sigma_{i,\,\nu \to f} = \frac{v_f}{c} \frac{\rho_f}{\rho_{i,\,\nu}} \sigma_{f \to i,\,\nu}\,, \tag{1}$$

where v_f is the velocity of the final electron and ρ_f and $\rho_{i,\nu}$ are the density of states for the two final states:

$$\begin{aligned} \rho_f &= 2 \times 4\pi \frac{m k_f}{(2\pi)^3 \hbar^2}\,, \\ \rho_{i,\,\nu} &= 2 \times 4\pi \frac{m k_i}{(2\pi)^3 \hbar^2}\, 2 \times 4\pi \frac{\nu^2}{2\pi \hbar c^2}\,, \end{aligned} \tag{2}$$

where the factors of 2 are for electron spin and photon polarizations and the free electrons are normalized to plane waves of unit density at infinity.

For bremsstrahlung, we have (Sommerfeld 1953; Biedenharn 1956)

$$d\sigma_{f\to i,\,\nu}^{\text{em}}=\frac{32}{3}Z^2\left(\frac{e^2}{\hbar c}\right)^3\frac{k_i k_f}{k_f^2}\frac{d(h\nu)}{h\nu}\sum_{l=0}^{\infty}\left[(l+1)\,\tau_{l+1,\,l}^2+l\,\tau_{l-1,\,l}^2\right], \tag{3}$$

where

$$\tau_{l',\,l}=\int_0^{\infty}r^2dr\,\psi_{l'}(r,E_i)\frac{1}{r^2}\psi_l(r,E_f), \tag{4}$$

and, asymptotically,

$$\psi_l\sim\frac{\sin(kr+\delta_l)}{kr}.$$

Equation (3) gives the total cross-section to emit an unpolarized photon in the energy range $h\nu$ to $h\nu + d(h\nu)$ for electrons of unit density. Thus, for absorption, using the same electron normalization,

$$\begin{aligned}
d\sigma_{i,\,\nu\to f}^{\text{ab}}&=\frac{\hbar^2k_f^2c^2}{4mk_i\nu^2}\,d\sigma_{f\to i,\,\nu}^{\text{em}}\\
&=\tfrac{8}{3}Z^2\left(\frac{e^2}{\hbar c}\right)^3\frac{\hbar^2c^2k_f}{m\nu^2}\frac{d(h\nu)}{h\nu}\sum_{l=0}^{\infty}\left[(l+1)\,\tau_{l+1,\,l}^2+l\,\tau_{l-1,\,l}^2\right]\\
&=\frac{2\pi^2\hbar^2}{mk_i}\,a_0^2\left\{\tfrac{64}{3}Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{Ry}{h\nu}\right)^3\frac{d(h\nu)}{Ry}\right.\\
&\qquad\left.\times\,k_ik_f\Sigma\left[(l+1)\,\tau_{l+1,\,l}^2+l\,\tau_{l-1,l}^2\right]\right\}.
\end{aligned} \tag{5}$$

The sum can be explicitly evaluated (Sommerfeld 1953; Biedenharn 1956), and we obtain a result in terms of hypergeometric functions, which can be expressed in several ways.

One convenient way is (Biedenharn 1956)

$$\begin{aligned}\Sigma=\big[\,&(k_i^2+k_f^2+2k_f^2\eta_f^2)(0,1;0)\\
&-2k_ik_f(1+\eta_i^2)^{1/2}(1+\eta_f^2)^{1/2}(0,1;1)\big](0,1;0),\end{aligned} \tag{6}$$

where

$$(0,1;l)=\int_0^{\infty}r^2dr\,\frac{1}{r}\,\psi_l(r,E_i)\,\psi_l(r,E_f), \tag{7}$$

$$\eta=\frac{Ze^2}{\hbar v}=\frac{Z}{ka_0}, \tag{8}$$

and again

$$\psi_l\sim\frac{\sin(kr+\delta)}{kr}.$$

For Coulomb wave functions,

$$(0,1;l)=\frac{1}{k_ik_f}I_l,$$

with

$$I_l = \frac{1}{4}\left[\frac{4k_ik_f}{(k_i-k_f)^2}\right]^{l+1} e^{\pi|\eta_i-\eta_f|/2}\, \frac{|\Gamma(l+1+i\eta_i)\,\Gamma(l+1+i\eta_f)|}{\Gamma(2l+2)}\, G_l\,, \tag{9}$$

and the real function, G_l, is given by

$$G_l = \left|\frac{k_f-k_i}{k_f+k_i}\right|^{i\eta_i+i\eta_f} {}_2F_1\!\left[l+1-i\eta_f,\ l+1-i\eta_i;\ 2l+2;\ -\frac{4k_ik_f}{(k_i-k_f)^2}\right]. \tag{10}$$

If the argument of the hypergeometric function is greater than 1, this can be rewritten as

$$\begin{aligned} G_l = \left|\frac{k_f-k_i}{k_f+k_i}\right|^{i\eta_i+i\eta_f} \Gamma(2l+2) \Bigg\{ & \frac{\Gamma(i\eta_f-i\eta_i)}{\Gamma(l+1-i\eta_i)\Gamma(l+1+i\eta_f)}\left[\frac{4k_fk_i}{(k_f-k_i)^2}\right]^{-l-1+i\eta_f} \\ & \times {}_2F_1\!\left[l+1-i\eta_f,\ -l-i\eta_f;\ 1+i\eta_i-i\eta_f;\ -\frac{(k_f-k_i)^2}{4k_fk_i}\right] \\ & + \frac{\Gamma(i\eta_i-i\eta_f)}{\Gamma(l+1+i\eta_i)\,\Gamma(l+1-i\eta_f)}\left[\frac{4k_fk_i}{(k_f-k_i)^2}\right]^{-l-1+i\eta_i} \\ & \times {}_2F_1\!\left[l+1-i\eta_i,\ -l-i\eta_i,\ 1+i\eta_f-i\eta_i;\ -\frac{(k_f-k_i)^2}{4k_fk_i}\right]\Bigg\}; \end{aligned} \tag{11}$$

in each of these cases it is possible to express the quantity G_l as a real series in the variable $4k_fk_i\,(k_f-k_i)^{-2}$ (see Appendix A).

Thus we have

$$\begin{aligned} d\sigma_{i,\,\nu\to f} = {} & \tfrac{64}{3}a_0^2Z^2\,\frac{e^2}{\hbar c}\left(\frac{Ry}{h\nu}\right)^3 d\left(\frac{h\nu}{Ry}\right)\frac{1}{k_fk_i}\,\frac{2\pi^2\hbar^2}{mk_i} \\ & \times\left[\left(k_f^2+k_i^2+2k_f^2\eta_f^2\right)I_0 - 2k_fk_i\left(1+\eta_i^2\right)^{1/2}\left(1+\eta_f^2\right)^{1/2}I_1\right]I_0\,. \end{aligned} \tag{12}$$

This cross-section assumes that the electrons are normalized to unit density. More conveniently, we can normalize the initial electron wave functions on the energy scale (i.e., in terms of the number per unit volume per unit energy range). The pertinent relation is

$$\sigma_{\text{energy scale}} = \sigma_{\text{density scale}}\cdot\frac{mk_i}{2\pi^2\hbar^2}\,. \tag{13}$$

Thus

$$\begin{aligned} d\sigma_{ff,\ \text{energy scale}} = {} & a_0^2Z^2\,\frac{64}{3}\,\frac{e^2}{\hbar c}\,\frac{Ry}{h\nu}\,\frac{1}{\eta_i\eta_f} \\ & \times\left[\left(\eta_i^2+\eta_f^2+2\eta_i^2\eta_f^2\right)I_0 - 2\eta_i\eta_f\left(1+\eta_i^2\right)^{1/2}\left(1+\eta_f^2\right)^{1/2}I_1\right]I_0\,. \end{aligned} \tag{14}$$

The classical cross-section for free-free absorption is given by (Kramers 1923)

$$d\sigma_{\text{class; energy scale}} = a_0^2Z^2\,\frac{32\pi}{3\sqrt{3}}\,\frac{e^2}{\hbar c}\left(\frac{Ry}{h\nu}\right)^3 d\left(\frac{h\nu}{Ry}\right). \tag{15}$$

Thus the Gaunt factor is

$$g_{ff}=\frac{d\sigma_{ff}}{d\sigma_{\text{class}}}=\frac{2\sqrt{3}}{\pi\eta_i\eta_f}\left[\left(\eta_i^2+\eta_f^2+2\eta_i^2\eta_f^2\right)I_0-2\eta_i\eta_f\left(1+\eta_i^2\right)^{1/2}\left(1+\eta_f^2\right)^{1/2}I_1\right]I_0, \tag{16}$$

where

$$\eta_i^2=\frac{Z^2Ry}{E_i}; \qquad \eta_f^2=\frac{Z^2Ry}{E_f}. \tag{17}$$

Values of g_{ff} are plotted in Figures 1 and 2 as functions of E_i/Z^2 and $h\nu/Z^2$.

To obtain the total absorption coefficient for a distribution of initial electron energies, we take the cross-section on the energy scale and multiply by the number of electrons/unit energy. For the case of the Boltzmann distribution, this is

$$\frac{dn}{dE_i}=\frac{1}{kT}\,e^{-E_i/kT}\times(\text{number/unit volume})\,. \tag{18}$$

For many cases of interest in opacity computations the temperatures and densities are such that it is permissible to neglect the degeneracy of the electrons. This is not

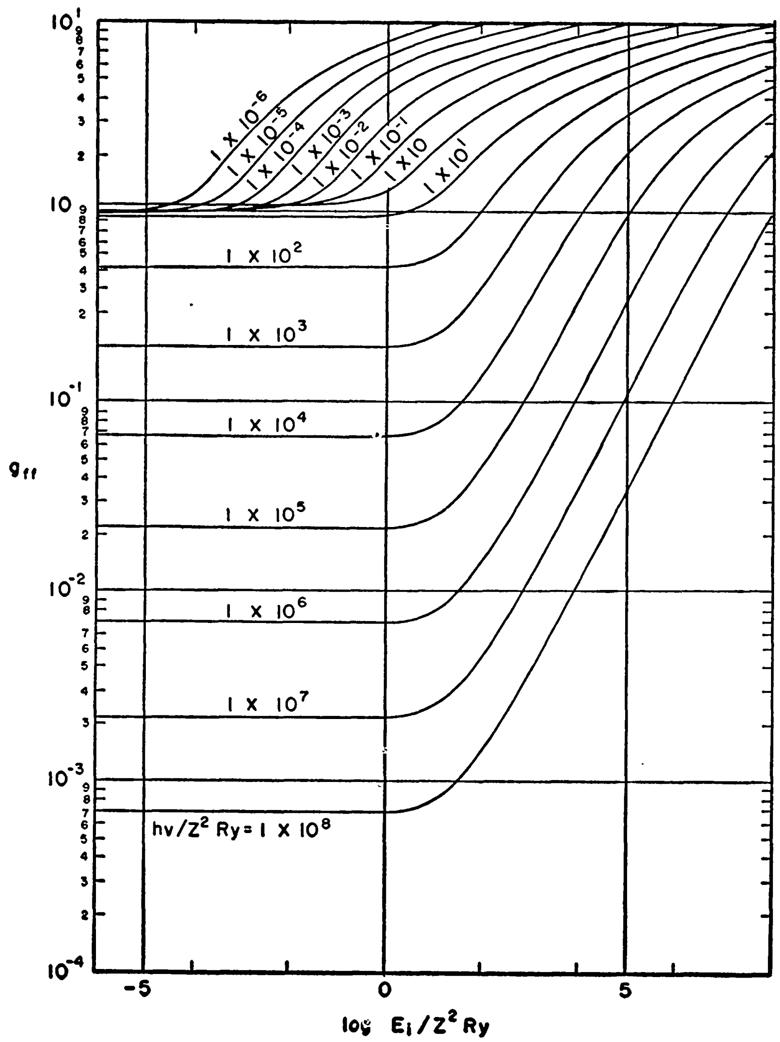

FIG. 1.—Free-free Gaunt factor versus incident electron energy for various photon energies

true, however, of such stellar matter as the white dwarfs, for instance. In general, one ought to use the Fermi-Dirac distribution function rather than the Boltzmann; it is a straightforward matter to do the integration for any particular case of interest. Because of the difficulty in presenting the results of such calculations (which add a third parameter—the free energy—to the frequency and temperature dependence on the Gaunt factor), we have limited ourselves to the Boltzmann distribution.

In this case the average cross-section at temperature T for absorption of photons of frequency $h\nu$ by free elctrons is

$$\langle d\sigma_{ff}\rangle = \frac{1}{kT}\int e^{-E_i/kT} dE_i d\sigma_{ff}(E_i, E_i + h\nu) . \quad (19)$$

In terms of the parameters η, we have

$$\frac{1}{\eta_f^2} = \frac{1}{\eta_i^2} + \frac{h\nu}{Z^2}, \quad (20)$$

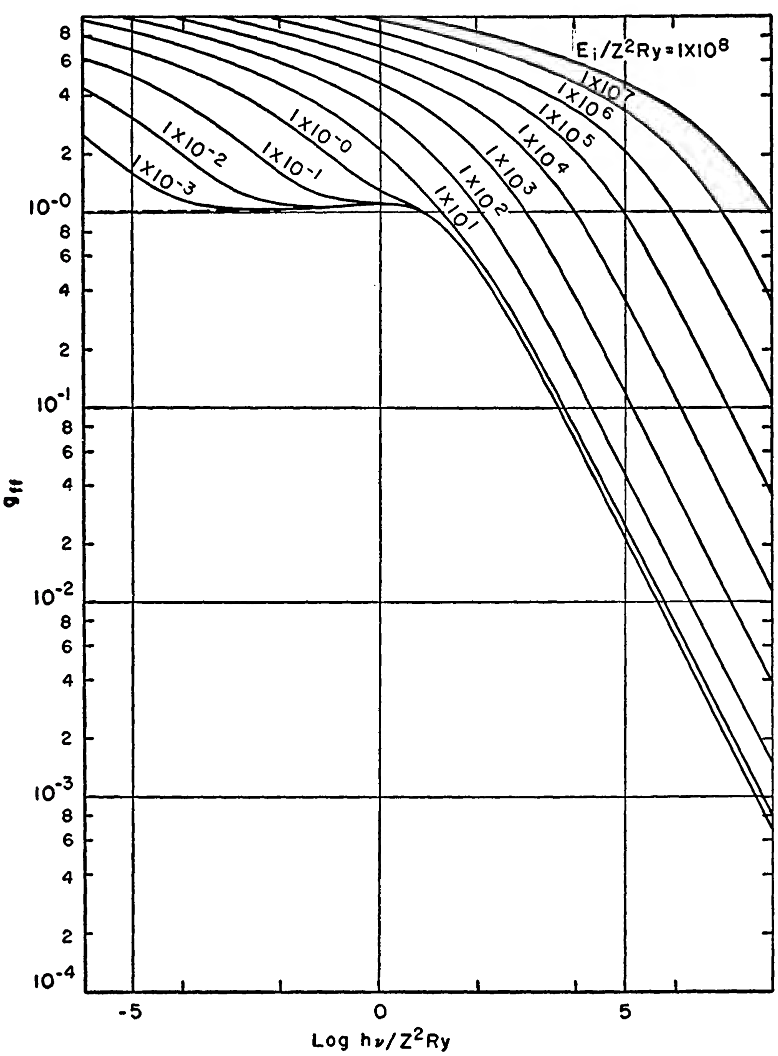

FIG. 2.—Free-free Gaunt factor versus photon energy for various incident electron energies

so that we obtain, for the temperature-averaged free-free Gaunt factor,

$$\langle g_{ff}(u, \gamma^2) \rangle = \int_0^\infty d x \, e^{-x} g_{ff}\left(\eta_i = \gamma x^{-1/2}, \frac{h\nu}{Z^2} = \frac{u}{\gamma^2}\right), \tag{21}$$

where

$$u = \frac{h\nu}{kT}, \qquad \gamma^2 = \frac{Z^2 R y}{kT}.$$

This temperature-averaged Gaunt factor is plotted in Figures 3, 4, and 5.

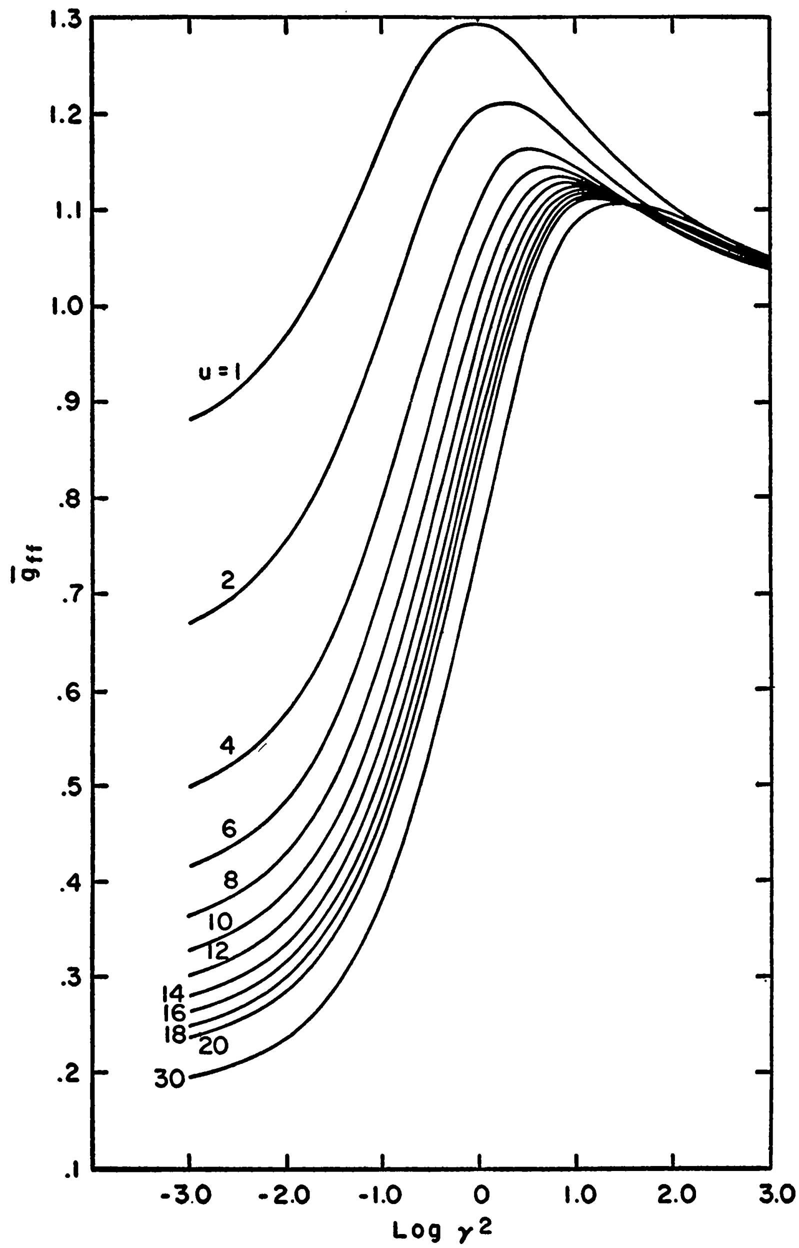

FIG. 3.—Temperature-averaged free-free Gaunt factor versus $\gamma^2 = Z^2Ry/kT$ for various values of $u = h\nu/kT$.

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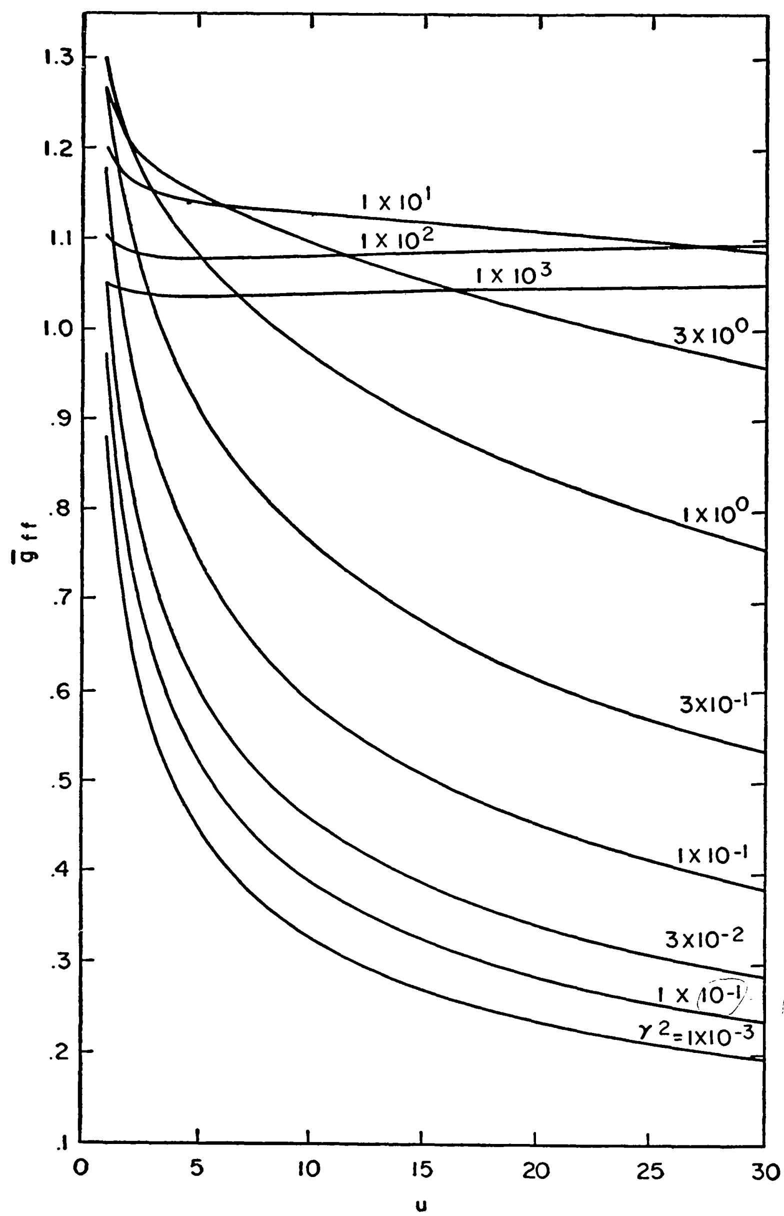
The Gaunt factors described above can also be used to compute the amount of energy emitted in the bremsstrahlung process. By comparing equations (3), (6), (9), and (16), we see that we can write the cross-section for an electron of energy E_0, normalized to unit density at infinity, to emit a photon in the energy range $h\nu$ to $h\nu + d(h\nu)$ in terms of the free-free Gaunt factor:

$$d\sigma^{\text{em}} = \frac{16\pi}{3\sqrt{3}} \left(\frac{e^2}{\hbar c}\right)^3 Z^2 \frac{\hbar^2}{2mE_0} \frac{d(h\nu)}{h\nu} g_{ff}(E_i = E_0 - h\nu, E_f = E_0) . \tag{22}$$

The energy emitted per unit time is found by multiplying by $h\nu$ and by the electron flux, $n_e v_0$:



1×10⁻²

FIG. 4.—Temperature-averaged free-free Gaunt factor versus $u = h\nu/kT$ for various values of $\gamma^2 = Z^2Ry/kT$. *Note:* The labeling of curve 2 from below should be 1×10^{-2} (*not* 1×10^{-1}).

$$W(\nu)\,d\nu = \frac{8\pi}{3}\left(\frac{e^2}{\hbar c}\right)^3 Z^2 n_e \frac{\hbar^2}{m}\left(\frac{2}{3m}\right)^{1/2} E_0^{-1/2} d(h\nu)\, g_{ff}$$
$$= \frac{128\pi^2}{3\sqrt{3}}\left(\frac{e^2}{\hbar c}\right)^3 Z^2 (n_e a_0^3)\left(\frac{E_0}{Ry}\right)^{-1/2} d\nu\, g_{ff} Ry\,. \tag{23}$$

To find the energy emitted per unit time by a Maxwellian distribution of electron energies, we average this spectrum:

$$\langle W(\nu)\rangle d\nu = \frac{\int_{h\nu}^{\infty} dE_0 E_0^{1/2} e^{-E_0/kT} W(\nu)\,d\nu}{\int_0^{\infty} dE_0 E_0^{1/2} e^{-E_0/kT}}\,. \tag{24}$$

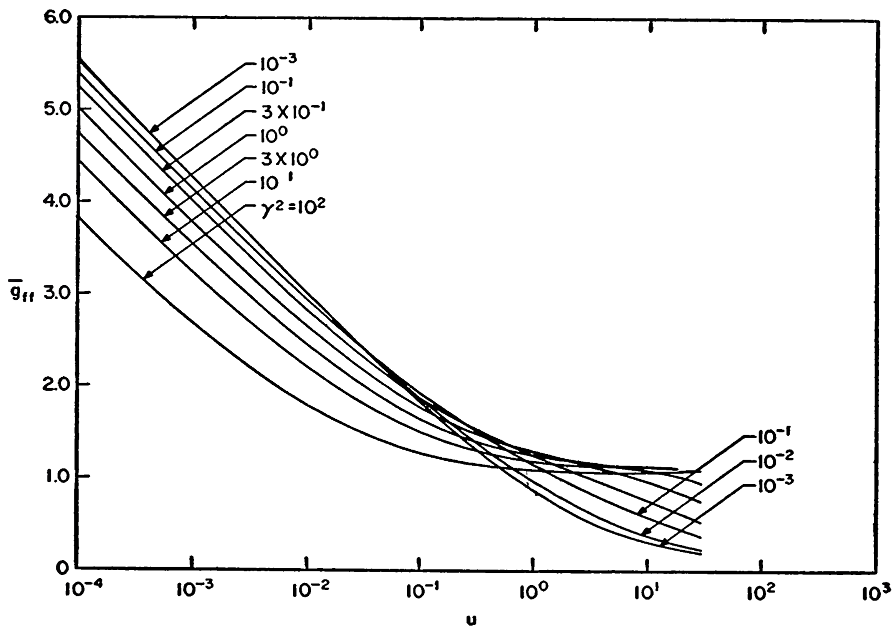


FIG. 5.—Temperature-averaged free-free Gaunt factor versus $u = h\nu/kT$ for various values of $\gamma^2 = Z^2Ry/kT$.

