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OBSERVATIONAL APPROACH TO EVOLUTION. III. SEMIEMPIRICAL EVOLUTION TRACKS FOR M67 AND M3

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ABSTRACT

A method is presented for obtaining the tracks of evolution for individual stars in the subgiant and giant regions of color-magnitude diagrams of star clusters. The method utilizes the evolutionary information contained in the observed luminosity functions and color-magnitude diagrams of clusters. It is applied to the galactic cluster M67 and to the globular cluster M3. The evolutionary tracks, the time scale for evolution along these tracks, and the fraction of the total mass exhausted of hydrogen have been computed for both clusters, and the results are tabulated in Tables 3–10. The computed fraction of the mass exhausted of hydrogen for stars in M3 is compared with the theoretical predictions of the Hoyle-Schwarzschild (H-S) models. Fair agreement is obtained It is shown that the H-S models are capable of predicting nearly the correct luminosity function for M3 except at the very top of the giant sequence. The time taken for stars to evolve along the horizontal branch in M3 from $B - V = 0.50$ to $B - V = -0.10$ is found to be $2\ 3 \times 10^8$ years. The lifetime of the RR Lyrae phase of the evolution is 8×10^7 years. The expected rate of change of the period of the RR Lyrae stars due to this evolution is $\Delta t/t = 2.4 \times 10^{-11}$, or 0.1 second per century, which is about a factor of 5 below the limit of detectability with the available data. The observed period changes for RR Lyrae stars in M3 average twenty times this value and are believed to be due to causes other than evolution.

With certain assumptions, knowledge of the color-magnitude (C-M) diagram and the luminosity function for a cluster of stars is sufficient to determine (1) the evolutionary tracks in the M_{bol}, $\log T_e$ plane for the individual member stars; (2) the time scale for traversing these tracks; and (3) the fraction of the mass exhausted of hydrogen at each evolutionary stage. The semiempirical procedure necessary to arrive at these results was developed following a basic suggestion by Schwarzschild in 1954, and I should like to acknowledge this discussion here. The method was first applied to M3, and a partial summary of the results was given in a communication to the Liège symposium on nuclear processes in the stars (Sandage 1954*a*). The purpose of the present paper is (1) to describe the procedure in detail, (2) to apply it to the galactic cluster M67, (3) to present the complete results for the globular cluster M3, and (4) to compare the evolutionary tracks for M67 with those of M3.

I. OBSERVED AND ORIGINAL LUMINOSITY FUNCTION FOR M67

The observed luminosity function $\phi(M_v)$ for M67 has been investigated both by van den Bergh (1957) and by Sandage (1957*a*, Paper I). The results are in satisfactory agreement. Table 1 gives the adopted $\phi(M_v)$. The M67 $\phi(M_v)$ differs greatly from the van Rhijn function valid for the general field. It has a maximum at $M_v = +4$ and then declines rapidly toward zero at about $M_v = +9$. Van den Bergh interprets the difference at the faint end to be the result of the escape of stars of low mass from the cluster in earlier times. By successive application of Chandrasekhar's (1942) evaporation ratio, $QT(m)/T(\bar{m})$, to an original luminosity function (assumed to be of the van Rhijn type), van den Bergh has shown that the peculiar form of the present-day $\phi(M_v)$ for M67 stars fainter than $M_v = +4$ could be produced by this preferential escape in 180 relaxation times. This is about 4×10^9 years. The lack of stars brighter than $M_v = 0$ is interpreted to be the effect of stellar evolution for main-sequence stars brighter than $M_v = +4$.

To obtain the evolutionary tracks from luminosity functions requires knowledge not only of the present $\phi(M)$ but also of the luminosity distribution as it would be if evolution were not present. We have seen in Paper I that, in the absence of preferential escape, this initial distribution is probably given by Salpeter's $\psi(M_v)$ function (Salpeter 1955). In clusters that are older than several relaxation times, the unevolved distribution will be $f(M_v)\psi(M_v)$, where $f(M_v)$ is the evaporation ratio. The $f(M_v)$'s after 180 relaxation times computed by van den Bergh for the specific case of M67 are tabulated in

TABLE 1

OBSERVED $\phi(M_v)$ FOR M67 AND SALPETER'S INITIAL $\psi(M_v)$ CORRECTED FOR EVAPORATION OF STARS IN 180 RELAXATION TIMES

M_v	M67 $\phi(M_v)$	f	$f\psi$	M_v	M67 $\phi(M_v)$	f	$f\psi$
−0 25–+0 25	2 0	1 000	24 0	+4 75–+5 25	29 6	0 691	50 3
+0 25–+0 75	3 3	1 000	27 0	+5 25–+5 75	23 6	0 538	44 5
+0 75–+1 25	4 6	1 000	30 7	+5 75–+6 25	19 4	0 350	33 0
+1 25–+1 75	6 4	1 000	34 7	+6 25–+6 75	16 1	0 225	23 0
+1 75–+2 25	9 0	1 000	39 0	+6 75–+7 25		0 127	13 6
+2 25–+2 75	15 6	1 000	43 5	+7 25–+7 75		0 075	8 3
+2 75–+3 25	32 6	1 000	48 0	+7 75–+8 25		0 045	5 1
+3 25–+3 75	39 2	1 000	53 4	+8 25–+8 75		0 032	3 8
+3 75–+4 25	40 6	0 910	53 6	+8 75–+9 25		0 017	2 2
+4 25–+4 75	35 6	0 810	53 3				