If we change to the integration variable $E_0 - h\nu$ in the numerator, we are evaluating exactly the same integral that gives the temperature-averaged Gaunt factor. Thus, after simplifying and rearranging the constants, we find

$$\langle W(\nu)\rangle d\nu = \frac{2^5\pi e^6}{3hmc^3} Z^2 n_e \left(\frac{2\pi kT}{3m}\right)^{1/2} du\, e^{-u} \langle g_{ff}(u,\gamma^2)\rangle$$
$$= 1.42\times10^{-27} Z^2 n_e T^{1/2} du\, e^{-u} \langle g_{ff}(u,\gamma^2)\rangle \text{ ergs/sec}\,, \tag{25}$$

where $u = h\nu/kT$; $\gamma^2 = Z^2Ry/kT$; and T is in degrees Kelvin in the second expression.

The energy emitted per unit volume per unit time is found by multiplying by the density of positive ions which serve as targets for the bremsstrahlung production. To find the total energy emitted, we integrate over the photon spectrum. Thus

$$\epsilon=\left(\frac{2\pi kT}{3m}\right)^{1/2}\frac{2^5\pi e^6}{3hmc^3}Z^2n_en_i\int_0^\infty du\, e^{-u}\langle g_{ff}(u,\gamma^2)\rangle$$

$$=1.42\times 10^{-27}Z^2n_en_iT^{1/2}\int_0^\infty du\, e^{-u}\langle g_{ff}(u,\gamma^2)\rangle \text{ergs/cm}^2\text{ sec} \qquad (26)$$

$$=\epsilon_0\langle g_{ff}(\gamma^2)\rangle\,,$$

where ϵ_0 is the factor multiplying the integral and is the rate of energy production, assuming $\langle g_{ff}\rangle = 1$ (see, e.g., formula [5-49] of Spitzer 1956).

The integral, $\langle g_{ff}(\gamma^2)\rangle = \epsilon/\epsilon_0$, is plotted as a function of γ^2 in Figure 6.

III. BOUND-FREE TRANSITIONS

The bound-free (photoelectric) absorption coefficients in a pure Coulomb field are readily computed directly by using the appropriate Coulomb wave functions. Alternatively, they may be obtained by analytic continuation of the bound-bound oscillator strength formulae. We shall use the latter method here.

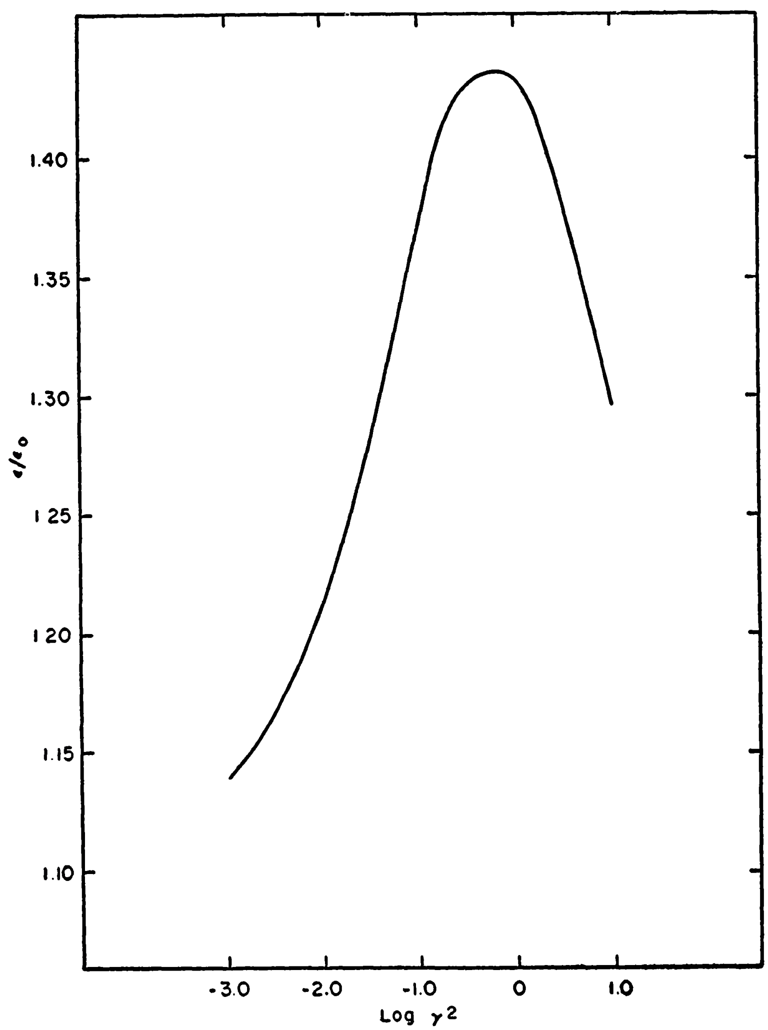


FIG. 6.—Ratio of bremsstrahlung energy production to that calculated with unit Gaunt factor plotted versus $\gamma^2 = Z^2Ry/kT$.

For a dipole electron transition from a bound state (n, l) to a free (continuum) state $(E = \hbar^2k^2/2m)$, the total cross-section for absorption of an unpolarized photon of frequency ν is given by (Gaunt 1930)

$$\sigma_{nl\to E} = \frac{16\pi^2}{3}\frac{e^2}{\hbar c}\frac{mk\nu}{\hbar}\left[\frac{l+1}{2l+1}\left(\tau_{nl}^{E,\,l+1}\right)^2 + \frac{l}{2l+1}\left(\tau_{nl}^{E,\,l-1}\right)^2\right], \tag{27}$$

where

$$\tau_{nl}^{E,\,l\pm1} = \int r^2 dr\, \psi_{E,\,l\pm1}(r)\, r\psi_{nl}(r), \tag{28}$$

with wave function normalizations such that

$$\int r^2 dr\, |\psi_{nl}(r)|^2 = 1,$$

and, asymptotically,

$$\psi_{E,\,l}(r) \sim \frac{\sin(kr+\delta_l)}{kr}. \tag{29}$$

We must therefore evaluate the matrix element $\tau_{nl}^{E,\,l\pm1}$. To do this, we first consider the radial matrix element between bound states:

$$\tau_{nl}^{n',\,l\pm1} = \int r^2 dr\, \psi_{n',\,l\pm1}\, r\psi_{nl}(r), \tag{30}$$

with the same normalization on bound wave functions as before.

An explicit formula for this matrix element was given by Gordon (1929) in computing the bound-bound oscillator strengths. We show in Appendix B that this expression may, by analytic continuation, be used also for the bound-free case.

In order to make this continuation, we first look at the form of the bound and free wave functions (Bethe and Salpeter 1956):

$$\psi_{nl}(r) = \left(\frac{2Z}{na_0}\right)^{3/2}\left[\frac{\Gamma(n+l+1)}{\Gamma(n-l)\,2n}\right]^{1/2}\frac{(2Zr/na_0)^l}{(2l+1)!} \times e^{-Zr/na_0}\,{}_1F_1\left(l+1-n;\,2l+2;\,\frac{2Zr}{na_0}\right), \tag{31}$$

$$\psi_{E,\,l}(r) = e^{\eta\pi/2}\frac{|\Gamma(l+1-i\eta)|}{(2l+1)!}(2kr)^l e^{ikr}{}_1F_1(l+1-i\eta;\,2l+2;\,-2ikr). \tag{32}$$

Here ${}_1F_1$ is the regular confluent hypergeometric function (a polynomial if the first parameter is a negative integer or zero), and

$$\eta = \frac{Z}{ka_0} = \left(\frac{Z^2Ry}{E}\right)^{1/2}. \tag{33}$$

As before, the bound wave functions have the usual normalization, while the free functions are asymptotically plane waves of unit density.

We note that, apart from constants, the free wave function is just the bound wave function with $n \to i\eta$. Specifically,

$$\psi_{E,\,l}(r) = \left[\frac{2ik^3}{\pi\eta}\left(1-e^{-2\pi\eta}\right)\right]^{-1/2}\psi^b_{i\eta,\,l}(r). \tag{34}$$

Thus we wish to write the bound-bound matrix element in a form for which we can make the continuation $n \to e^{i\pi/2}\eta$. This form is (Gordon 1929)

$$\tau_{nl}^{n',\,l-1}=\tau_{n',\,l-1}^{nl}=\frac{a_0}{Z}\frac{2^{2l}}{(2l-1)\,!}\left[\frac{\Gamma(n+l+1)\,\Gamma(n'+l)}{\Gamma(n-l)\,\Gamma(n'-l+1)}\right]^{1/2}$$

$$\times\,(nn')^{l+1}(n+n')^{-n-n'}(n-n')^{n-2-l}(n'-n)^{n'-l}\Big\{{}_2F_1\Big[l+1-n,\,l-n';\,2l;\tag{35}$$

$$-\frac{4nn'}{(n-n')^2}\Big]-\left(\frac{n-n'}{n+n'}\right)^2{}_2F_1\Big[l-1-n,\,l-n';\,2l;-\frac{4nn'}{(n-n')^2}\Big]\Big\}.$$

In Appendix B we show that this expression is valid if n is an integer $\geq l+1$ and $Re\ n/n'<1$, or, conversely, if n' is an integer $\geq l$ and $Re\ n'/n<1$.

Thus, in the first case, the formula is valid everywhere in the complex n'-plane outside a circle of radius $n/2$ centered at $n'=n/2$. In particular, therefore, we may use it for $n'=i\eta$. Similarly, we may interchange n and n', let $l\to l+1$, and use the formula for the transition $l\to l+1$. These operations are carried out in Appendix C. We quote here the final results:

$$\sigma_{nl\to E,\,l-1}=\frac{\pi e^2}{m\,cv}\frac{2^{4l}}{3}\frac{l^2\,(n+l)\,!\{[1^2+\eta^2][2^2+\eta^2]\dots[(l-1)^2+\eta^2]\}}{(2l+1)\,!\,(2l-1)\,!\,(n-l-1)\,!}$$

$$\times\frac{\exp[-4\eta\cot^{-1}\rho]}{1-e^{-2\pi\eta}}\frac{\rho^{2l+2}}{(1+\rho^2)^{2n-2}}\tag{36}$$

$$\times\,[G_l(l+1-n;\,\eta;\,\rho)-(1+\rho^2)^{-2}G_l(l-1-n;\,\eta;\,\rho)]^2$$

(here the first expression in braces is set equal to 1 if $l=1$);

$$\sigma_{nl\to E,\,l+1}=\frac{\pi e^2}{m\,cv}\frac{2^{4l+6}}{3}\frac{(l+1)^2\,(n+l)\,!\,(1^2+\eta^2)(2^2+\eta^2)\dots[(l+1)^2+\eta^2]}{(2l+1)(2l+1)\,!\,(2l+2)\,!\,(n-l-1)\,!\,[(l+1)^2+\eta^2]^2}$$

$$\times\frac{\exp[-4\eta\cot^{-1}\rho]}{1-e^{-2\pi\eta}}\frac{\rho^{2l+4}\eta^2}{(1+\rho^2)^{2n}}\tag{37}$$

$$\times\Big[(l+1-n)\,G_{l+1}(l+1-n;\,\eta;\,\rho)+\frac{l+1+n}{1+\rho^2}G_{l+1}(l-n;\,\eta;\,\rho)\Big]^2.$$

Here $\rho=\eta/n$, and G_l and G_{l+1} are real polynomials in ρ, described in Appendix C.

The total cross-section is given by

$$\sigma_{nl\to E}=\sigma_{nl\to E,\,l-1}+\sigma_{nl\to E,\,l+1}\,.\tag{38}$$

Kramers' semiclassical expression for the bound-free cross-section is

$$\sigma_{nl\to E}^{K}=\frac{2^4}{3\sqrt{3}}\frac{e^2}{m\,cv}\frac{1}{n}\left(\frac{\rho^2}{1+\rho^2}\right)^2.\tag{39}$$

The Gaunt factor is defined as the ratio of the cross-section to Kramers' cross-section (Gaunt 1930):

$$g_{bf}=\frac{\sigma_{bf}}{\sigma_{bf}^{K}}.\tag{40}$$

These dimensionless Gaunt factors are most commonly used in opacity calculations. Values for the Gaunt factors from bound states $n=1$ to $n=6$ (and all pertinent l) and bound states $n=1$ to $n=15$ (averaged over l) to free states covering $E/Z^2\,Ry$ from zero to 10^{12} are given in Table 1 and plotted in Figures 7–21.

TABLE 1

BOUND-FREE GAUNT FACTORS

$n = 1, n = 2$

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	Level, 1s	Photon Energy, $h\nu/Z^2Ry$	Level 2s	Level 2p	$\bar{2}$
	$n = 1$		$n = 2$			
.1000 (13)*	.1000 (13)	.6928 (−5)	.1000 (13)	.2771 (−4)	.6928 (−17)	.6128 (−5)
.1111 (12)	.1111 (12)	.2078 (−4)	.1111 (12)	.8314	.1871 (−15)	.2078 (−4)
.1000 (11)	.1000 (11)	.6928	.1000 (11)	.2771 (−3)	.6928 (−14)	.6928
1111 (10)	.1111 (10)	.2078 (−3)	.1111 (10)	.8313	.1870 (−12)	.2078 (−3)
1000 (9)	.1000 (9)	.6926	.1000 (9)	.2770 (−2)	.6926 (−11)	.6926
.1111 (8)	.1111 (8)	.2076 (−2)	.1111 (8)	.8306	.1869 (−9)	.2076 (−2)
.1000 (7)	.1000 (7)	.6906	.1000 (7)	.2763 (−1)	.6906 (−8)	.6906
.1111 (6)	.1111 (6)	.2059 (−1)	.1111 (6)	.8236	.1853 (−6)	.2059 (−1)
.4000 (5)	.4000 (5)	.3410	.4000 (5)	.1364 (0)	.8525	.3410
.2041	.2041	.4745	.2041	.1898	.2325 (−5)	.4745
.1000	.1000	.6715	.1000	.2686	.6714	.6715
.4444 (4)	.4445 (4)	.9917	.4445 (4)	.3966	.2231 (−4)	.9918
.2500	.2501	.1302 (0)	.2500	.5207	.5206	.1302 (0)
.1111	.1112	.1894	.1111	.7572	.1703 (−3)	.1894
.4000 (3)	.4010 (3)	.2971	.4002 (3)	.1187 (1)	.7411	.2972
.2041	.2051	.3918	.2043	.1563	.1912 (−2)	.3923
.1000	.1010	.5129	.1002	.2042	.5087	.5142
.4444 (2)	.4544 (2)	.6687	.4469 (2)	.2645	.1477 (−1)	.6724
.2500	.2600	.7800	.2525	.3059	.3019	.7875
.1600	.1700	.8585	.1625	.3331	.5100	.8711
.1111	.1211	.9129	.1136	.3497	.7642	.9316
.6250 (1)	.7250 (1)	.9729	.6500 (1)	.3612	.1373 (0)	.1006 (1)
.4000	.5000	.9939	.4250	.3553	.2055	.1043
.2778	.3778	.9948	.3028	.3408	.2752	.1058
.2041	.3041	.9856	.2291	.3225	.3423	.1063
.1562	.2562	.9721	.1812	.3031	.4045	.1061
.1235	.2235	.9571	.1485	.2842	.4607	.1056
.1000	.2000	.9423	.1250	.2664	.5106	.1049
.6944 (0)	.1694	.9157	.9444 (0)	.2354	.5925	.1033
.4444	.1444	.8850	.6944	.2000	.6784	.1009
.2500	.1250	.8531	.5000	.1626	.7587	.9755 (0)
.1111	.1111	.8246	.3611	.1279	.8192	.9342
.4000 (−1)	.1040	.8076	.2900	.1067	.8459	.9011
.2041	.1020	.8026	.2704	.1003	.8518	.8896
.1000	.1010	.7999	.2600	.9686 (0)	.8545	.8830
.4444 (−2)	.1004	.7985	.2541	.9498	.8558	.8793
.2500	.1002	.7980	.2525	.9431	.8562	.8779
.1111	.1001	.7976	.2511	.9384	.8565	.8770
.1000 (−3)	.1000	.7973	.2501	.9349	.8567	.8763
.1111 (−4)	.1000	.7973	.2500	.9346	.8567	.8761
.1000 (−5)	.1000	.7973	.2500	.9346	.8567	.8761
.1111 (−6)	.1000	.7973	.2500	.9346	.8567	.8761
.1000 (−7)	.1000	.7973	.2500	.9346	.8567	.8761
.1111 (−8)	.1000	.7973	.2500	.9346	.8567	.8761
.1000 (−9)	.1000	.7973	.2500	.9346	.8567	.8761
.1111 (−10)	.1000	.7973	.2500	.9346	.8567	.8761
.1000 (−11)	.1000	.7973	.2500	.9346	.8567	.8761
.1111 (−12)	.1000	.7973	.2500	.9346	.8567	.8761
.1000 (−13)	.1000	.7973	.2500	.9346	.8567	.8761
.1111 (−14)	.1000	.7973	.2500	.9346	.8567	.8761
.1000 (−15)	.1000	.7973	.2500	.9346	.8567	.8761

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 1—continued

$n = 3$

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	Level 3s	3p	3d	$\bar{3}$
.1000 (13)*	.1000 (13)	.6235 (−4)	.1848 (−16)	.1369 (−29)	.6928 (−5)
.1111 (12)	.1111 (12)	.1871 (−3)	.4988 (−15)	.3326 (−27)	.2078 (−4)
.1000 (11)	.1000 (11)	.6235	.1847 (−13)	.1368 (−24)	.6928
.1111 (10)	.1111 (10)	.1870 (−2)	.4988 (−12)	.3325 (−22)	.2078 (−3)
.1000 (9)	.1000 (9)	.6233	.1847 (−10)	.1368 (−19)	.6926
.1111 (8)	.1111 (8)	.1869 (−1)	.4984 (−9)	.3322 (−17)	.2076 (−2)
.1000 (7)	.1000 (7)	.6216	.1842 (−7)	.1364 (−14)	.6906
.1111 (6)	.1111 (6)	.1853 (0)	.4942 (−6)	.3294 (−12)	.2059 (−1)
.4000 (5)	.4000 (5)	.3069	.2273 (−5)	.4210 (−11)	.3410
.2041	.2041	.4270	.6199	.2250 (−10)	.4745
.1000	.1000	.6043	.1790 (−4)	.1326 (−9)	.6715
.4444 (4)	.4445 (4)	.8924	.5949	.9915	.9918
.2500	.2500	.1171 (1)	.1388 (−3)	.4113 (−8)	.1302 (0)
.1111	.1111	.1704	.4541	.3027 (−7)	.1894
.4000 (3)	.4001 (3)	.2670	.1976 (−2)	.3658 (−6)	.2973
.2041	.2042	.3516	.5097	.1849 (−5)	.3924
.1000	.1001	.4589	.1355 (−1)	.1003 (−4)	.5144
.4444 (2)	.4456 (2)	.5939	.3930	.6542	.6731
.2500	.2511	.6858	.8022	.2372 (−3)	.7889
.1600	.1611	.7453	.1352 (0)	.6239	.8735
.1111	.1122	.7804	2021	.1340 (−2)	.9352
.6250 (1)	.6361 (1)	.8009	.3608	.4236	.1013 (1)
.4000	.4111	.7818	.5357	.9765	.1053
.2778	.2889	.7428	.7101	.1849 (−1)	.1072
.2041	.2152	.6954	.8732	.3064	.1081
.1562	.1673	.6458	.1019 (1)	.4611	.1083
.1235	.1346	.5976	.1144	.6460	.1081
.1000	.1111	.5524	.1249	.8563	.1078
.6944 (0)	.8056 (0)	.4736	.1402	.1332 (0)	.1068
.4444	.5556	.3834	.1523	.2107	.1051
.2500	.3611	.2875	.1559	.3332	.1024
.1111	.2222	.1976	.1454	.5059	.9854 (0)
.4000 (−1)	.1511	.1416	.1277	.6531	.9458
.2041	.1315	.1246	.1199	.7042	.9293
.1000	.1211	.1153	.1150	.7333	.9189
.4444 (−2)	.1156	.1102	.1122	.7495	.9128
.2500	.1136	.1085	.1111	.7552	.9105
.1111	.1122	.1072	.1104	.7593	.9089
.1000 (−3)	.1112	.1062	.1098	.7623	.9076
.1111 (−4)	.1111	.1062	.1098	.7626	.9075
.1000 (−5)	.1111	.1062	.1098	.7626	.9075
.1111 (−6)	.1111	.1062	.1098	.7626	.9075
.1000 (−7)	.1111	.1062	.1098	.7626	.9075
.1111 (−8)	.1111	.1062	.1098	.7626	.9075
.1000 (−9)	.1111	.1062	.1098	.7626	.9075
.1111 (−10)	.1111	.1062	.1098	.7626	.9075
.1000 (−11)	.1111	.1062	.1098	.7626	.9075
.1111 (−12)	.1111	.1062	.1098	.7626	.9075
.1000 (−13)	.1111	.1062	.1098	.7626	.9075
.1111 (−14)	.1111	.1062	.1098	.7626	.9075
.1000 (−15)	.1111	.1062	.1098	.7626	.9075

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 1—continued

$n = 4$

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	Level				$\bar{4}$
		4s	4p	4d	4f	
.1000 (13)*	.1000 (13)	.1109 (−3)	.3464 (−16)	.3464 (−29)		.6928 (−5)
.1111 (12)	.1111 (12)	.3326	.9353 (−15)	..8418 (−27)		.2078 (−4)
.1000 (11)	.1000 (11)	.1108 (−2)	.3464 (−13)	.3464 (−24)	.1155 (−35)	.6928
.1111 (10)	.1111 (10)	.3325	.9352 (−12)	.8417 (−22)	.2525 (−32)	.2078 (−3)
.1000 (9)	.1000 (9)	.1108 (−1)	.3463 (−10)	.3463 (−19)	.1154 (−28)	.6926
.1111 (8)	.1111 (8)	.3322	.9344 (−9)	.8410 (−17)	.2523 (−25)	.2076 (−2)
.1000 (7)	.1000 (7)	.1105 (0)	.3453 (−7)	.3453 (−14)	.1151 (−21)	.6906
.1111 (6)	.1111 (6)	.3294	.9265 (−6)	.8339 (−12)	.2502 (−18)	.2059 (−1)
.4000 (5)	.4000 (5)	.5456	.4263 (−5)	.1066 (−10)	.8881 (−17)	.3410
.2041	.2041	.7591	.1162 (−4)	.5696	.9303 (−16)	.4745
.1000	.1000	.1074 (1)	.3357	.3357 (−9)	.1119 (−14)	.6715
.4444 (4)	.4444 (4)	.1586	.1115 (−3)	.2510 (−8)	.1882 (−13)	.9918
.2500	.2500	.2083	.2603	.1041 (−7)	.1388 (−12)	.1302 (0)
.1111	.1111	.3028	.8515	.7663	.2299 (−11)	.1894
.4000 (3)	.4001 (3)	.4746	.3704 (−2)	.9260 (−6)	.7715 (−10)	.2973
.2041	.2042	.6250	.9555	.4681 (−5)	.7643 (−9)	.3924
.1000	.1001	.8156	.2540 (−1)	.2539 (−4)	.8457 (−8)	.5145
.4444 (2)	.4451 (2)	.1055 (2)	.7364	.1655 (−3)	.1239 (−6)	.6733
.2500	.2506	.1218	.1502 (0)	.5994	.7973	.7894
.1600	.1606	.1322	.2530	.1575 (−2)	.3270 (−5)	.8745
.1111	.1117	.1383	.3778	.3381	.1009 (−4)	.9365
.6250 (1)	.6312 (1)	.1417	.6730	.1066 (−1)	.5629	.1015 (1)
.4000	.4062	.1379	.9962	.2448	.2010 (−3)	.1056
.2778	.2840	.1306	.1316 (1)	.4617	.5422	.1077
.2041	.2103	.1217	.1611	.7613	.1207 (−2)	.1087
.1562	.1625	.1125	.1871	.1140 (0)	.2340	.1091
.1235	.1297	.1036	.2090	.1587	.4083	.1091
.1000	.1062	.9526 (1)	.2267	.2090	.6566	.1089
.6944 (0)	.7569 (0)	.8069	.2512	.3204	.1413 (−1)	.1082
.4444	.5069	.6397	.2667	.4945	.3258	.1069
.2500	.3125	.4614	.2617	.7469	.7987	.1047
.1111	.1736	.2929	.2240	.1035 (1)	.1992 (0)	.1014
.4000 (−1)	.1025	.1865	.1734	.1169	.3808	.9736 (0)
.2041	.8291 (−1)	.1539	.1521	.1173	.4721	.9542
.1000	.7250	.1358	.1388	.1159	.5335	.9408
.4444 (−2)	.6694	.1259	.1311	.1146	.5708	.9323
.2900	.6500	.1225	.1283	.1140	.5848	.9291
.1111	.6361	.1200	.1263	.1135	.5950	.9267
.1000 (−3)	.6260	.1181	.1248	.1131	.6026	.9249
.1111 (−4)	.6251	.1180	.1246	.1131	.6033	.9248
.1000 (−5)	.6250	.1179	.1246	.1131	.6034	.9247
.1111 (−6)	.6250	.1179	.1246	.1131	.6034	.9247
.1000 (−7)	.6250	.1179	.1246	.1131	.6034	.9247
.1111 (−8)	.6250	.1179	.1246	.1131	.6034	.9247
.1000 (−9)	.6250	.1179	.1246	.1131	.6034	.9247
.1111 (−10)	.6250	.1179	.1246	.1131	.6034	.9247
.1000 (−11)	.6250	.1179	.1246	.1131	.6034	.9247
.1111 (−12)	.6250	.1179	.1246	.1131	.6034	.9247
.1000 (−13)	.6250	.1179	.1246	.1131	.6034	.9247
.1111 (−14)	.6250	.1179	.1246	.1131	.6034	.9247
.1000 (−15)	.6250	.1179	.1246	.1131	.6034	.9247

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 1—continued

$n = 5$

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	Level					$\bar{5}$
		$5s$	$5p$	$5d$	$5f$	$5g$	
.1000 (13)*	.1000 (13)	.1732 (−3)	.5543 (−16)	.6208 (−29)			.6928 (−5)
.1111 (12)	.1111 (12)	.5196	.1496 (−14)	.1508 (−26)			.2078 (−4)
.1000 (11)	.1000 (11)	.1732 (−2)	.5542 (−13)	.6207 (−24)	.3027 (−35)		.6928
.1111 (10)	.1111 (10)	.5196	.1496 (−11)	.1508 (−21)	.6619 (−32)		.2078 (−3)
.1000 (9)	.1000 (9)	.1732 (−1)	.5541 (−10)	.6206 (−19)	.3026 (−28)		.6926
.1111 (8)	.1111 (8)	.5191	.1495 (−8)	.1507 (−16)	.6614 (−25)	.1020 (−33)	.2076 (−2)
.1000 (7)	.1000 (7)	.1727 (0)	.5525 (−7)	.6188 (−14)	.3017 (−21)	.5173 (−29)	.6906
.1111 (6)	.1111 (6)	.5147	.1482 (−5)	.1494 (−11)	.6558 (−18)	.1012 (−24)	.2059 (−1)
.4000 (5)	.4000 (5)	.8525	.6820	.1910 (−10)	.2328 (−16)	.9977 (−23)	.3410
.2041	.2041	.1186 (1)	.1860 (−4)	.1021 (−9)	.2439 (−15)	.2048 (−21)	.4745
.1000	.1000	.1679	.5371	.6016	.2933 (−14)	.5029 (−20)	.6715
.4444 (4)	.4444 (4)	.2479	.1785 (−3)	.4497 (−8)	.4934 (−13)	.1903 (−18)	.9918
.2500	.2500	.3254	.4165	.1866 (−7)	.3639 (−12)	.2495 (−17)	.1302 (0)
.1111	.1111	.4732	.1362 (−2)	.1373 (−6)	.6026 (−11)	.9297 (−16)	.1894
.4000 (3)	.4000 (3)	.7415	.5927	.1659 (−5)	.2022 (−9)	.8666 (−14)	.2973
.2041	.2041	.9765	.1529 (−1)	.8387	.2003 (−8)	.1682 (−12)	.3924
.1000	.1000	.1274 (2)	.4064	.4549 (−4)	.2217 (−7)	.3798 (−11)	.5146
.4444 (2)	.4448 (2)	.1648	.1178 (0)	.2964 (−3)	.3247 (−6)	.1251 (−9)	.6734
.2500	.2504	.1902	.2402	.1073 (−2)	.2088 (−5)	.1430 (−8)	.7897
.1600	.1604	.2064	.4045	.2820	.8561	.9149	.8747
.1111	.1115	.2158	.6037	.6050	.2640 (−4)	.4059 (−7)	.9371
.6250 (1)	.6290 (1)	.2208	.1074 (1)	.1904 (−1)	.1471 (−3)	.4009 (−6)	.1016 (1)
.4000	.4040	.2146	.1588	.4368	.5243	.2225 (−5)	.1058
.2778	.2818	.2029	.2094	.8221	.1412 (−2)	.8588	.1080
.2041	.2081	.1888	.2559	.1352 (0)	.3136	.2584 (−4)	.1090
.1562	.1602	.1742	.2964	.2019	.6061	.6484	.1095
.1235	.1275	.1600	.3302	.2804	.1054 (−1)	.1418 (−3)	.1095
.1000	.1040	.1467	.3573	.3681	.1690	.2785	.1094
.6944 (0)	.7344	.1235	.3933	.5603	.3610	.8425	.1089
.4444	.4844	.9689 (1)	.4127	.8539	.8212	.2906 (−2)	.1078
.2500	.2900	.6845	.3960	.1258 (1)	.1961 (0)	.1159 (−1)	.1060
.1111	.1511	.4145	.3220	.1650	.4616	.5234	.1030
.4000 (−1)	.8000 (−1)	.2422	.2275	.1690	.7969	.1706 (0)	.9925 (0)
.2041	.6041	.1887	.1878	.1590	.9242	.2618	.9719
.1000	.5000	.1588	.1631	.1492	.9890	.3383	.9563
.4444 (−2)	.4444	.1424	.1488	.1421	.1018 (1)	.3918	.9458
.2500	.4250	.1366	.1435	.1392	.1027	.4132	.9416
.1111	.4111	.1324	.1397	.1370	.1033	.4295	.9385
.1000 (−3)	.4010	.1293	.1369	.1354	.1036	.4418	.9361
.1111 (−4)	.4001	.1290	.1366	.1353	.1037	.4429	.9358
.1000 (−5)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1111 (−6)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1000 (−7)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1111 (−8)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1000 (−9)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1111 (−10)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1000 (−11)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1111 (−12)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1000 (−13)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1111 (−14)	.4000	.1290	.1366	.1352	.1037	.4431	.9358
.1000 (−15)	.4000	.1290	.1366	.1352	.1037	.4431	.9358

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 1—continued

$n = 6$

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	Level						$\bar{6}$
		$6s$	$6p$	$6d$	$6f$	$6g$	$6h$	
.1000 (13)*	.1000 (13)	.2494 (−3)	.8083 (−16)	.9580 (−29)				.6928 (−5)
.1111 (12)	.1111 (12)	.7482	.2182 (−14)	.2328 (−26)				.2078 (−4)
.1000 (11)	.1000 (11)	.2494 (−2)	.8083 (−13)	.9579 (−24)	.5474 (−35)			.6928
.1111 (10)	.1111 (10)	.7482	.2182 (−11)	.2328 (−21)	.1197 (−31)			.2078 (−3)
.1000 (9)	.1000 (9)	.2493 (−1)	.8080 (−10)	.9577 (−19)	.5472 (−28)			.6926
.1111 (8)	.1111 (8)	.7475	.2180 (−8)	.2326 (−16)	.1196 (−24)	.2848 (−33)		.2076 (−2)
.1000 (7)	.1000 (7)	.2486 (0)	.8058 (−7)	.9550 (−14)	.5457 (−21)	.1444 (−28)		.6906
.1111 (6)	.1111 (6)	.7412	.2162 (−5)	.2306 (−11)	.1186 (−17)	.2824 (−24)	.2510 (−31)	.2059 (−1)
.4000 (5)	.4000 (5)	.1228 (1)	.9946	.2947 (−10)	.4210 (−16)	.2784 (−22)	.6875 (−29)	.3410
.2041	.2041	.1708	.2712 (−4)	.1575 (−9)	.4410 (−15)	.5717 (−21)	.2767 (−27)	.4745
.1000	.1000	.2417	.7833	.9284	.5305 (−14)	.1403 (−19)	.1386 (−25)	.6715
.4444 (4)	.4444 (4)	.3570	.2603 (−3)	.6940 (−8)	.8923 (−13)	.5311 (−18)	.1180 (−23)	.9918
.2500	.2500	.4686	.6073	.2879 (−7)	.6581 (−12)	.6964 (−17)	.2751 (−22)	.1302 (0)
.1111	.1111	.6814	.1987 (−2)	.2119 (−6)	.1090 (−10)	.2595 (−15)	.2306 (−20)	.1894
.4000 (3)	.4000 (3)	.1068 (2)	.8643	.2560 (−5)	.3657 (−9)	.2419 (−13)	.5971 (−18)	.2973
.2041	.2041	.1406	.2228 (−1)	.1294 (−4)	.3623 (−8)	.4695 (−12)	.2272 (−16)	.3924
.1000	.1000	.1835	.5926	.7019	.4008 (−7)	.1060 (−10)	.1047 (−14)	.5146
.4444 (2)	.4447 (2)	.2373	.1717 (0)	.4573 (−3)	.5871 (−6)	.3492 (−9)	.7754 (−13)	.6735
.2500	.2503	.2737	.3502	.1656 (−2)	.3775 (−5)	.3989 (−8)	.1574 (−11)	.7898
.1600	.1603	.2971	.5895	.4349	.1547 (−4)	.2552 (−7)	.1572 (−10)	.8750
.1111	.1114	.3106	.8798	.9328	.4770	.1132 (−6)	.1003 (−9)	.9374
.6250 (1)	.6278 (1)	.3176	.1564 (1)	.2934 (−1)	.2656 (−3)	.1117 (−5)	.1757 (−8)	.1017 (1)
.4000	.4028	.3085	.2311	.6724	.9458	.6880	.1518 (−7)	.1059
.2778	.2806	.2913	.3045	.1264 (0)	.2543 (−2)	.2387 (−4)	.8404	.1081
.2041	.2069	.2709	.3717	.2077	.5643	.7172	.3424 (−6)	.1092
.1562	.1590	.2496	.4300	.3097	.1089 (−1)	.1797 (−3)	.1116 (−5)	.1097
.1235	.1263	.2289	.4783	.4294	.1891	.3924	.8070	.1098
.1000	.1028	.2096	.5167	.5627	.3025	.7691	.7391	.1097
.6944 (0)	.7222 (0)	.1759	.5667	.8531	.6434	.2316 (−2)	.3167 (−4)	.1092
.4444	.4722	.1371	.5908	.1291 (1)	.1453 (0)	.7926	.1657 (−3)	.1083
.2500	.2778	.9570 (1)	.5593	.1875	.3417	.3113 (−1)	.1106 (−2)	.1067
.1111	.1389	.5626	.4406	.2374	.7753	.1353 (0)	.9618	.1041
.4000 (−1)	.6778 (−1)	.3091	.2916	.2267	.1243 (1)	.4086	.5947 (−1)	.1006
.2041	.4819	.2295	.2288	.2021	.1362	.5918	.1211 (0)	.9851 (0)
.1000	.3778	.1847	.1897	.1804	.1384	.7250	.1892	.9682
.4444 (−2)	.3222	.1599	.1667	.1653	.1369	.8057	.2464	.9560
.2500	.3028	.1510	.1584	.1594	.1357	.8348	.2717	.9509
.1111	.2889	.1446	.1522	.1549	.1346	.8556	.2918	.9470
.1000 (−3)	.2788	.1399	.1477	.1514	.1337	.8707	.3077	.9439
.1111 (−4)	.2779	.1395	.1473	.1511	.1336	.8720	.3091	.9436
.1000 (−5)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1111 (−6)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1000 (−7)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1111 (−8)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1000 (−9)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1111 (−10)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1000 (−11)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1111 (−12)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1000 (−13)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1111 (−14)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436
.1000 (−15)	.2778	.1395	.1473	.1511	.1336	.8722	.3093	.9436

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 1—continued

$n = 7, 8, 9, 10$

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	$\bar{7}$	Photon Energy, $h\nu/Z^2Ry$	$\bar{8}$	Photon Energy, $h\nu/Z^2Ry$	$\bar{9}$	Photon Energy, $h\nu/Z^2Ry$	$\overline{10}$
.1000 (13)*	.1000 (13)	.6928 (−5)	.1000 (13)	.6928 (−5)	.1000 (13)	.6928 (−5)	.1000 (13)	.6728 (−5)
.1111 (12)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)
.1000 (11)	.1000 (11)	.6928	.1000 (11)	.6928	.1000 (11)	.6928	.1000 (11)	.6928
.1111 (10)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)
.1000 (9)	.1000 (9)	.6926	.1000 (9)	.6926	.1000 (9)	.6926	.1000 (9)	.6926
.1111 (8)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)
.1000 (7)	.1000 (7)	.6906	.1000 (7)	.6906	.1000 (7)	.6906	.1000 (7)	.6906
.1111 (6)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)
.4000 (5)	.4000 (5)	.3410	.4000 (5)	.3410	.4000 (5)	.3410	.4000 (5)	.3410
.2041	.2041	.4745	.2041	.4745	.2041	.4745	.2041	.4745
.1000	.1000	.6715	.1000	.6715	.1000	.6715	.1000	.6715
.4444 (4)	.4444 (4)	.9918	.4444 (4)	.9918	.4444 (4)	.9918	.4444 (4)	.9918
.2500	.2500	.1302 (0)	.2500	.1302 (0)	.2500	.1302 (0)	.2500	.1302 (0)
.1111	.1111	.1894	.1111	.1894	.1111	.1894	.1111	.1894
.4000 (3)	.4000 (3)	.2973	.4000 (3)	.2973	.4000 (3)	.2973	.4000 (3)	.2973
.2041	.2041	.3924	.2041	.3925	.2041	.3925	.2041	.3925
.1000	.1000	.5146	.1000	.5146	.1000	.5146	.1000	.5146
.4444 (2)	.4446 (2)	.6735	.4446 (2)	.6735	.4446 (2)	.6736	.4445 (2)	.6736
.2500	.2502	.7899	.2502	.7899	.2501	.7899	.2501	.7900
.1600	.1602	.8751	.1602	.8752	.1601		.1601	.8753
.1111	.1113	.9376	.1113	.9377	.1112	.9378	.1112	.9379
.6250 (1)	.6270 (1)	.1017 (1)	.6266 (1)	.1017 (1)	.6262 (1)		.6260 (1)	.1018 (1)
.4000	.4020	.1060	.4016	.1060	.4012		.4010	.1060
.2778	.2798	.1082	.2794	.1082	.2790		.2788	.1083
.2041	.2061	.1093	.2057	.1094	.2053		.2051	.1095
.1562	.1582	.1098	.1578	.1099	.1575		.1572	.1100
.1235	.1255	.1099	.1251	.1100	.1247		.1245	.1102
.1000	.1020	.1099	.1016	.1100	.1012	.1101	.1010	.1101
.6944 (0)	.7149 (0)	.1095	.7101 (0)	.1096	.7068 (0)		.7044 (0)	.1098
.4444	.4649	.1086	.4601	.1088	.4568		.4544	.1091
.2500	.2704	.1071	.2656	.1074	.2623		.2600	.1078
.1111	.1315	.1047	.1267	.1052	.1235	.1056	.1211	.1058
.4000 (−1)	.6041 (−1)	.1015	.5562 (−1)	.1022	.5235 (−1)		.5000 (−1)	.1032
.2041	.4082	.9952 (0)	.3603	.1003	.3275		.3041	.1014
.1000	.3041	.9776	.2562	.9853 (0)	.2235	.9917 (0)	.2000	.9970 (0)
.4444 (−2)	.2485	.9641	.2007	.9708	.1679		.1444	.9814
.2500	.2291	.9582	.1812	.9642	.1485		.1250	.9737
.1111	.2152	.9535	.1674	.9588	.1346	.9632	.1111	.9666
.1000 (−3)	.2051	.9498	.1572	.9544	.1245	.9582	.1010	.9613
.1111 (−4)	.2042	.9495	.1564	.9540	.1236	.9577	.1001	.9607
.1000 (−5)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1111 (−6)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1000 (−7)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1111 (−8)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1000 (−9)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1111 (−10)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1000 (−11)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1111 (−12)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1000 (−13)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1111 (−14)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606
.1000 (−15)	.2041	.9494	.1563	.9540	.1235	.9576	.1000	.9606

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 1—continued

n = 11, 12, 13, 14, 15

Electron Energy, E/Z^2Ry	Photon Energy, $h\nu/Z^2Ry$	$\overline{11}$	Photon Energy, $h\nu/Z^2Ry$	$\overline{12}$	Photon Energy, $h\nu/Z^2Ry$	$\overline{13}$	Photon Energy, $h\nu/Z^2Ry$	$\overline{14}$	Photon Energy, $h\nu/Z^2Ry$	$\overline{15}$
.1000 (13)*	.1000 (13)	.6928 (−5)	.1000 (13)	.6928 (−5)	.1000 (13)	.6928 (−5)	.1000 (13)	.6928 (−5)	.1000 (13)	.6928 (−5)
.1111 (12)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)	.1111 (12)	.2078 (−4)
.1000 (11)	.1000 (11)	.6928	.1000 (11)	.6928	.1000 (11)	.6928	.1000 (11)	.6928	.1000 (11)	.6928
.1111 (10)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)	.1111 (10)	.2078 (−3)
.1000 (9)	.1000 (9)	.6926	.1000 (9)	.6926	.1000 (9)	.6926	.1000 (9)	.6926	.1000 (9)	.6926
.1111 (8)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)	.1111 (8)	.2076 (−2)
.1000 (7)	.1000 (7)	.6906	.1000 (7)	.6906	.1000 (7)	.6906	.1000 (7)	.6906	.1000 (7)	.6906
.1111 (6)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)	.1111 (6)	.2059 (−1)
.1000 (5)	.1000 (5)	.6715	.1000 (5)	.6715	.1000 (5)	.6715	.1000 (5)	.6715	.1000 (5)	.6715
.1111 (4)	.1111 (4)	.1894 (0)	.1111 (4)	.1894 (0)	.1111 (4)	.1894 (0)	.1111 (4)	.1894 (0)	.1111 (4)	.1894 (0)
.1000 (3)	.1000 (3)	.5146	.1000 (3)	.5146	.1000 (3)	.5146	.1000 (3)	.5146	.1000 (3)	.5146
.1111 (2)	.1112 (2)	.9379	.1112 (2)	.9379	.1112 (2)	.9380	.1112 (2)	.9380	.1112 (2)	.9380
.1000 (1)	.1008 (1)	.1102 (1)	.1007 (1)	.1102 (1)	.1006 (1)	.1102 (1)	.1005 (1)	.1103 (1)	.1004 (1)	.1103 (1)
.1111 (0)	.1194 (0)	.1060	.1181 (0)	.1062	.1170 (0)	.1063	.1162 (0)	.1064	.1156 (0)	.1064
.1000 (−1)	.1826 (−1)	.1002	.1694 (−1)	.1005	.1592 (−1)	.1008	.1510 (−1)	.1011	.1444 (−1)	.1014
.1111 (−2)	.9376 (−2)	.9703 (0)	.8056 (−2)	.9732 (0)	.7028 (−2)	.9758 (0)	.6213 (−2)	.9780 (0)	.5556 (−2)	.9803 (0)
.1000 (−3)	.8364	.9639	.7044	.9661	.6017	.9681	.5202	.9698	.4544	.9714
.1111 (−4)	.8276	.9633	.6956	.9654	.5928	.9673	.5113	.9690	.4456	.9704
.1000 (−5)	.8265	.9632	.6945	.9654	.5918	.9672	.5103	.9689	.4445	.9703
.1111 (−6)	.8265	.9632	.6945	.9653	.5917	.9672	.5102	.9689	.4445	.9703
.1000 (−7)	.8264	.9632	.6944	.9653	.5917	.9672	.5102	.9689	.4444	.9703
.1111 (−8)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1000 (−9)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1111 (−10)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1000 (−11)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1111 (−12)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1000 (−13)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1111 (−14)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703
.1000 (−15)	.8264	.9632	.6944	.9653	.5917	.9672	.5103	.9689	.4444	.9703

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

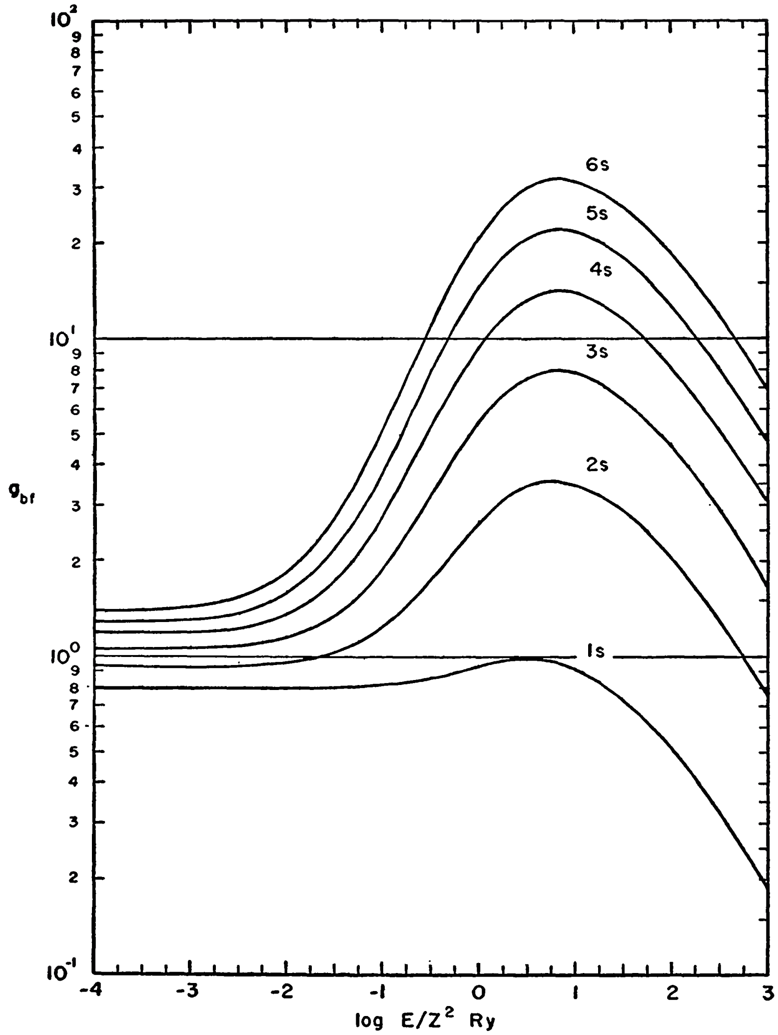


FIG. 7.—Bound-free Gaunt factor versus free electron energy for $l = 0$, $n = 1$–6

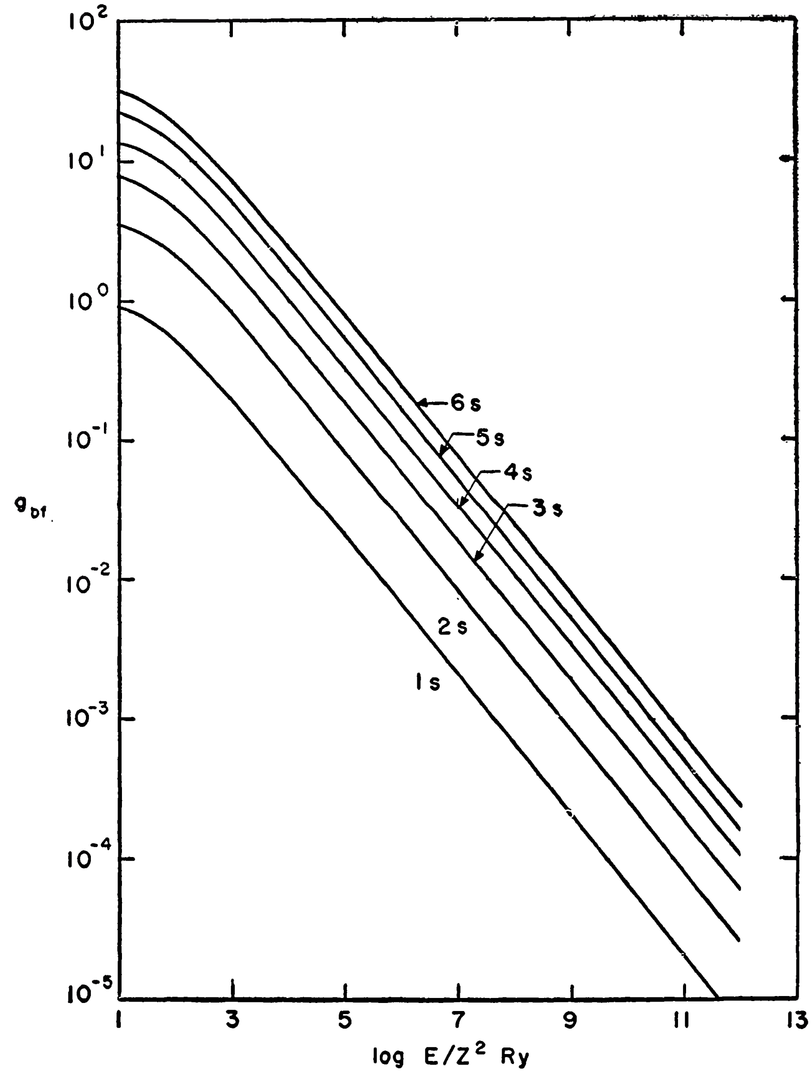


FIG. 8.—Bound-free Gaunt factor versus free electron energy for $l = 0$, $n = 1$–6

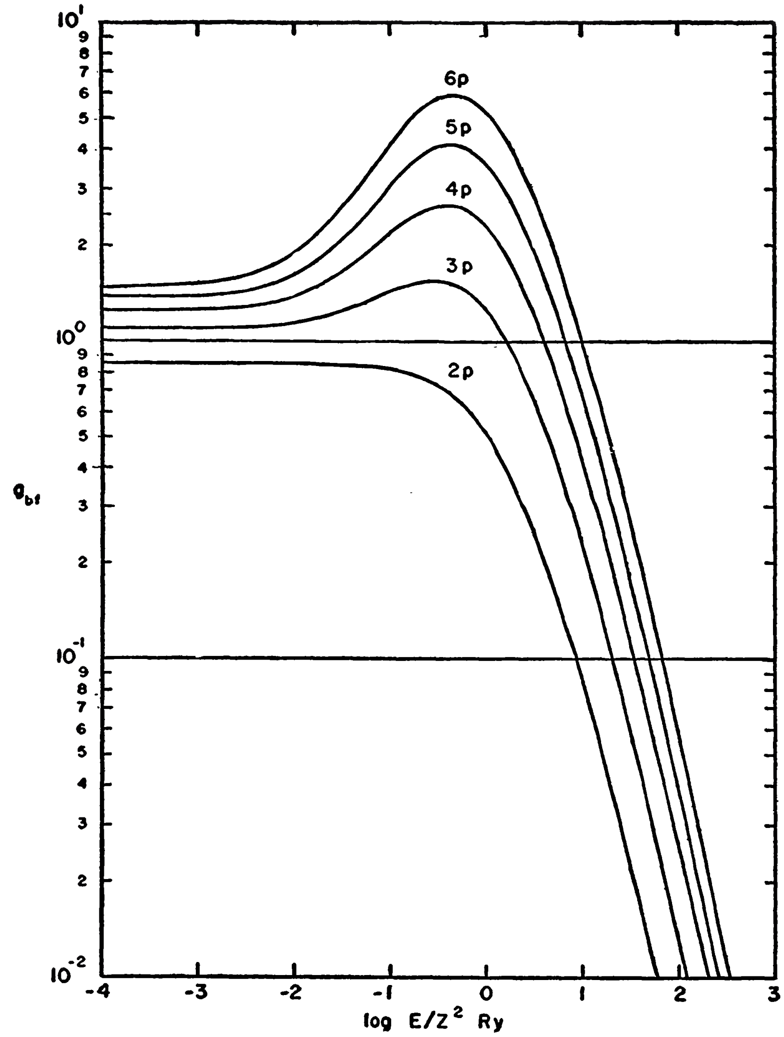


FIG. 9.—Bound-free Gaunt factor versus free electron energy for $l = 1$, $n = 2$–6

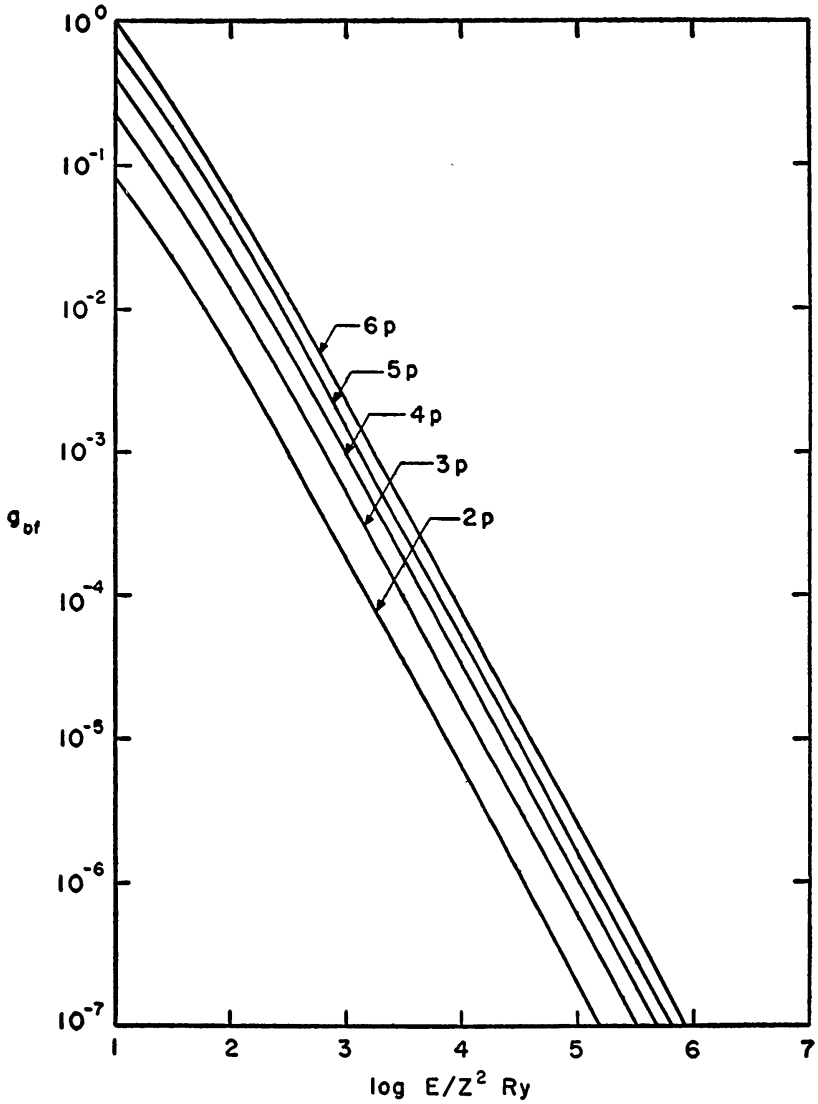


FIG. 10.—Bound-free Gaunt factor versus free electron energy for $l = 1$, $n = 2$–6

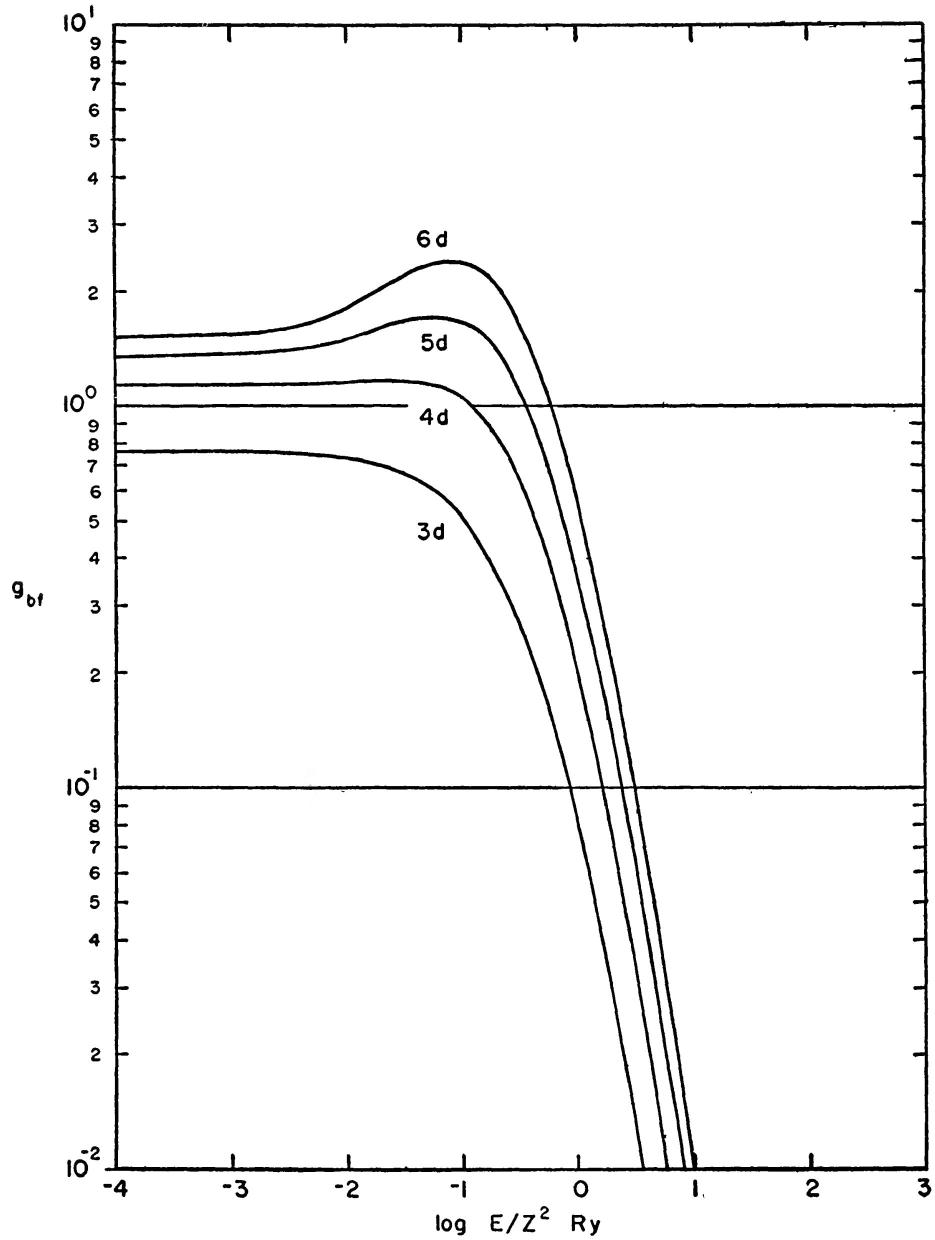


FIG. 11.—Bound-free Gaunt factor versus free electron energy for $l = 2$, $n = 3$–6

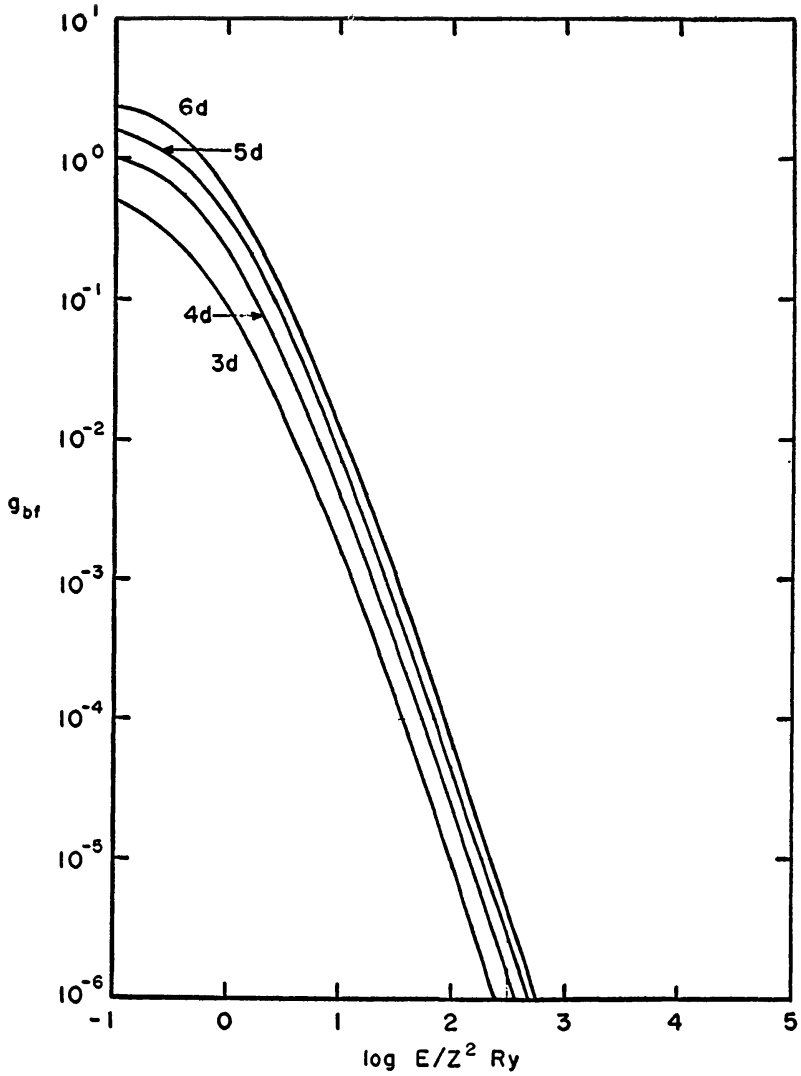


FIG. 12.—Bound-free Gaunt factor versus free electron energy for $l = 2$, $n = 3$–6

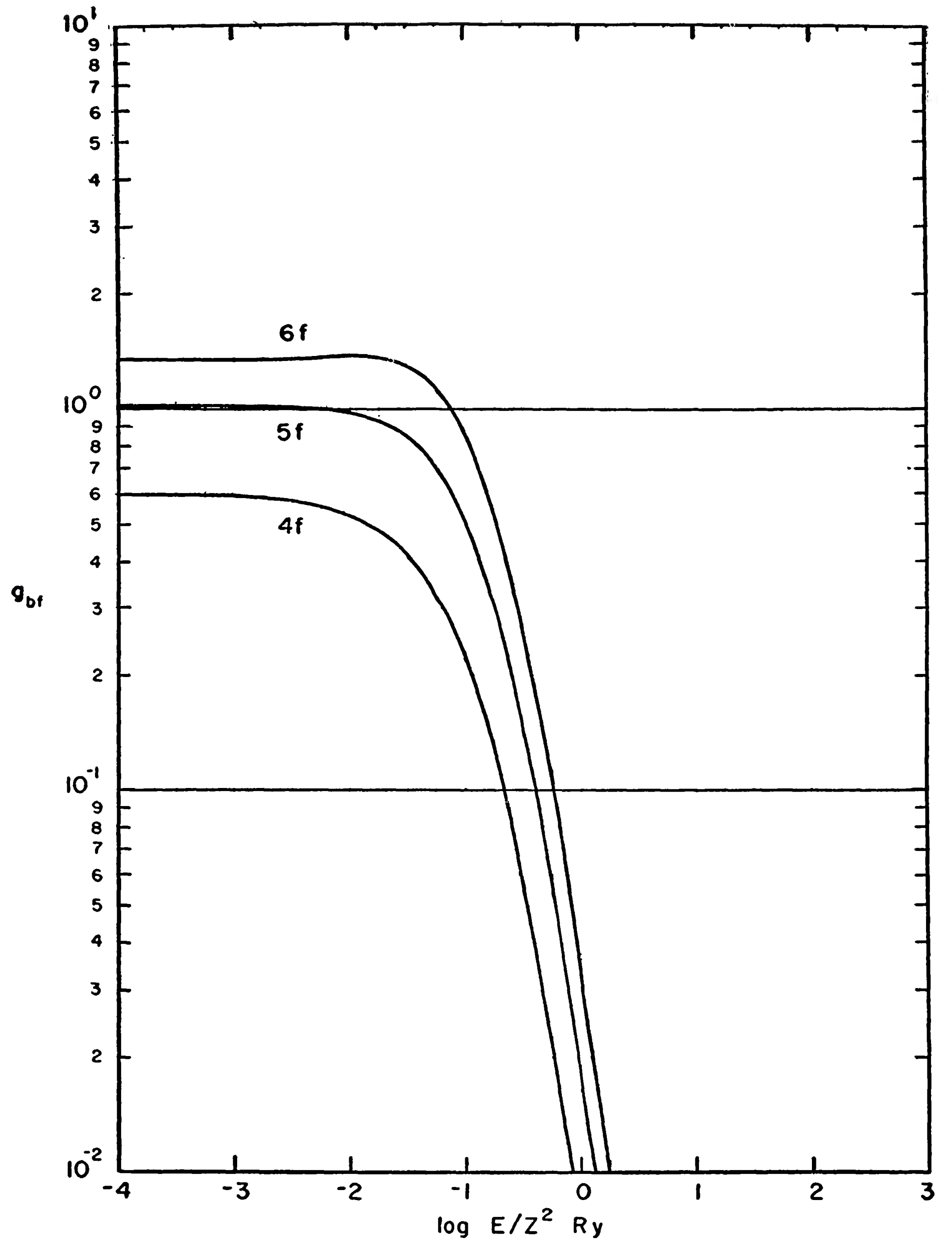


FIG. 13.—Bound-free Gaunt factor versus free electron energy for $l = 3$, $n = 4$–6

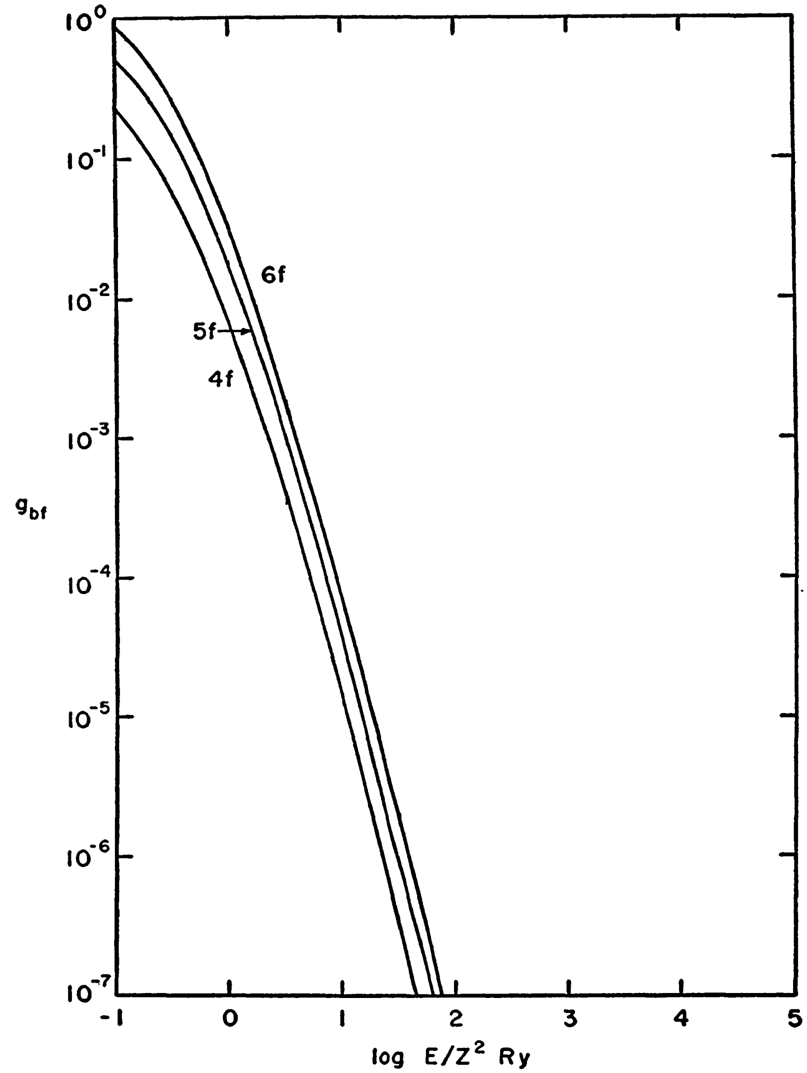


FIG. 14.—Bound-free Gaunt factor versus free electron energy for $l = 3$, $n = 4$–6

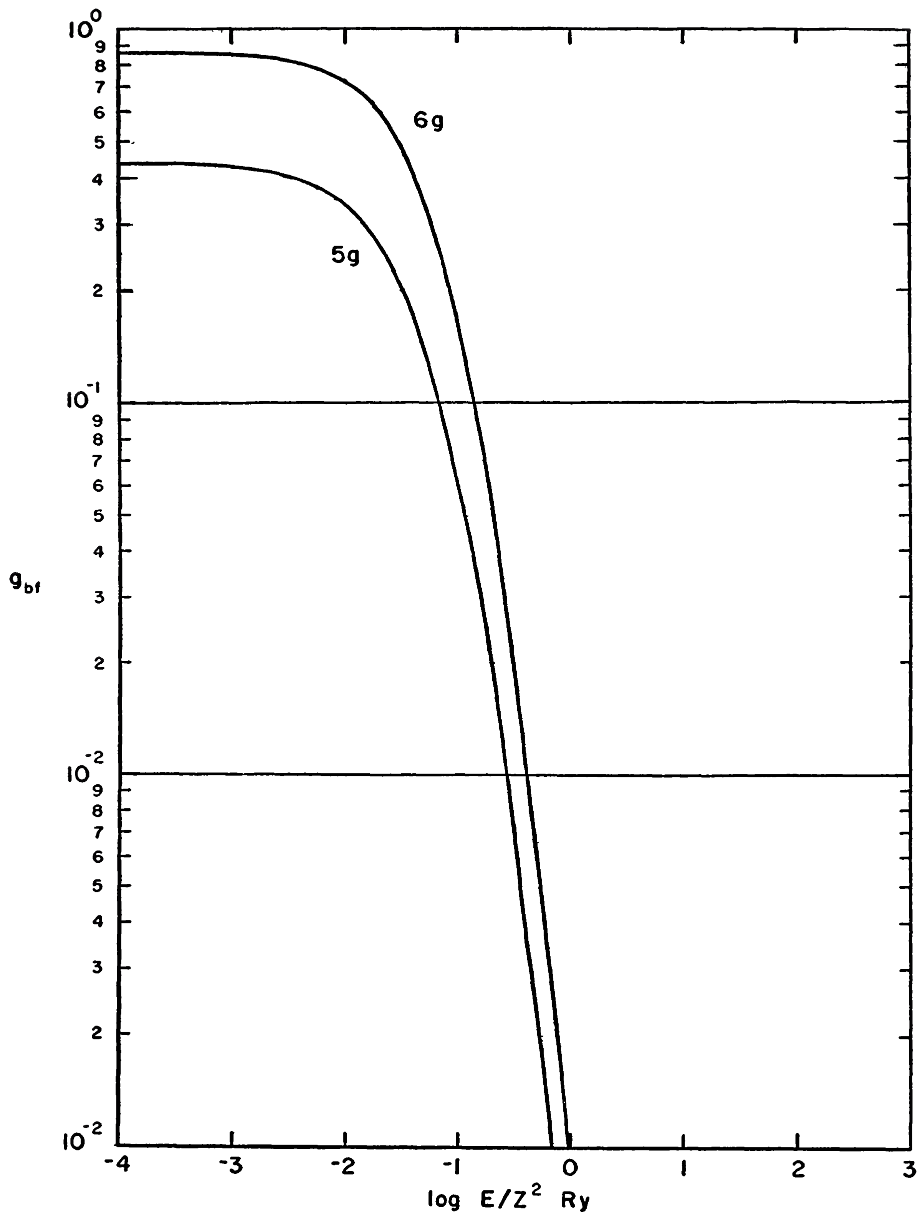


FIG. 15.—Bound-free Gaunt factor versus free electron energy for $l = 4$, $n = 5, 6$

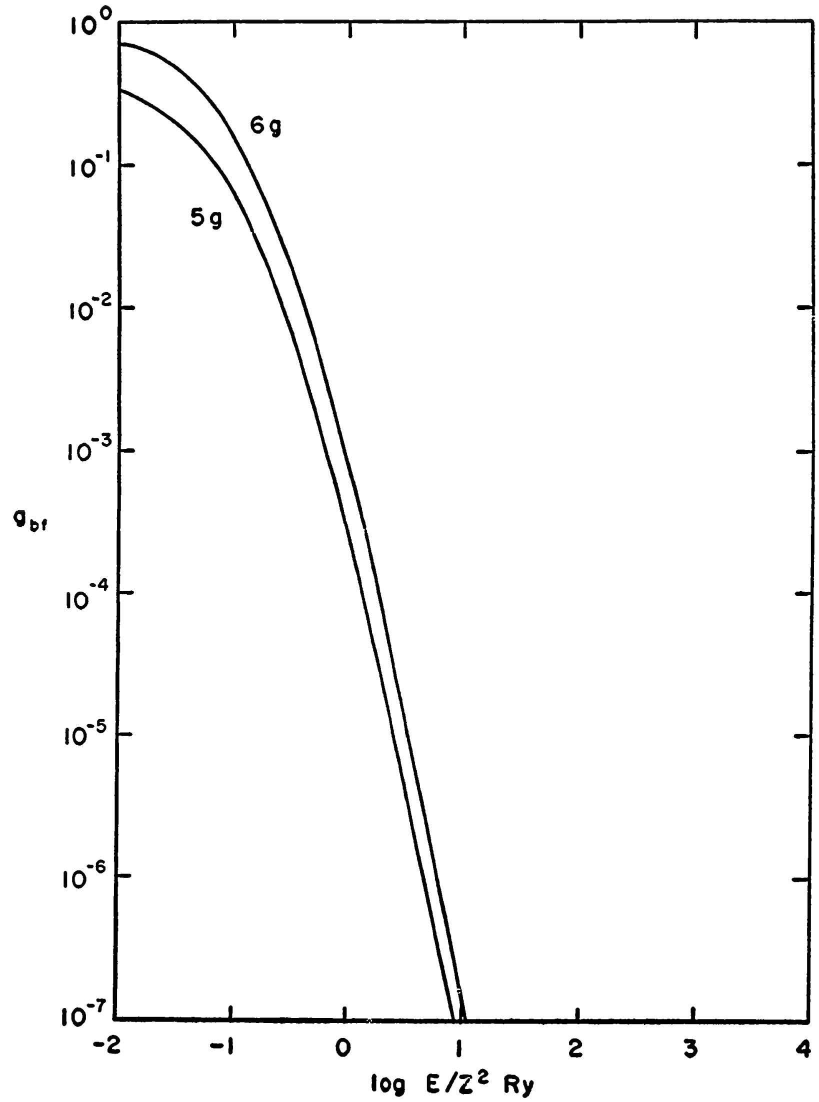


FIG. 16.—Bound-free Gaunt factor versus free electron energy for $l = 4$, $n = 5, 6$

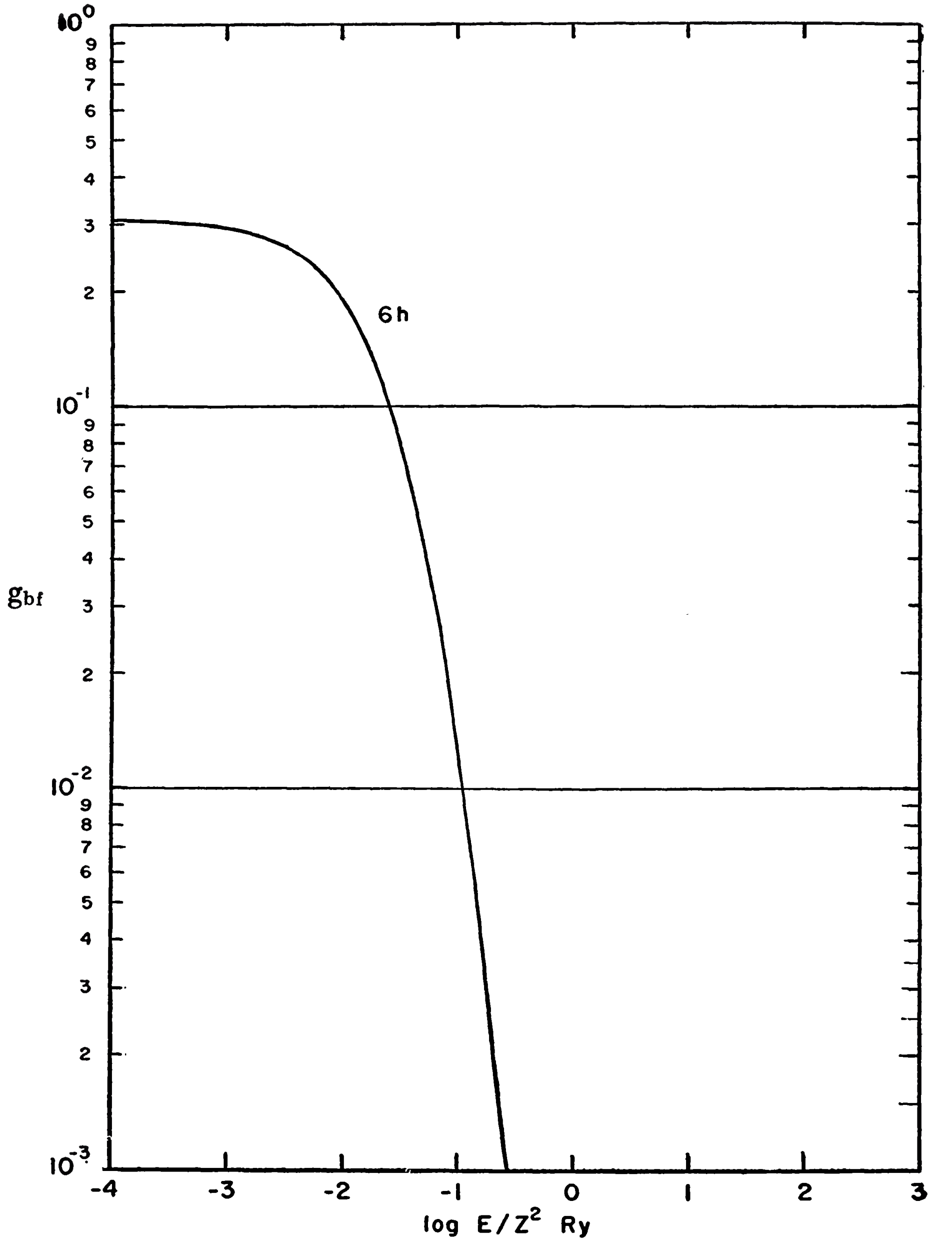


FIG. 17.—Bound-free Gaunt factor versus free electron energy for $l = 5$, $n = 6$

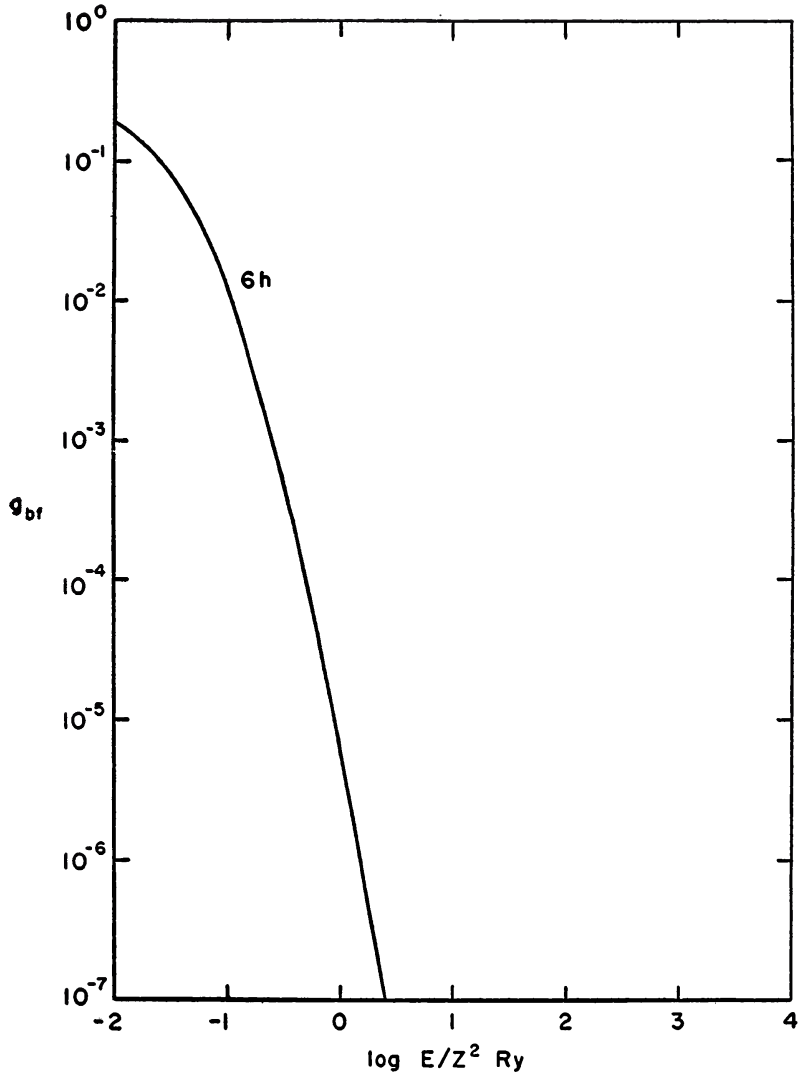


FIG. 18.—Bound-free Gaunt factor versus free electron energy for $l = 5$, $n = 6$

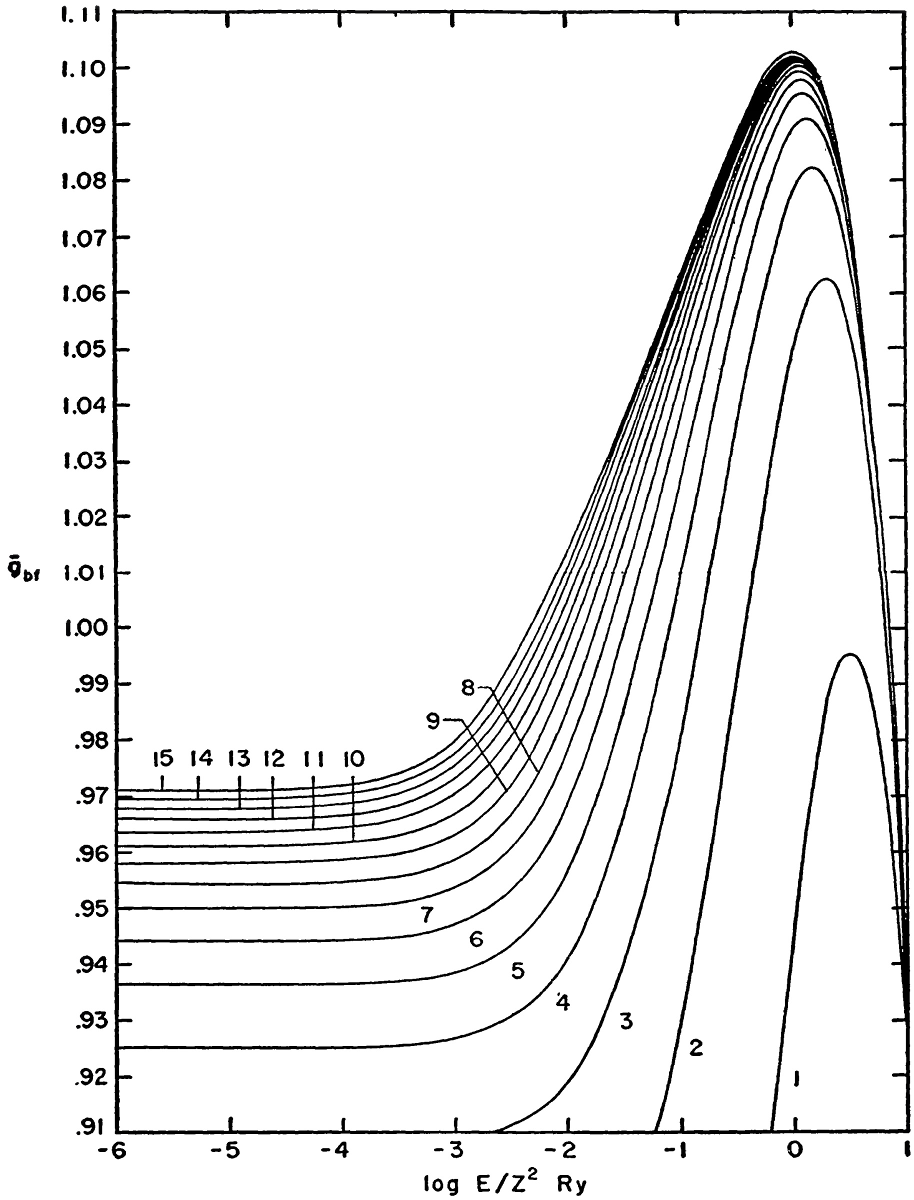


FIG. 19.—Bound-free Gaunt factor averaged over a shell versus free electron energy for n = 1–15

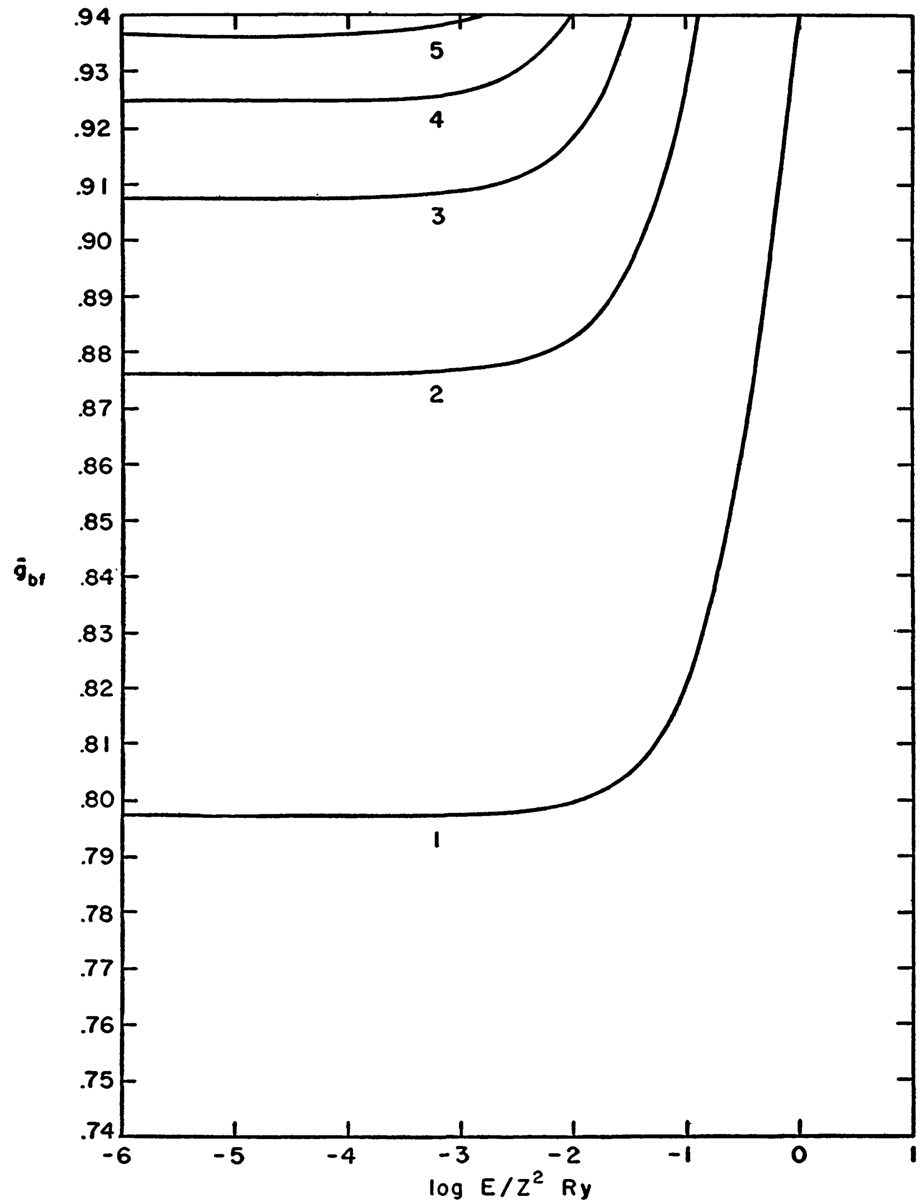


FIG. 20.—Bound-free Gaunt factor averaged over a shell versus free electron energy for $n = 1$–5

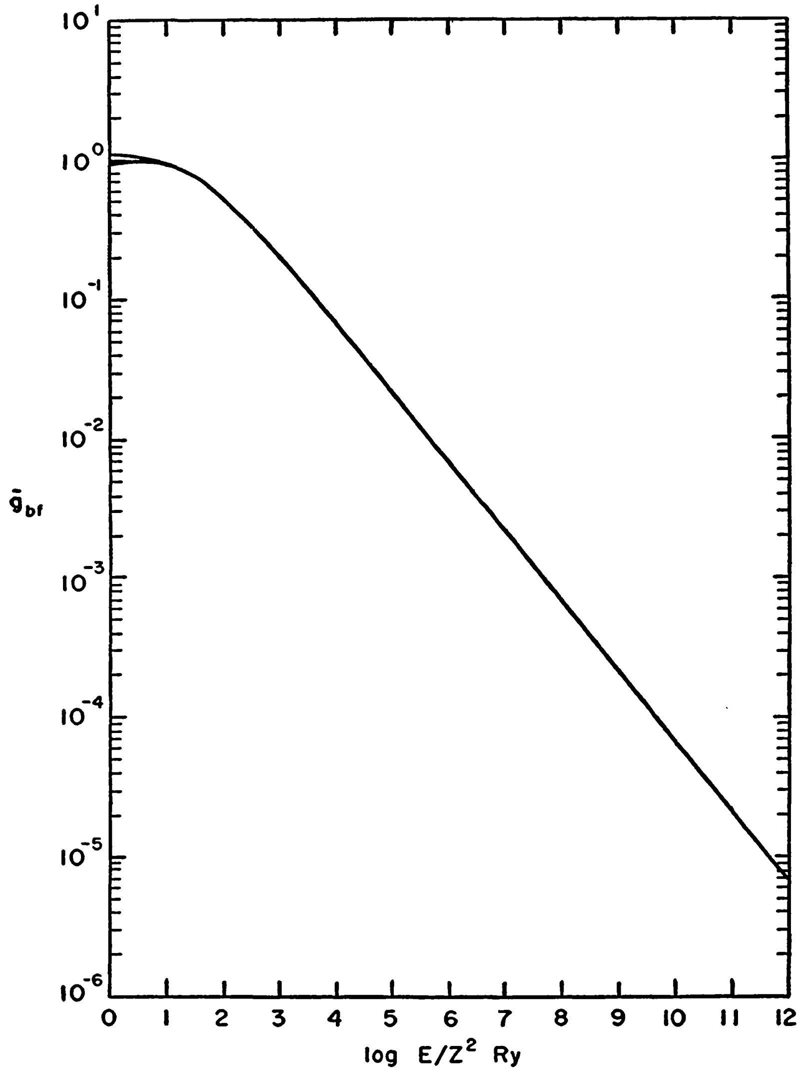


FIG. 21.—Bound-free Gaunt factor averaged over a shell versus free electron energy for n = 1–15

TABLE 2

BOUND-BOUND OSCILLATOR STRENGTHS

$l = 0$

$n \backslash n'l'$	2,1	3,1	4,1	5,1	6,1	7,1	8,1	9,1
1	.4162 (0)*	.7910 (−1)	.2899 (−1)	.1394 (−1)	.7799 (−2)	.4184 (−2)	.3183 (−2)	.2216 (−2)
2	.0000	.4349 (0)	.1028 (0)	.4193	.2163 (−1)	.1274 (−1)	.8180	.5583
3	−.4077 (−1)	.0000	.4847	.1210 (0)	.5139	.2737	.1655 (−1)	.1086 (−1)
4	−.9135 (−2)	−.9675 (−1)	.0000	.5442	.1381 (0)	.5965	.3227	.1980
5	−.3640	−.2228	−.1587	.0000	.6078	.1549 (0)	.6737	.3675
6	−.1854	−.9097 (−2)	−.3701 (−1)	−.2236	.0000	.6736	.1715 (0)	.7482
7	−.1084	−.4736	−.1528	−.5248 (−1)	−.2902	.0000	.7406	.1881 (0)
8	−.6925 (−3)	−.2824	−.8047 (−2)	−.2180	−.6835 (−1)	−.3577	.0000	.8086
9	−.4711	−.1836	−.4850	−.1155	−.2849	−.8446 (−1)	−.4259	.0000
10	−.3357	−.1268	−.3186	−.7008 (−2)	−.1516	−.3528	−.1007	−.4946
11	−.2480	−.9161 (−3)	−.2222	−.4633	−.9231 (−2)	−.1881	−.4212 (−1)	−.1171
12	−.1887	−.6852	−.1619	−.3252	−.6128	−.1149	−.2249	−.4900 (−1)
13	−.1469	−.5268	−.1221	−.2384	−.4319	−.7647 (−2)	−.1376	−.2619
14	−.1167	−.4143	−.9459 (−3)	−.1808	−.3180	−.5405	−.9178 (−2)	−.1604
15	−.9433 (−4)	−.3321	−.7491	−.1408	−.2421	−.3991	−.6500	−.1071
16	−.7733	−.2704	−.6042	−.1120	−.1893	−.3048	−.4811	−.7599 (−2)

$n \backslash n'l'$	10,1	11,1	12,1	13,1	14,1	15,1	16,1	
1	.1605 (−2)	.1200 (−2)	.9214 (−3)	.7227 (−3)	.5774 (−3)	.4686 (−3)	.3856 (−3)	
2	.3988	.2952	.2248 (−2)	.1753 (−2)	.1394 (−2)	.1127 (−2)	.9244	
3	.7554	.5484	.4117	.3175	.2502	.2009	.1639 (−2)	
4	.1316 (−1)	.9264	.6797	.5152	.4007	.3184	.2574	
5	.2273	.1524 (−1)	.1081 (−1)	.7986	.6093	.4769	.3811	
6	.4098	.2548	.1717	.1224 (−1)	.9088	.6967	.5478	
7	.8211	.4508	.2810	.1900	.1359 (−1)	.1013 (−1)	.7791	
8	.2047 (0)	.8931	.4907	.3065	.2076	.1488	.1112 (−1)	
9	.8770	.2213 (0)	.9645	.5300	.3313	.2248	.1614	
10	.0000	.9459	.2380 (0)	.1036 (0)	.5689	.3557	.2415	
11	−.5636	.0000	.1015 (1)	.2546	.1106 (0)	.6074	.3798	
12	−.1335	−.6329	.0000	.1084 (1)	.2712	.1177 (0)	.6456	
13	−.5590 (−1)	−.1500	−.7023 (0)	.0000	.1154 (1)	.2878	.1247 (0)	
14	−.2990	−.6282	−.1665	−.7719 (0)	.0000	.1224 (1)	.3045	
15	−.1832	−.3361 (−1)	−.6975 (−1)	−.1830	−.8416 (0)	.0000	.1293 (1)	
16	−.1225	−.2061	−.3732	−.7669 (−1)	−.1996	−.9114 (0)	.0000	

$l = 1$

$n \backslash n'l'$	1,0	2,0	3,0	3,2	4,0	4,2	5,0	5,2
2	−.1387 (0)	.0000	.1359 (−1)	.6958 (0)	.3045 (−2)	.1218 (0)	.1213 (−2)	.4437 (−1)
3	−.2637 (−1)	−.1450 (0)	.0000	.0000	.3225 (−1)	.6183	.7428	.1392 (0)
4	−.9664 (−2)	−.3425 (−1)	−.1616 (0)	−.1832 (−1)	.0000	.0000	.5291 (−1)	.6093
5	−.4646	−.1398	−.4034 (−1)	−.3683 (−2)	−.1814 (0)	−.4637 (−1)	.0000	.0000
6	−.2600	−.7210 (−2)	−.1713	−.1403	−.4604 (−1)	−.9729 (−2)	−.2026 (0)	−.7999 (−1)
7	−.1605	−.4247	−.9123 (−2)	−.7019 (−3)	−.1988	−.3790	−.5163 (−1)	−.1722
8	−.1061	−.2727	−.5516	−.4079	−.1076	−.1925	−.2246	−.6804 (−2)
9	−.7387 (−3)	−.1861	−.3621	−.2607	−.6599 (−2)	−.1133	−.1225	−.3491
10	−.5351	−.1329	−.2518	−.1779	−.4389	−.7318	−.7578 (−2)	−.2071
11	−.4002	−.9840 (−3)	−.1828	−.1274	−.3088	−.5041	−.5080	−.1347
12	−.3071	−.7494	−.1372	−.9463 (−4)	−.2266	−.3641	−.3602	−.9342 (−3)
13	−.2409	−.5843	−.1058	−.7238	−.1717	−.2726	−.2662	−.6787
14	−.1925	−.4646	−.8342 (−3)	−.5668	−.1336	−.2100	−.2031	−.5110
15	−.1562	−.3756	−.6698	−.4528	−.1061	−.1655	−.1590	−.3956
16	−.1285	−.3081	−.5463	−.3678	−.8582 (−3)	−.1330	−.1270	−.3134

$n \backslash n'l'$	6,0	6,2	7,0	7,2	8,0	8,2	9,0	9,2
2	.6180 (−3)	.2163 (−1)	.3613 (−3)	.1233 (−1)	.2308 (−3)	.7756 (−2)	.1570 (−3)	.5220 (−2)
3	.3032 (−2)	.5614	.1579 (−2)	.2901	.9412	.1721 (−1)	.6119	.1115 (−1)
4	.1234 (−1)	.1485 (0)	.5094	.6255	.2682 (−2)	.3330	.1617 (−2)	.2021
5	.7454	.6247	.1749 (−1)	.1570 (0)	.7267	.6755	.3851	.3656
6	.0000	.0000	.9672	.6514	.2278 (−1)	.1658 (0)	.9498	.7212
7	−.2245 (0)	−.1171	.0000	.0000	.1192 (0)	.6843	.2815 (−1)	.1750 (0)
8	−.5718 (−1)	−.2564 (−1)	−.2469 (0)	−.1564	.0000	.0000	.1420 (0)	.7210
9	−.2494	−.1023	−.6271 (−1)	−.3468 (−1)	−.2695	−.1973	.0000	.0000
10	−.1366	−.5283 (−2)	−.2737	−.1393	−.6825 (−1)	−.4415 (−1)	−.2923	−.2394
11	−.8493 (−2)	−.3152	−.1502	−.7230 (−2)	−.2977	−.1782	−.7378 (−1)	−.5394 (−1)
12	−.5723	−.2061	−.9368 (−2)	−.4331	−.1636	−.9287 (−2)	−.3215	−.2186
13	−.4079	−.1438	−.6334	−.2842	−.1022	−.5580	−.1767	−.1142
14	−.3029	−.1047	−.4530	−.1986	−.6921 (−2)	−.3671	−.1104	−.6878 (−2)
15	−.2322	−.7918 (−3)	−.3376	−.1454	−.4961	−.2572	−.7492 (−2)	−.4535
16	−.1826	−.6154	−.2597	−.1102	−.3707	−.1888	−.5379	−.3184

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 2—continued

$l = 1$

n \ n'l'	10,0	10,2	11,0	11,2	12,0	12,2	13,0
2	.1119 (−3)*	.3693 (−2)	.8268 (−4)	.2714 (−2)	.6289 (−4)	.2056 (−2)	.4898 (−4)
3	.4227	.7686	.3054 (−3)	.5543	.2284 (−3)	.4140	.1756 (−3)
4	.1062 (−1)	.1333 (−1)	.7406	.9329	.5398	.6815	.4070
5	.2336	.2248	.1544 (−2)	.1501 (−1)	.1084 (−2)	.1061 (−1)	.7947
6	.5052	.3940	.3077	.2443	.2043	.1643	.1440 (−2)
7	.1176 (−1)	.7653	.6270	.4203	.3829	.2620	.2549
8	.3357	.1846 (0)	.1404 (−1)	.8091	.7497	.4457	.4586
9	.1649 (0)	.7600	.3902	.1944 (0)	.1633 (−1)	.8529	.8729
10	.0000	.0000	.1879 (0)	.8008	.4450	.2044 (0)	.1863 (−1)
11	−.3153	−.2823	.0000	.0000	.2110 (0)	.8427	.5000
12	−.7932 (−1)	−.6396 (−1)	−.3383	−.3259	.0000	.0000	.2341 (0)
13	−.3452	−.2601	−.8486 (−1)	−.7416 (−1)	−.3614	−.3699	.0000
14	−.1896	−.1362	−.3687	−.3023	−.9040 (−1)	−.8450 (−1)	−.3846
15	−.1186	−.8215 (−2)	−.2025	−.1586	−.3922	−.3452	−.9595 (−1)
16	−.8051 (−2)	−.5425	−.1266	−.9580 (−2)	−.2152	−.1813	−.4157

n \ n'l'	13,2	14,0	14,2	15,0	15,2	16,0	16,2
2	.1596 (−2)	.3892 (−4)	.1265 (−2)	.3144 (−4)	.1020 (−2)	.2578 (−4)	.8347 (−3)
3	.3180	.1381 (−3)	.2499	.1107 (−3)	.2001	.9015	.1629 (−2)
4	.5148	.3153	.3993	.2497	.3166	.2014 (−3)	.2555
5	.7818	.6026	.5952	.4693	.4650	.3735	.3711
6	.1169 (−1)	.1060 (−2)	.8670	.8070	.6640	.6310	.5215
7	.1171	.1802	.1266 (−1)	.1330 (−2)	.9432	.1016 (−2)	.7253
8	.2788	.3059	.1890	.2167	.1356 (−1)	.1604	.1013 (−1)
9	.4706	.5346	.2949	.3571	.2004	.2533	.1441
10	.8968	.9965	.4952	.6108	.3107	.4084	.2115
11	.2146 (0)	.2094 (−1)	.9409	.1120 (−1)	.5196	.6870	.3262
12	.8855	.5550	.2250 (0)	.2325	.9852	.1244 (−1)	.5440
13	.0000	.2573 (0)	.9290	.6102	.2354 (0)	.2556	.1030 (0)
14	−.4144	.0000	.0000	.2805 (0)	.9730	.6654	.2460
15	−.9494 (−1)	−.4078	−.4592	.0000	.0000	.3038 (0)	.1017 (1)
16	−.3886	−.1015	−.1055	−.4311	−.5042	.0000	.0000

$l = 2$

n \ n'l'	2,1	3,1	4,1	4,3	5,1	5,3	6,1
3	−.4175 (0)	.0000	.1099 (−1)	.1018 (1)	.2210 (−2)	.1566 (0)	.8420 (−3)
4	−.7308 (−1)	−.3710 (0)	.0000	.0000	.2782 (−1)	.8902	.5837 (−2)
5	−.2662	−.8354 (−1)	−.3656 (0)	−.1242 (−1)	.0000	.0000	.4800 (−1)
6	−.1298	−.3368	−.8908 (−1)	−.2216 (−2)	−.3748 (0)	−.3259 (−1)	.0000
7	−.7399 (−2)	−.1741	−.3753	−.7902 (−3)	−.9419 (−1)	−.6205 (−2)	−.3908 (0)
8	−.4654	−.1033	−.1998	−.3802	−.4053	−.2282	−.9948 (−1)
9	−.3132	−.6692 (−2)	−.1212	−.2158	−.2194	−.1118	−.4327
10	−.2216	−.4611	−.8000 (−2)	−.1360	−.1349	−.6425 (−3)	−.2364
11	−.1628	−.3326	−.5597	−.9199 (−4)	−.9005 (−2)	−.4086	−.1466
12	−.1233	−.2484	−.4089	−.6553	−.6364	−.2786	−.9858 (−2)
13	−.9576 (−3)	−.1908	−.3089	−.4855	−.4691	−.1998	−.7014
14	−.7589	−.1499	−.2396	−.3709	−.3571	−.1489	−.5202
15	−.6119	−.1201	−.1899	−.2905	−.2790	−.1143	−.3984
16	−.5008	−.9774 (−3)	−.1533	−.2322	−.2226	−.8998 (−4)	−.3129

n \ n'l'	6,3	7,1	7,3	8,1	8,3	9,1	9,3
3	.5389 (−1)	.4212 (−3)	.2559 (−1)	.2448 (−3)	.1442 (−1)	.1564 (−3)	.9034 (−2)
4	.1862 (0)	.2274 (−2)	.7224	.1155 (−2)	.3659	.6797	.2149 (−1)
5	.8443	.1033 (−1)	.1968 (0)	.4082	.8082	.2094 (−2)	.4241
6	.0000	.7024	.8329	.1539 (−1)	.2038 (0)	.6137	.8623
7	−.5788 (−1)	.0000	.0000	.9385	.8393	.2081 (−1)	.2103 (0)
8	−.1149	−.4106 (0)	−.8674 (−1)	.0000	.0000	.1184 (0)	.8558
9	−.4314 (−2)	−.1050	−.1771	−.4326 (0)	−.1181	.0000	.0000
10	−.2141	−.4592 (−1)	−.6754 (−2)	−.1107	−.2463 (−1)	−.4560	−.1515
11	−.1241	−.2522	−.3383	−.4855 (−1)	−.9503 (−2)	−.1166	−.3208 (−1)
12	−.7950 (−3)	−.1572	−.1975	−.2674	−.4796	−.5117 (−1)	−.1249
13	−.5451	−.1063	−.1272	−.1673	−.2815	−.2824	−.6339 (−2)
14	−.3928	−.7597 (−2)	−.8757 (−3)	−.1134	−.1819	−.1770	−.3736
15	−.2940	−.5659	−.6335	−.8136 (−2)	−.1257	−.1203	−.2423
16	−.2267	−.4352	−.4758	−.6081	−.9122 (−3)	−.8646 (−2)	−.1679

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 2—continued

$l = 2$

n \ $n'l'$	10,1	10,3	11,1	11,3	12,1	12,3	13,1
3	.1067 (−3)*	.6081 (−2)	.7643 (−4)	.4311 (−2)	.5678 (−4)	.3179 (−2)	.4343 (−4)
4	.4391	.1386 (−1)	.3025 (−3)	.9537	.2184 (−3)	.6881	.1635 (−3)
5	.1242 (−2)	.2552	.8084	.1677 (−1)	.5605	.1171 (−1)	.4072
6	.3170	.4619	.1891 (−2)	.2824	.1236 (−2)	.1878	.8613
7	.8358	.9049	.4338	.4909	.2599	.3031	.1705 (−2)
8	.2649 (−1)	.2171 (0)	.1070 (−1)	.9431	.5572	.5157	.3348
9	.1436 (0)	.8788	.3236	.2242 (0)	.1312 (−1)	.9798	.6853
10	.0000	.0000	.1694 (0)	.9061	.3838	.2318 (0)	.1560 (−1)
11	−.4805	−.1862	.0000	.0000	.1955 (0)	.9364	.4450
12	−.1227	−.3994 (−1)	−.5056	−.2221	.0000	.0000	.2220 (0)
13	−.5381 (−1)	−.1566	−.1288	−.4812 (−1)	−.5313	−.2588	.0000
14	−.2971	−.7986 (−2)	−.5646 (−1)	−.1898	−.1350	−.5655 (−1)	−.5574
15	−.1864	−.4723	−.3118	−.9716 (−2)	−.5912 (−1)	−.2241	−.1412
16	−.1269	−.3071	−.1957	−.5763	−.3264	−.1151	−.6179 (−1)

n \ $n'l'$	13,3	14,1	14,3	15,1	15,3	16,1	16,3
3	.2418 (−2)	.3401 (−4)	.1886 (−2)	.2717 (−4)	.1501 (−2)	.2207 (−4)	.1216 (−2)
4	.5148	.1260 (−3)	.3964	.9932	.3123	.7982	.2509
5	.8554	.3066	.6466	.2374 (−3)	.5023	.1881 (−3)	.3989
6	.1325 (−1)	.6285	.9762	.4751	.7436	.3693	.5815
7	.2033	.1192 (−2)	.1445 (−1)	.8724	.1071 (−1)	.6615	.8205 (−1)
8	.3206	.2203	.2163	.1544 (−2)	.1545	.1133 (−2)	.1151
9	.5384	.4127	.3362	.2721	.2278	.1911	.1633
10	.1016 (0)	.8170	.5601	.4929	.3508	.3255	.2383
11	.2397	.1814 (−1)	.1052 (0)	.9514	.5812	.5748	.3647
12	.9689	.5070	.2479	.2071 (−1)	.1089 (0)	.1088 (−1)	.6021
13	.0000	.2486 (0)	.1003 (1)	.5697	.2564	.2331	.1126 (0)
14	−.2963	.0000	.0000	.2755 (0)	.1038 (1)	.6329	.2650
15	−.6519 (−1)	−.5838	−.3344 (0)	.0000	.0000	.3025 (0)	.1075 (1)
16	−.2594	−.1478	−.7400 (−1)	−.6104	−.3729 (0)	.0000	.0000

$l = 3$

n \ $n'l'$	3,2	4,2	5,2	5,4	6,2	6,4	7,2	7,4	8,2
4	−.7268 (0)	.0000	.8871 (−2)	.1346 (1)	.1583 (−2)	.1824 (0)	.5644 (−3)	.5838 (−1)	.2716 (−3)
5	−.1119	−.6359 (0)	.0000	.0000	.2328 (−1)	.1185 (1)	.4432 (−2)	.2287 (0)	.1630 (−2)
6	−.3849 (−1)	−.1330	−.6031 (0)	−.9483 (−2)	.0000	.0000	.4135 (−1)	.1110 (1)	.8205
7	−.1828	−.5160 (−1)	−.1406	−.1509	−.5950 (0)	−.2546 (−1)	.0000	.0000	.6195 (−1)
8	−.1030	−.2614	−.5773 (−1)	−.5007 (−3)	−.1455	−.4417 (−2)	−.5995 (0)	−.4611 (−1)	.0000
9	−.6453 (−2)	−.1535	−.3029	−.2297	−.6159 (−1)	−.1529	−.1502	−.8456 (−2)	−.6113 (0)
10	−.4343	−.9899 (−2)	−.1823	−.1263	−.3299	−.7184 (−3)	−.6464 (−1)	−.3014	−.1550
11	−.3080	−.6812	−.1198	−.7777 (−4)	−.2017	−.4010	−.3506	−.1441	−.6737 (−1)
12	−.2271	−.4915	−.8366 (−2)	−.5178	−.1342	−.2497	−.2165	−.8139 (−3)	−.3683
13	−.1727	−.3677	−.6110	−.3647	−.9465 (−2)	−.1676	−.1452	−.5109	−.2290
14	−.1347	−.2831	−.4619	−.2679	−.6973	−.1188	−.1032	−.3451	−.1545
15	−.1072	−.2231	−.3588	−.2035	−.5311	−.8776 (−4)	−.7651 (−2)	−.2459	−.1103
16	−.8683 (−3)	−.1792	−.2849	−.1587	−.4153	−.6696	−.5860	−.1824	−.8219 (−2)

n \ $n'l'$	8,4	9,2	9,4	10,2	10,4	11,2	11,4	12,2	12,4
4	.2655 (−1)	.1542 (−3)	.1457 (−1)	.9713 (−4)	.8979 (−2)	.6571 (−4)	.5981 (−2)	.4681 (−4)	.4213 (−2)
5	.8450	.7985	.4148	.4589 (−3)	.2386 (−1)	.2919 (−3)	.1519 (−1)	.1990 (−3)	.1036 (−1)
6	.2453 (0)	.3081 (−2)	.9733	.1529 (−2)	.4990	.8867	.2956	.5678	.1922
7	.1075 (1)	.1265 (−1)	.2538 (0)	.4824	.1048 (0)	.2417 (−2)	.5518	.1411 (−2)	.3332
8	.0000	.8439	.1063 (1)	.1760 (−1)	.2601	.6788	.1099 (0)	.3425	.5888
9	−.7021 (−1)	.0000	.0000	.1082 (0)	.1065 (1)	.2292 (−1)	.2658	.8920	.1141 (0)
10	−.1339	−.6277 (0)	−.9695 (−1)	.0000	.0000	.1330 (0)	.1076 (1)	.2853 (−1)	.2717
11	−.4876 (−2)	−.1602	−.1903	−.6472	−.1257 (0)	.0000	.0000	.1586 (0)	.1093 (1)
12	−.2363	−.6999 (−1)	−.7046 (−2)	−.1656	−.2524 (−1)	−.6689	−.1561 (0)	.0000	.0000
13	−.1346	−.3846	−.3451	−.7258 (−1)	−.9466 (−2)	−.1712	−.3191 (−1)	−.6921	−.1878 (0)
14	−.8508 (−3)	−.2401	−.1980	−.4001	−.4675	−.7517 (−1)	−.1209	−.1771	−.3894 (−1)
15	−.5776	−.1627	−.1258	−.2505	−.2698	−.4152	−.6013 (−2)	−.7779 (−1)	−.1489
16	−.4132	−.1166	−.8574 (−3)	−.1702	−.1721	−.2605	−.3487	−.4301	−.7445 (−2)

n \ $n'l'$	13,2	13,4	14,2	14,4	15,2	15,4	16,2	16,4
4	.3468 (−4)	.3095 (−2)	.2649 (−4)	.2349 (−2)	.2075 (−4)	.1830 (−2)	.1658 (−4)	.1456 (−2)
5	.1427 (−3)	.7428	.1063 (−3)	.5536	.8167	.4252	.6427	.3347
6	.3893	.1332 (−1)	.2806	.9678	.2100 (−3)	.7290	.1620 (−3)	.5650
7	.9083	.2197	.6255	.1541 (−1)	.4525	.1130 (−1)	.3399	.8581
8	.2010 (−2)	.3602	.1300 (−2)	.2400	.8980	.1697	.6516	.1253 (−1)
9	.4528	.6177	.2669	.3812	.1731 (−2)	.2558	.1199 (−2)	.1820
10	.1119 (−1)	.1177 (0)	.5705	.6421	.3374	.3988	.2194	.2690
11	.3437	.2779	.1355 (−1)	.1211 (0)	.6940	.6641	.4116	.4142
12	.1849 (0)	.1115 (1)	.4039	.2845	.1601 (−1)	.1244 (0)	.8223	.6847
13	.0000	.0000	.2117 (0)	.1140 (1)	.4656	.2914	.1853 (−1)	.1277 (0)
14	−.7165	−.2206 (0)	.0000	.0000	.2388 (0)	.1167 (1)	.5286	.2985
15	−.1831	−.4628 (−1)	−.7417	−.2542 (0)	.0000	.0000	.2664 (0)	.1196 (1)
16	−.8044 (−1)	−.1782	−.1893	−.5388 (−1)	−.7677	−.2886 (0)	.0000	.0000

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 2—continued

$l = 4$

n \ $n'l'$	4,3	5,3	6,3	6,5	7,3	7,5	8,3	8,5
5	−.1047 (1)*	.0000	.7376 (−2)	.1676 (1)	.1174 (−2)	.2018 (0)	.3895 (−3)	.5992 (−1)
6	−.1418 (0)	−.9220 (0)	.0000	.0000	.1980 (−1)	.1492 (1)	.3435 (−2)	.2657 (0)
7	−.4541 (−1)	−.1779	−.8632 (0)	−.7693 (−2)	.0000	.0000	.3586 (−1)	.1392 (1)
8	−.2065	−.6572 (−1)	−.1908	−.1101	−.8363 (0)	−.2099	.0000	.0000
9	−.1133	−.3226	−.7570 (−1)	−.3398 (−3)	−.1974	−.3336 (−2)	−.8270 (0)	−.3854 (−1)
10	−.6984 (−2)	−.1856	−.3881	−.1482	−.8151 (−1)	−.1086	−.2023	−.6558 (−2)
11	−.4652	−.1181	−.2299	−.7847 (−4)	−.4291	−.4887 (−3)	−.8552 (−1)	−.2218
12	−.3277	−.8055 (−2)	−.1495	−.4703	−.2591	−.2641	−.4580	−.1021
13	−.2407	−.5777	−.1036	−.3067	−.1709	−.1604	−.2801	−.5599 (−3)
14	−.1827	−.4306	−.7527 (−2)	−.2126	−.1198	−.1056	−.1866	−.3437
15	−.1423	−.3307	−.5670	−.1543	−.8790 (−2)	−.7378 (−4)	−.1320	−.2281
16	−.1133	−.2603	−.4395	−.1161	−.6674	−.5387	−.9747 (−2)	−.1603

n \ $n'l'$	9,3	9,5	10,3	10,5	11,3	11,5	12,3	12,5
5	.1787 (−3)	.2592 (−1)	.9821 (−4)	.1375 (−1)	.6049 (−4)	.8270 (−2)	.4027 (−4)	.5415 (−2)
6	.1189 (−2)	.9304	.5588 (−3)	.4400	.3119 (−3)	.2466 (−1)	.1942 (−3)	.1539 (−1)
7	.6577	.2909 (0)	.2344 (−2)	.1110 (0)	.1121 (−2)	.5530	.6330	.3209
8	.5461 (−1)	.1337 (1)	.1042 (−1)	.3031	.3793	.1215 (0)	.1838 (−2)	.6261
9	.0000	.0000	.7540	.1308 (1)	.1480 (−1)	.3107	.5480	.1285 (0)
10	−.8285 (0)	−.5937 (−1)	.0000	.0000	.9779	.1296 (1)	.1963 (−1)	.3166
11	−.2068	−.1061	−.8370 (0)	−.8281 (−1)	.0000	.0000	.1214 (0)	.1295 (1)
12	−.8871 (−1)	−.3689 (−2)	−.2113	−.1534	−.8503 (0)	−.1083 (0)	.0000	.0000
13	−.4804	−.1727	−.9153 (−1)	−.5451 (−2)	−.2162	−.2065 (−1)	−.8671	−.1356 (0)
14	−.2965	−.9587 (−3)	−.4994	−.2588	−.6418 (−1)	−.7461 (−2)	−.2213	−.2643 (−1)
15	−.1990	−.5935	−.3102	−.1450	−.5165	−.3582	−.9677 (−1)	−.9682 (−2)
16	−.1415	−.3964	−.2092	−.9037 (−3)	−.3222	−.2023	−.5326	−.4692

n \ $n'l'$	13,3	13,5	14,3	14,5	15,3	15,5	16,3	16,5
5	.2836 (−4)	.3767 (−2)	.2084 (−4)	.2742 (−2)	.1582 (−4)	.2067 (−2)	.1234 (−4)	.1603 (−2)
6	.1303 (−3)	.1035 (−1)	.9239	.7345	.6825	.5431	.5208	.4147
7	.3974	.2054	.2684 (−3)	.1407 (−1)	.1912 (−3)	.1013 (−1)	.1419 (−3)	.7582
8	.1047 (−2)	.3719	.6617	.2422	.4492	.1682	.3214	.1225 (−1)
9	.2684	.6764	.1540 (−2)	.4084	.9783	.2693	.6668	.1889
10	.7363	.1336 (0)	.3636	.7138	.2099 (−2)	.4359	.1339 (−2)	.2901
11	.2482 (−1)	.3220	.9404	.1377 (0)	.4677	.7435	.2712	.4579
12	.1461 (0)	.1302 (1)	.3029 (−1)	.3275	.1158 (−1)	.1413 (0)	.5791	.7687
13	.0000	.0000	.1716 (0)	.1315 (1)	.3600	.3331	.1386 (−1)	.1446 (0)
14	−.8864	−.1642 (0)	.0000	.0000	.1977 (0)	.1333 (1)	.4190	.3390
15	−.2266	−.3261 (−1)	−.9076	−.1940 (0)	.0000	.0000	.2245 (0)	.1354 (1)
16	−.9934 (−1)	−.1208	−.2322	−.3913 (−1)	−.9304	−.2248 (0)	.0000	.0000

$l = 5$

n \ $n'l'$	5,4	6,4	7,4	7,6	8,4	8,6	9,4	9,6
6	−.1372 (1)	.0000	.6294 (−2)	.2008 (1)	.9010 (−3)	.2169 (0)	.2781 (−3)	.5979 (−1)
7	−.1651 (0)	−.1221 (1)	.0000	.0000	.1718 (−1)	.1806 (1)	.2729 (−2)	.2977 (0)
8	−.4902 (−1)	−.2174 (0)	−.1139 (1)	−.6479 (−2)	.0000	.0000	.3153 (−1)	.1685 (1)
9	−.2121	−.7613 (−1)	−.2380 (0)	−.8412 (−3)	−.1094 (1)	−.1789 (−1)	.0000	.0000
10	−.1125	−.3600	−.9079 (−1)	−.2420	−.2480 (0)	−.2620 (−2)	−.1070 (1)	−.3318 (−1)
11	−.6767 (−2)	−.2017	−.4525	−.1002	−.9944 (−1)	−.8041 (−3)	−.2542 (0)	−.5262 (−2)
12	−.4430	−.1259	−.2626	−.5100 (−4)	−.5122	−.3460	−.1051	−.1691
13	−.3082	−.8467 (−2)	−.1680	−.2965	−.3043	−.1806	−.5534 (−1)	−.7483 (−3)
14	−.2243	−.6009	−.1151	−.1888	−.1982	−.1068	−.3341	−.3982
15	−.1691	−.4444	−.8291 (−2)	−.1285	−.1376	−.6887 (−4)	−.2203	−.2386
16	−.1311	−.3393	−.6204	−.9184 (−5)	−.1002	−.4728	−.1546	−.1553

n \ $n'l'$	10,4	10,6	11,4	11,6	12,4	12,6	13,4	13,6
6	.1212 (−3)	.2455 (−1)	.6421 (−4)	.1253 (−1)	.3848 (−4)	.7326 (−2)	.2509 (−4)	.4694 (−2)
7	.8889	.9866	.3999 (−3)	.4482	.2161 (−3)	.2437 (−1)	.1312 (−3)	.1487 (−1)
8	.5366 (−2)	.3328 (0)	.1815 (−2)	.1218 (0)	.8352	.5881	.4581	.3331
9	.4858 (−1)	.1611 (1)	.8679	.3503	.3018 (−2)	.1360 (0)	.1413 (−2)	.6834
10	.0000	.0000	.6775 (−1)	.1567 (1)	.1255 (−1)	.3605	.4460	.1454 (0)
11	−.1060 (1)	−.5158 (−1)	.0000	.0000	.8864	.1541 (1)	.1690 (−1)	.3675
12	−.2590 (0)	−.8662 (−2)	−.1060 (1)	−.7251 (−1)	.0000	.0000	.1109 (0)	.1529 (1)
13	−.1093	−.2878	−.2635 (0)	−.1271	−.1066 (1)	−.9553 (−1)	.0000	.0000
14	−.5840 (−1)	−.1302	−.1127	−.4336 (−2)	−.2679 (0)	−.1732	−.1076 (1)	−.1203 (0)
15	−.3567	−.7031 (−3)	−.6083 (−1)	−.1995	−.1156	−.6032 (−2)	−.2725 (0)	−.2240 (−1)
16	−.2374	−.4259	−.3746	−.1090	−.6289 (−1)	−.2815	−.1183	−.7936 (−2)

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 2—*continued*

$l = 5$

n \ $n'l'$	14,4	14,6	15,4	15,6	16,4	16,6	
6	.1739 (−4)*	.3212 (−2)	.1262 (−4)	.2308 (−2)	.9496 (−5)	.1723 (−2)	
7	.8644	.9820	.6037	.6874	.4408 (−4)	.5028	
8	.2812 (−3)	.2093 (−1)	.1866 (−3)	.1413 (−1)	.1311 (−3)	.1006 (−1)	
9	.7844	.3981	.4856	.2553	.3243	.1751	
10	.2118 (−2)	.7503	.1186 (−2)	.4459	.7394	.2903	
11	.6104	.1520 (0)	.2931	.7994	.1655 (−2)	.4819	
12	.2163 (−1)	.3732	.7922	.1571 (0)	.3839	.8374	
13	.1344 (0)	.1527 (1)	.2668 (−1)	.3784	.9887	.1613 (0)	
14	.0000	.0000	.1588 (0)	.1531 (1)	.3202 (−1)	.3836	
15	−.1090 (1)	−.1465 (0)	.0000	.0000	.1840 (0)	.1541 (1)	
16	−.2774 (0)	−.2790 (−1)	−.1107 (1)	−.1740 (0)	.0000	.0000	

$l = 6$

n \ $n'l'$	6,5	7,5	8,5	8,7	9,5	9,7	10,5
7	−.1699 (1)	.0000	.5482 (−2)	.2340 (1)	.7118 (−3)	.2289 (0)	.2048 (−3)
8	−.1835 (0)	−.1528 (1)	.0000	.0000	.1514 (−1)	.2124 (1)	.2217 (−2)
9	−.5059 (−1)	−.2519 (0)	−.1426 (1)	−.5599 (−2)	.0000	.0000	.2808 (−1)
10	−.2077	−.8348 (−1)	−.2816 (0)	−.6645 (−3)	−.1363 (1)	−.1561 (−1)	.0000
11	−.1060	−.3792	−.1030	−.1787	−.2964 (0)	−.2117 (−2)	−.1326 (1)
12	−.6199 (−2)	−.2062	−.4976 (−1)	−.7024 (−4)	−.1151	−.6135 (−3)	−.3050 (0)
13	−.3972	−.1258	−.2819	−.3436	−.5782 (−1)	−.2525	−.1230
14	−.2718	−.8309 (−2)	−.1771	−.1935	−.3369	−.1273	−.6348 (−1)
15	−.1953	−.5816	−.1195	−.1201	−.2160	−.7320 (−4)	−.3773
16	−.1458	−.4254	−.8509 (−2)	−.8001 (−5)	−.1482	−.4613	−.2456

n \ $n'l'$	10,7	11,5	11,7	12,5	12,7	13,5	13,7
7	.5876 (−1)	.8475 (−4)	.2288 (−1)	.4316 (−4)	.1122 (−1)	.2509 (−4)	.6359 (−2)
8	.3253 (0)	.6804 (−3)	.1021 (0)	.2927 (−3)	.4449	.1528 (−3)	.2342 (−1)
9	.1986 (1)	.4453 (−2)	.3709	.1430 (−2)	.1300 (0)	.6332	.6076
10	.0000	.4365 (−1)	.1896 (1)	.7329	.3947	.2435 (−2)	.1482 (0)
11	−.2918 (−1)	.0000	.0000	.6136 (−1)	.1836 (1)	.1076 (−1)	.4085
12	−.4329 (−2)	−.1304 (1)	−.4568 (−1)	.0000	.0000	.8083	.1798 (1)
13	−.1323	−.3109 (0)	−.7230 (−2)	−.1294 (1)	−.6461 (−1)	.0000	.0000
14	−.5632 (−3)	−.1286	−.2298	−.3158 (0)	−.1074	−.1292 (1)	−.8559 (−1)
15	−.2905	−.6764 (−1)	−.1004	−.1329	−.3521 (−2)	−.3202 (0)	−.1479
16	−.1698	−.4078	−.5275 (−3)	−.7086 (−1)	−.1570	−.1365	−.4968 (−2)

n \ $n'l'$	14,5	14,7	15,5	15,7	16,5	16,7	
7	.1598 (−4)	.3977 (−2)	.1087 (−4)	.2669 (−2)	.7771 (−5)	.1889 (−2)	
8	.9039	.1393 (−1)	.5828	.9018	.4001 (−4)	.6211	
9	.3369 (−3)	.3353	.2019 (−3)	.2063 (−1)	.1314 (−3)	.1369 (−1)	
10	.1102 (−2)	.7249	.5949	.4133	.3603	.2605	
11	.3669	.1604 (0)	.1688 (−2)	.8098	.9227	.4728	
12	.1466 (−1)	.4175	.5104	.1690 (0)	.2382 (−2)	.8729	
13	.1018 (0)	.1775 (1)	.1896 (−1)	.4242	.6715	.1754 (0)	
14	.0000	.0000	.1240 (0)	.1763 (1)	.2361 (−1)	.4298	
15	−.1296 (1)	−.1083 (0)	.0000	.0000	.1472 (0)	.1759 (1)	
16	−.3246 (0)	−.1931 (−1)	−.1304 (1)	−.1325 (0)	.0000	.0000	

$l = 7$

n \ $n'l'$	7,6	8,6	9,6	9,8	10,6	10,8	
8	−.2028 (1)	.0000	.4853 (−2)	.2672 (1)	.5759 (−3)	.2387 (0)	
9	−.1984 (0)	−.1840 (1)	.0000	.0000	.1353 (−1)	.2445 (1)	
10	−.5092 (−1)	−.2819 (0)	−.1721 (1)	−.4931 (−2)	.0000	.0000	
11	−.1983	−.8847 (−1)	−.3215 (0)	−.5385 (−3)	−.1643 (1)	−.1385 (−1)	
12	−.9723 (−2)	−.3856	−.1127	−.1358	−.3421 (0)	−.1748 (−2)	
13	−.5511	−.2030	−.5266 (−1)	−.5076 (−4)	−.1284	−.4795 (−3)	
14	−.3447	−.1207	−.2906	−.2385	−.6282 (−1)	−.1889	
15	−.2313	−.7816 (−2)	−.1788	−.1300	−.3582	−.9195 (−4)	
16	−.1637	−.5383	−.1187	−.7853 (−5)	−.2258	−.5137	

n \ $n'l'$	11,6	11,8	12,6	12,8	13,6	13,8	
8	.1549 (−3)	.5724 (−1)	.6088 (−4)	.2115 (−1)	.2978 (−4)	.9957 (−2)	
9	.1835 (−2)	.3493 (0)	.5317 (−3)	.1039 (0)	.2188 (−3)	.4343 (−1)	
10	.2529 (−1)	.2292 (1)	.3752 (−2)	.4055	.1146 (−2)	.1362 (0)	
11	.0000	.0000	.3959 (−1)	.2187 (1)	.6266	.4363	
12	−.1593 (1)	−.2605 (−1)	.0000	.0000	.5599 (−1)	.2115 (1)	
13	−.3541 (0)	−.3630 (−2)	−.1559 (1)	−.4102 (−1)	.0000	.0000	
14	−.1390	−.1057	−.3619 (0)	−.6139 (−2)	−.1539 (1)	−.5832 (−1)	
15	−.7018 (−1)	−.4330 (−3)	−.1464	−.1869	−.3677 (0)	−.9218 (−2)	
16	−.4097	−.2165	−.7565 (−1)	−.7892 (−3)	−.1520	−.2906	

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 2—continued

$l = 7$

n \ $n'l'$	14,6	14,8	15,6	15,8	16,6	16,8		
8	.1677 (−4)*	.5463 (−2)	.1041 (−4)	.3329 (−2)	.6934 (−5)	.2187 (−2)		
9	.1103 (−3)	.2211 (−1)	.6344	.1280 (−1)	.3998 (−4)	.8110		
10	.4881	.6153	.2518 (−3)	.3303	.1471 (−3)	.1987 (−1)		
11	.1992 (−2)	.1582 (0)	.8704	.7527	.4571	.4195		
12	.9310	.4545	.3051 (−2)	.1734 (0)	.1361 (−2)	.8559		
13	.7418 (−1)	.2065 (1)	.1282 (−1)	.4662	.4305	.1843 (0)		
14	.0000	.0000	.9387	.2031 (1)	.1673 (−1)	.4745		
15	−.1528 (1)	−.7761 (−1)	.0000	.0000	.1149 (0)	.2010 (1)		
16	−.3725 (0)	−.1281	−.1525 (1)	−.9863 (−1)	.0000	.0000		

$l = 8$

n \ $n'l'$	8,7	9,7	10,7	10,9	11,7	11,9	12,7	12,9
9	−.2358 (1)	.0000	.4351 (−2)	.3005 (1)	.4752 (−3)	.2468 (0)	.1199 (−3)	.5547 (−1)
10	−.2106 (0)	−.2157 (1)	.0000	.0000	.1222 (−1)	.2768 (1)	.1543 (−2)	.3703 (0)
11	−.5050 (−1)	−.3082 (0)	−.2022 (1)	−.4407 (−2)	.0000	.0000	.2299 (−1)	.2603 (1)
12	−.1866	−.9169 (−1)	−.3578 (0)	−.4455 (−3)	−.1930 (1)	−.1245 (−1)	.0000	.0000
13	−.8785 (−2)	−.3832	−.1202	−.1057	−.3850 (0)	−.1470 (−2)	−.1866 (1)	−.2355 (−1)
14	−.4821	−.1951	−.5429 (−1)	−.3761 (−4)	−.1396	−.3823 (−3)	−.4010 (0)	−.3091 (−2)
15	−.2937	−.1130	−.2915	−.1697	−.6642 (−1)	−.1443	−.1530	−.8590 (−3)
16	−.1930	−.7156 (−2)	−.1753	−.8950 (−5)	.3701	−.6781 (−4)	−.7552 (−1)	−.3388

n \ $n'l'$	13,7	13,9	14,7	14,9	15,7	15,9	16,7	16,9
9	.4479 (−4)	.1948 (−1)	.2104 (−4)	.8803 (−2)	.1147 (−4)	.4673 (−2)	.6929 (−5)	.2771 (−2)
10	.4231 (−3)	.1046 (0)	.1667 (−3)	.4193 (−1)	.8113	.2064 (−1)	.4532 (−4)	.1162 (−1)
11	.3203 (−2)	.4369	.9326	.1407 (0)	.3820 (−3)	.6141	.1910 (−3)	.3205
12	.3619 (−1)	.2485 (1)	.5417 (−2)	.4750	.1649 (−2)	.1664 (0)	.6963	.7693
13	.0000	.0000	.5146 (−1)	.2400 (1)	.8133	.4982	.2564 (−2)	.1847 (0)
14	−.1822 (1)	−.3725 (−1)	.0000	.0000	.6848 (−1)	.2339 (1)	.1130 (−1)	.5132
15	−.4114 (0)	−.5285 (−2)	−.1792 (1)	−.5318 (−1)	.0000	.0000	.8702	.2295 (1)
16	−.1627	−.1543	−.4187 (0)	−.8009 (−2)	−.1773 (1)	−.7105 (−1)	.0000	.0000

$l = 9$

n \ $n'l'$	9,8	10,8	11,8	11,10	12,8	12,10	13,8	13,10
10	−.2689 (1)	.0000	.3943 (−2)	.3338 (1)	.3986 (−3)	.2537 (0)	.9458 (−4)	.5361 (−1)
11	−.2208 (0)	−.2476 (1)	.0000	.0000	.1114 (−1)	.3093 (1)	.1315 (−2)	.3888 (0)
12	−.4963 (−1)	−.3314 (0)	−.2329 (1)	−.3983 (−2)	.0000	.0000	.2107 (−1)	.2916 (1)
13	−.1743	−.9358 (−1)	−.3909 (0)	−.3748 (−3)	−.2223 (1)	−.1131 (−1)	.0000	.0000
14	−.7876 (−2)	−.3751	−.1259	−.8391 (−4)	−.4250 (0)	−.1254 (−2)	−.2147 (1)	−.2149 (−1)
15	−.4181	−.1847	−.5494 (−1)	−.2847	−.1489	−.3100 (−3)	−.4458 (0)	−.2667 (−2)
16	−.2479	−.1040	−.2868	−.1235	−.6883 (−1)	−.1122	−.1653	−.7083 (−3)

n \ $n'l'$	14,8	14,10	15,8	15,10	16,8	16,10		
10	.3365 (−4)	.1792 (−1)	.1519 (−4)	.7776 (−2)	.8008 (−5)	.3993 (−2)		
11	.3421 (−3)	.1044 (0)	.1291 (−3)	.4019 (−1)	.6068 (−4)	.1913 (−1)		
12	.2766 (−2)	.4654	.7686	.1438 (0)	.3032 (−3)	.6064		
13	.3333 (−1)	.2787 (1)	.4729 (−2)	.5111	.1381 (−2)	.1729 (0)		
14	.0000	.0000	.4758 (−1)	.2691 (1)	.7166	.5396		
15	−.2092 (1)	−.3412 (−1)	.0000	.0000	.6357 (−1)	.2619 (1)		
16	−.4592 (0)	−.4602 (−2)	−.2053 (1)	−.4890 (−1)	.0000	.0000		

$l = 10$

n \ $n'l'$	10,9	11,9	12,9	12,11	13,9	13,11		
11	−.3020 (1)	.0000	.3604 (−2)	.3671 (1)	.3391 (−3)	.2595 (0)		
12	−.2295 (0)	−.2798 (1)	.0000	.0000	.1023 (−1)	.3419 (1)		
13	−.4851 (−1)	−.3518 (0)	−.2639 (1)	−.3635 (−2)	.0000	.0000		
14	−.1621	−.9449 (−1)	−.4211 (0)	−.3197 (−3)	−.2521 (1)	−.1036 (−1)		
15	−.7036 (−2)	−.3636	−.1301	−.6774 (−4)	−.4624 (0)	−.1082 (−2)		
16	−.3612	−.1730	−.5487 (−1)	−.2195	−.1565	−.2549 (−3)		

n \ $n'l'$	14,9	14,11	15,9	15,11	16,9	16,11		
11	.7592 (−4)	.5174 (−1)	.2576 (−4)	.1648 (−1)	.1118 (−4)	.6875 (−2)		
12	.1134 (−2)	.4052 (0)	.2804 (−3)	.1037 (0)	.1015 (−3)	.3833 (−1)		
13	.1944 (−1)	.3233 (1)	.2413 (−2)	.4913	.6408	.1458 (0)		
14	.0000	.0000	.3087 (−1)	.3093 (1)	.4164 (−2)	.5445		
15	−.2434 (1)	−.1976 (−1)	.0000	.0000	.4424 (−1)	.2986 (1)		
16	−.4882 (0)	−.2325 (−2)	−.2369 (1)	−.3149 (−1)	.0000	.0000		

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

TABLE 2—*continued*

$l = 11$

n \ $n'l'$	11,10	12,10	13,10	13,12	14,10			
12	−.3352 (1)*	.0000	.3319 (−2)	.4004 (1)	.2919 (−3)			
13	−.2370 (0)	−.3122 (1)	.0000	.0000	.9462 (−2)			
14	−.4724 (−1)	−.3700 (0)	−.2952 (1)	−.3342 (−2)	.0000			
15	−.1505	−.9468 (−1)	−.4486 (0)	−.2760 (−3)	−.2824 (1)			
16	−.6277 (−2)	−.3500	−.1331	−.5547 (−4)	−.4972 (0)			

n \ $n'l'$	14,12	15,10	15,12	16,10	16,12			
12	.2646 (0)	.6185 (−4)	.4990 (−1)	.2004 (−4)	.1518 (−1)			
13	.3746 (1)	.9882 (−3)	.4198 (0)	.2327 (−3)	.1025 (0)			
14	.0000	.1804 (−1)	.3552 (1)	.2123 (−2)	.5150			
15	−.9564 (−2)	.0000	.0000	.2875 (−1)	.3402 (1)			
16	−.9441 (−3)	−.2726 (1)	−.1830 (−1)	.0000	.0000			

$l = 12$

n \ $n'l'$	12,11	13,11	14,11	14,13	15,11	15,13	16,11	16,13
13	−.3683 (1)	.0000	.3075 (−2)	.4337 (1)	.2539 (−3)	.2690 (0)	.5104 (−4)	.4812 (−1)
14	−.2434 (0)	−.3447 (1)	.0000	.0000	.8799 (−2)	.4074 (1)	.8686 (−3)	.4329 (0)
15	−.4590 (−1)	−.3862 (0)	−.3267 (1)	−.3093 (−2)	.0000	.0000	.1683 (−1)	.3872 (1)
16	−.1396	−.9434 (−1)	−.4738 (0)	−.2407 (−3)	−.3130 (1)	−.8880 (−2)	.0000	.0000

$l = 13$

n \ $n'l'$	13,12	14,12	15,12	15,14	16,12	16,14		
14	−.4015 (1)	.0000	.2864 (−2)	.4670 (1)	.2229 (−3)	.2729 (0)		
15	−.2491 (0)	−.3773 (1)	.0000	.0000	.8222 (−2)	.4403 (1)		
16	−.4455 (−1)	−.4009 (0)	−.3585 (1)	−.2879 (−2)	.0000	.0000		

$l = 14$

n \ $n'l'$	14,13	15,13	16,13	16,15
15	−.4348 (1)	.0000	.2681 (−2)	.5003 (1)
16	−.2540 (0)	−.4099 (1)	.0000	.0000

$l = 15$

n \ $n'l'$	15,14	16,14	
16	−.4680 (1)	.0000	

*The number in parentheses indicates the power of ten multiplying the entry and succeeding entries below.

IV. BOUND-BOUND TRANSITIONS

For completeness we also calculated the bound-bound (line) oscillator strengths.[1] The (dimensionless) dipole oscillator strength is given by

$$f_{nl}^{n',\, l-1} = \frac{1}{3} \frac{\max(l,\, l')}{2l+1} \left(\frac{1}{n'^2} - \frac{1}{n^2}\right) \left(\frac{Z}{a_0}\, \tau_{nl}^{n',\, l'}\right)^2, \tag{41}$$

with the matrix elements defined by equation (35).

Since the matrix elements are real polynomials, the evaluation is straightforward. Results are given in Table 2 for all dipole transitions between levels up to and including those in the $n = 16$ shell.

APPENDIX A

Evaluation of Free-free Matrix Element G_l

We consider the real function,

$$G_l = \left| \frac{\eta_i - \eta_f}{\eta_i + \eta_f} \right|^{i\eta_i + i\eta_f} {}_2F_1\left[l+1 - i\eta_f,\, l+1 - i\eta_1;\, 2l+2; -\frac{4\eta_i\eta_f}{(\eta_i - \eta_f)^2} \right], \tag{A.1}$$

where

$${}_2F_1(a,\, b;\, c;\, x) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\,\Gamma(b+m)\,\Gamma(c)}{\Gamma(c+m)\,\Gamma(a)\,\Gamma(b)} \frac{x^m}{m!}.$$

Let

$$x = -\frac{4\eta_i\eta_f}{(\eta_i - \eta_f)^2},$$

and thus

$$1 - x = \left(\frac{\eta_i + \eta_f}{\eta_i - \eta_f}\right)^2. \tag{A.2}$$

Let

$$a = l+1 - i\eta_f\,, \qquad b = l+1 - i\eta_i\,, \qquad c = 2l+2\,, \qquad d = \tfrac{1}{2}(a+b-c)\,.$$

Then

$$G = (1-x)^d F(a,\, b;\, c;\, x)\,, \tag{A.3}$$

or

$$F = (1-x)^{-d} G\,. \tag{A.4}$$

The hypergeometric function, F, satisfies the differential equation

$$x(1-x)F'' + [c - (a+b+1)x]F' - abF = 0\,; \tag{A.5}$$

so G satisfies

$$\begin{aligned} &x(1-x)^2 G'' + (1-x)[c + x(2d - a - b - 1)]G' \\ &\qquad + \{x[d^2 + ab - d(a+b)] - ab + dc\}G = 0\,, \end{aligned} \tag{A.6}$$

[1] After these calculations were completed, extensive tabulations of the squares of radial matrix elements were published which cover the same range of levels (Green, Rush, and Chandler 1957). The present results are in complete agreement with this earlier work.

Substituting for a, b, c, d, we have

$$x(1-x)^2 G'' + (1-x)[2l+2-x(2l+3)]G' + \left\{x\left[\left(\frac{\eta_i-\eta_f}{2}\right)^2 + (l+1)^2\right] + \eta_1\eta_2 - (l+1)^2\right\}G = 0\,. \tag{A.7}$$

The regular solution to this differential equation can be expressed in the usual manner as a power series. The result is

$$G = \sum_{n=0}^{\infty} a_n x^n\,, \tag{A.8}$$

with

$$a_0 = 1\,, \qquad a_1 = \frac{(l+1)^2 - \eta_i\eta_f}{2l+2}\,,$$

$$a_n = \frac{1}{n(2l+1+n)}\left\{a_{n-1}[(n-1)(2n+4l+1) + (l+1)^2 - \eta_i\eta_f] - a_{n-2}\left[(n-2)(n+2l) + (l+1)^2 + \frac{\eta_i\eta_f}{2}\right]^2\right\},$$

and

$$x = -\frac{4\eta_i\eta_f}{(\eta_i-\eta_f)^2}\,. \tag{A.9}$$

For $|x| > 1$ we find the series solution of the differential equation in terms of a series in inverse powers. With $y = -1/x$, we can write

$$G = y^{l+1}[\cos(\lambda \ln y)A(y) + \sin(\lambda \ln y)B(y)]\,, \tag{A.10}$$

where

$$\lambda = \tfrac{1}{2}(\eta_i - \eta_f)$$

and A and B are regular series in y. Substituting into the differential equation, we obtain two coupled differential equations:

$$y(1+y)^2 A'' + (1+y)(1+2y)A' + 2\lambda(1+y)^2 B' - (1+y)(l^2+l+\lambda^2)A - (\lambda^2+\eta_i\eta_f)A + \lambda(1+y)B = 0$$

and

$$y(1+y)^2 B'' + (1+y)(1+2y)B' - 2\lambda(1+y)^2 A' - (1+y)(l^2+l+\lambda^2)B - (\lambda^2+\eta_i\eta_f)B - \lambda(1+y)A = 0\,. \tag{A.11}$$

As before, the solutions are of the form

$$A = \sum_{n=0}^{\infty} a_n y^n\,, \qquad B = \sum_{n=0}^{\infty} b_n y^n\,, \tag{A.12}$$

with

$$a_0 = 2Re\left[\frac{\Gamma(2l+2)\Gamma(i\eta_f - i\eta_i)}{\Gamma(l+1-i\eta_i)\Gamma(l+1+i\eta_f)}\right],$$

$$b_0 = -2Im\left[\frac{\Gamma(2l+2)\Gamma(i\eta_f - i\eta_i)}{\Gamma(l+1-i\eta_i)\Gamma(l+1+i\eta_f)}\right],$$

and with the following recursion relations:

$$n(4\lambda^2+n^2)\,a_n = -\lambda(3n-2+2\alpha)\,b_{n-1} - \lambda(3n-4+2\beta)\,b_{n-2}$$
$$+\,[n\alpha - n(n-1)(2n-1) - 2\lambda^2(4n-3)]\,a_{n-1}$$
$$+\,[n\beta - n(n-1)(n-2) - 2\lambda^2(2n-3)]\,a_{n-2}$$

and

$$n(4\lambda^2+n)\,b_n = \lambda(3n-2+2\alpha)\,a_{n-1} + \lambda(3n-4+2\beta)\,a_{n-2}$$
$$+\,[n\alpha - n(n-1)(2n-1) - 2\lambda^2(4n-3)]\,b_{n-1} \quad \text{(A.13)}$$
$$+\,[n\beta - n(n-1)(n-2) - 2\lambda^2(2n-3)]\,b_{n-2}\,,$$

where

$$\alpha = l(l+1) + 2\lambda^2 + \eta_i\eta_f\,, \qquad \beta = l(l+1) + \lambda^2\,, \qquad \lambda = \tfrac{1}{2}(\eta_i - \eta_f)\,.$$

APPENDIX B

Evaluation of Bound-bound Matrix Element

We here rederive Gordon's (1929) expression for the dipole matrix element between bound states, paying particular attention to those factors for which an ambiguity might result in the continuation to the bound-free state:

$$r_{nl}^{n'l-1} = \int r^2 dr\,\psi_{nl}(r)\; r\,\psi_{n'l-1}(r)$$
$$= \left\{\left(\frac{2Z}{a_0}\right)^3 \frac{1}{2(nn')^2}\left[\frac{\Gamma(n+l+1)\,\Gamma(n'+l)}{\Gamma(n-l)\,\Gamma(n'-l+1)}\right]^{1/2} \frac{1}{\Gamma(2l)\,\Gamma(2l+2)}\right\}$$
$$\times \int r^3 dr\, e^{-Zr/a_0 n} e^{-Zr/a_0 n'} \left(\frac{2Zr}{a_0 n}\right)^l \left(\frac{2Zr}{a_0 n'}\right)^{l-1} \quad \text{(B.1)}$$
$$\times\, {}_1F_1\left(l+1-n;\, 2l+2;\, \frac{2Zr}{a_0 n}\right) {}_1F_1\left(l-n';\, 2l;\, \frac{2Zr}{a_0 n'}\right).$$

For arbitrary n and n' this integral does not converge because the hypergeometric functions, in general, grow exponentially in their argument for large r. Specifically,

$$ {}_1F_1(a;\, c;\, x) \sim \frac{\Gamma(c)}{\Gamma(a)}\, e^x x^{a-c} + \frac{\Gamma(c)}{\Gamma(c-a)}\, e^{\pm i\pi a} x^{-a} \quad \text{(B.2)}$$

as

$$x \to \infty\,, \qquad -\pi < \arg x < \pi\,.$$

For $a = 0$ or a negative integer, the first term vanishes, and the behavior is that of a polynomial. Thus, if $l + 1 - n$ is 0 or a negative integer, the integrand behaves for large r like

$$r^{1+n-n'} e^{Zr/a_0 n'} e^{-Zr/a_0 n}\,,$$

and the region of convergence is $Re(1/n - 1/n') > 0$. This is the region in the complex n'-plane outside the circle of radius $n/2$ centered at $n' = n/2$. In particular, therefore, the integral exists for pure imaginary $n' = i\eta$.

To calculate the integral, we need the result (Erdelyi *et al.* 1953)

$$\int_0^\infty e^{-st} t^{c-1}\, {}_1F_1(a;\, c;\, t)\, {}_1F_1(a';\, c{:}\; \lambda t)\, dt$$
$$= \Gamma(c)(s-1)^{-a} s^{a+a'-c} F\left[a,\, a';\, c;\, \frac{\lambda}{(s-1)(s-\lambda)}\right], \quad \text{(B.3)}$$

where F is the ordinary hypergeometric function. The region of validity for this formula indicated is $Re\, c > 0$, $Re\, s > \lambda + 1$; however, for the case $a = 0$ or a negative integer, it is readily extended to $Re\, s > Re\, \lambda$.

Thus, if we can convert, say, ${}_1F_1(l + 1 - n, 2l + 2, 2Zr/a_0n)$ to some linear sum of terms of the form $r^{-3}\, {}_1F_1(a; 2l; 2Zr/a_0n)$, we can evaluate the integral with this formula. To do this we use (three times) the relation (Erdelyi *et al.* 1953)

$$x\,{}_1F_1(a;\ c+1;\ x) = c\,[{}_1F_1(a;\ c;\ x) - {}_1F_1(a-1;\ c;\ x)] \tag{B.4}$$

and once the relation

$$(c-1)\,{}_1F_1(a;\ c-1;\ x) = a\,{}_1F_1(a+1;\ c;\ x) + (c-a-1)\,{}_1F_1(a;\ c;\ x) \tag{B.5}$$

to obtain the identity

$$\begin{aligned}{}_1F_1(l+1-n;\ 2l+2;\ x) &= 2l(2l+1)\,x^{-3}\,[(l+1-n)\,{}_1F_1(l+2-n;\ 2l;\ x) \\ &\quad + (4n-2l-2)\,{}_1F_1(l+1-n;\ 2l;\ x) - 6n\,{}_1F_1(l-n;\ 2l;\ x) \\ &\quad + (4n+2l+2)\,{}_1F_1(l-1-n;\ 2l;\ x) - (l+1+n)\,{}_1F_1(l-2-n;\ 2l;\ x)]\,.\end{aligned} \tag{B.6}$$

We next evaluate the integral, using equation (B.3), and find a sum of five hypergeometric functions with first parameters $l + 2 - n$, $l + 1 - n$, $l - n$, $l - 1 - n$, and $l - 2 - n$.

We now make use of the relation

$$a(1-x)\,F(a+1) + [c - 2a - (b-a)\,x]\,F - (c-a)\,F(a-1) = 0\,, \tag{B.7}$$

which, because of the particular coefficients involved, simplifies the sum of five terms to a sum of two hypergeometric functions.

The result of these operations is equation (35) for the matrix element $\tau_{nl}^{n'\,l-1}$.

APPENDIX C

Evaluation of Bound-free Matrix Element G_l

Here we derive the bound-free matrix element and put it into a simplified real form.

We wish to extend formula (35) to the case $n' = i\eta$. To do this, we note first that the integral exists and is defined for this case, as are the hypergeometric and elementary functions on the right side of the equation. Thus we need only substitute $n' = i\eta$.

First we look at the behavior of the terms

$$(n+n')^{-n-n'}\,(n'-n)^{n'-l}\,(n-n')^{n-2-l}$$

for $n' \to i\eta$. As long as n and l are integers, it does not matter whether $n' \to \eta e^{i\pi/2}$ or $n' \to \eta e^{-3\pi i/2}$. The result differs only by a factor $e^{2\pi i(l+n)}$. We find, for the product,

$$(n^2+\eta^2)^{-l-1}\,e^{2\eta(\theta-\pi/2)}\,e^{-i\pi l}\,e^{2i\theta(l+1-n)}\,,$$

where $\theta = \tan^{-1}\eta/n$, $0 < \theta < \pi/2$.

After substituting $n' = i\eta$ into the formula for $\tau_{nl}^{n'\,l-1}$, simplifying the complex gamma functions, and multiplying by the factor of equation (34), we find

$$\begin{aligned}\tau_{nl}^{E,\,l-1} &= \frac{a_0}{Z}\,\frac{2^{2l}}{(2l-1)!}\left[\frac{(n+l)!}{(n-l-1)!}\right]^{1/2}\left[\frac{\pi\eta^2\exp(-4\eta\cot^{-1}\rho)}{2k^3(1-e^{-2\pi\eta})}\right]^{1/2} \\ &\quad \times \{(1^2+\eta^2)(2^2+\eta^2)\ldots[(l-1)^2+\eta^2]\}\,\rho^{l+1}(1+\rho^2)^{-n} \\ &\quad \times \Big[\!\Big[\,[(1-i\rho)^{2n-2l-2}]\,\Big\{F\left[l+1-n,\ l-i\eta;\ 2l;\ -\frac{4i\rho}{(1-i\rho)^2}\right] \\ &\quad - \left(\frac{1-i\rho}{1+i\rho}\right)^2 F\left[l-1-n,\ l-i\eta;\ 2l;\ -\frac{4i\rho}{(1-i\rho)^2}\right]\Big\}\Big]\!\Big]\,,\end{aligned} \tag{C.1}$$

where $\rho = \eta/n$ and the first expression in braces is set equal to 1 if $l = 1$.

We first show that the last expression in double brackets is real, by using the transformation

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right). \tag{C.2}$$

Here

$$z = -\frac{4i\rho}{(1-i\rho)^2}, \qquad 1-z = \left(\frac{1+i\rho}{1-i\rho}\right)^2, \qquad \frac{z}{z-1} = \frac{4i\rho}{(1+i\rho)^2} = z^*, \tag{C.3}$$

and

$$\begin{aligned} H = {} & (1-i\rho)^{2n-2l-2}\left\{F\left[l+1-n, l-i\eta; 2l; -\frac{4i\rho}{(1-i\rho)^2}\right]\right. \\ & \left. -\left(\frac{1-i\rho}{1+i\rho}\right)^2 F\left[l-1-n, l-i\eta; 2l; -\frac{4i\rho}{(1-i\rho)^2}\right]\right\} \\ = {} & (1-i\rho)^{2n-2l-2}\left\{\left(\frac{1+i\rho}{1-i\rho}\right)^{2n-2l-2} F\left[l+1-n, l+i\eta; 2l; \frac{4i\rho}{(1+i\rho)^2}\right]\right. \\ & \left. -\left(\frac{1-i\rho}{1+i\rho}\right)^2\left(\frac{1+i\rho}{1-i\rho}\right)^{2n-2l+2} F\left[l-1-n, l+i\eta; 2l; \frac{4i\rho}{(1+i\rho)^2}\right]\right\} \\ = {} & (1+i\rho)^{2n-2l-2}\left\{F\left[l-1-n, l+i\eta; 2l; \frac{4i\rho}{(1+i\rho)^2}\right]\right. \\ & \left. -\left(\frac{1+i\rho}{1-i\rho}\right)^2 F\left[l-1-n, l+i\eta; 2l; \frac{4i\rho}{(1+i\rho)^2}\right]\right\} = H^*. \end{aligned} \tag{C.4}$$

Thus H is real. Furthermore, each of the two terms in H is real, and a polynomial in ρ. So let

$$G_l(-m; \eta; \rho) = \sum_{s=0}^{2m} b_s \rho^s \equiv (1-i\rho)^{2m} F\left[-m, l-i\eta; 2l; -\frac{4i\rho}{(1-i\rho)^2}\right]. \tag{C.5}$$

To obtain the coefficients of the polynomial in the simplest way, we look for the regular solution of the differential equation in ρ satisfied by G.

With the variable $z = -[4i\rho/(1-i\rho)^2]$, F satisfies the hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2} + [2l - (l+1-m-i\eta)z]\frac{dF}{dz} + m(l-i\eta)F = 0. \tag{C.6}$$

We transform to the variable ρ and find the differential equation satisfied by G_l:

$$\begin{aligned} & \rho(1+\rho^2)\frac{d^2G_l}{d\rho^2} + 2[l + 2\eta\rho + \rho^2(1-l-2m)]\frac{dG_l}{d\rho} \\ & \qquad - 2m[2\eta + \rho(1-2m-2l)]G_l = 0. \end{aligned} \tag{C.7}$$

The polynomial solution of this equation is

$$G_l(-m, \eta, \rho) = \sum_{s=0}^{2m} b_s \rho^s,$$

with

$$b_0 = 1\,, \qquad b_1 = \frac{2m\eta}{l}\,,$$

$$b_s = -\frac{1}{s(s+2l-1)}\,[4\eta(s-1-m)\,b_{s-1} + (2m+2-s)(2m+2l+1-s)\,b_{s-2}]\,. \tag{C.8}$$

An interesting feature of the solution is that the coefficients are antisymmetric about the middle term; that is,

$$b_{m+p} = (-)^p b_{m-p}\,. \tag{C.9}$$

This is most easily seen by investigating the differential equation satisfied by $\rho^{-m} G_l$ and noting that it is invariant to the change of variable $\rho \rightarrow -1/\rho$. Thus $\rho^{-m}G = I(\rho) = I(-1/\rho)$. As a result, only half of the coefficients of the polynomial need be computed from the recurrence relation.

An exactly analogous procedure is used for the case $n, l \rightarrow E, l+1$, and the same polynomials are involved in the solution. The results are quoted in equations (36) and (37).

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