Table 1. Also tabulated are the $f\psi$ values, where $\psi(M_v)$ has been taken from Table 2 of Paper I. These have been normalized by the condition.

$$\sum_{M_i=6\ 5}^{M_i=3\ 5} \phi\,(M_i) = \sum_{M_i=6\ 5}^{M_i=4.5} f\,(M_i)\,\psi\,(M_i)\,. \tag{1}$$

This condition expresses the fact that, because of evolution away from the main sequence, all stars which were originally contained along the main sequence from $M_v = 6.5$ to $M_v = 4.5$ are now contained between 6.5 and 3.5 on the observed C-M diagram if we adopt the Schönberg-Chandrasekhar (S-C) models (1942) for the first stages of the evolution. It should be remarked that although further development of the physics of the theory of evolution has produced models which differ from those of Schönberg and Chandrasekhar, all give nearly the same results concerning the C-M diagrams. It would therefore seem that a close approximation to the truth can be obtained by using the S-C models as the basis of the initial evolution.

Figure 1 shows the observed $\phi(M_v)$ and the initial $f(M_v)\psi(M_v)$ normalized according to equation (1). The difference between ϕ and $f\psi$ is due entirely to the effects of evolution, and use of this difference together with the C-M diagrams allows the evolutionary tracks in the subgiant and giant regions to be constructed.

II. SEMIEMPIRICAL TRACKS FOR M67

To determine the evolutionary tracks over the face of the M_{bol}, log T_e plane requires knowledge of the early stages of the evolution close to the main sequence. The S-C models provide the beginning evolutionary history of main-sequence stars until a limiting configuration is reached where the star has consumed all its hydrogen in an inner core containing a fraction, q_L, of the total mass. This limiting fraction is $q_L = 0.12$ for a jump of mean molecular weight of 2 between the core and the envelope. The fraction

q_L varies with the ratio of the mean molecular weights but in such a way that Xq_L maintains a nearly constant value of 0.07, where X is the fractional initial hydrogen abundance (see Sandage and Schwarzschild 1952, p. 474, for a discussion of the approximate constancy of Xq_L for the S-C models). The tracks of evolution from the main sequence to the S-C limit are obtained from the data in Table 2 of Schönberg and Chandrasekhar's paper (1942) and are shown in Figure 2 extending from the line labeled "original main sequence" to the line marked "*SC*." The appendix to the present paper gives the details of these tracks.

The heavy line in Figure 2 cutting across these tracks represents the observed C-M diagram for M67 (Johnson and Sandage 1955) but converted to the M_{bol}, log T_e plane

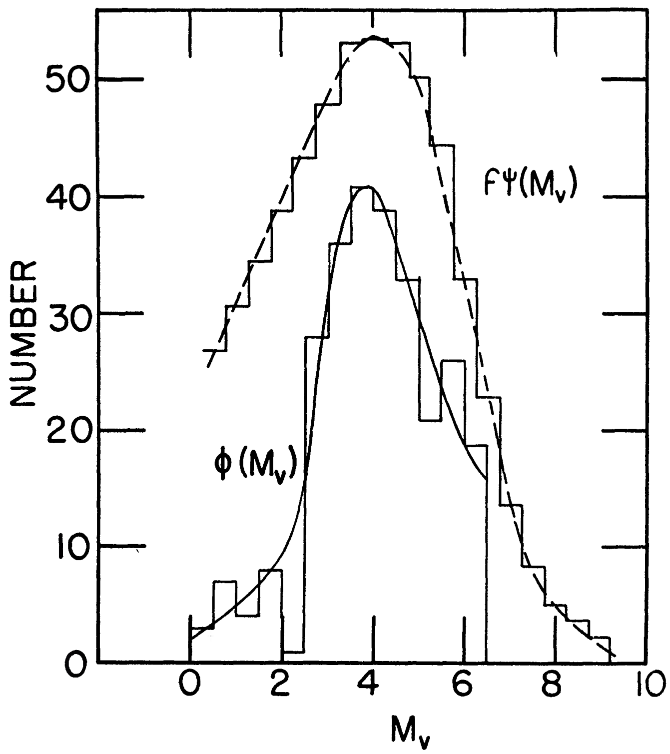

FIG. 1.—The observed luminosity function for M67 is shown as a solid line and is marked $\phi(M_v)$. The Salpeter original luminosity function corrected for the effect of the escape of stars of small mass is shown as a dashed line and is marked $f\psi(M_v)$.

by the usual procedure of going through the $B - V = f(\text{sp. type}) = g(T_e)$ relations of Johnson and Morgan (1953) and Morgan and Keenan (1951) and the $\Delta M_{\text{bol}} = h(\text{sp. type})$ relation of Kuiper (1938). Massive stars evolve rapidly along the appropriate paths because of their higher luminosity, and they travel farther toward the S-C limit in the lifetime of the cluster than do faint stars, which evolve more slowly. The sharp turnoff from the main sequence in the observed C-M diagram is usually identified in this theory with stars which have just reached the S-C limit in the age of the cluster. This limiting track is labeled *1* in Figure 2.

It is clear that use of the theoretical tracks to the S-C limit allows us to identify the points along the main sequence of the observed C-M diagram with points along the original main sequence from which the stars came. The first fourteen entries of Table 2 give the mapping data to the S-C limit computed from the details of the theory. For these computations the break point for M67 was assumed to be at $M_{\text{bol}} = +3.46$, and

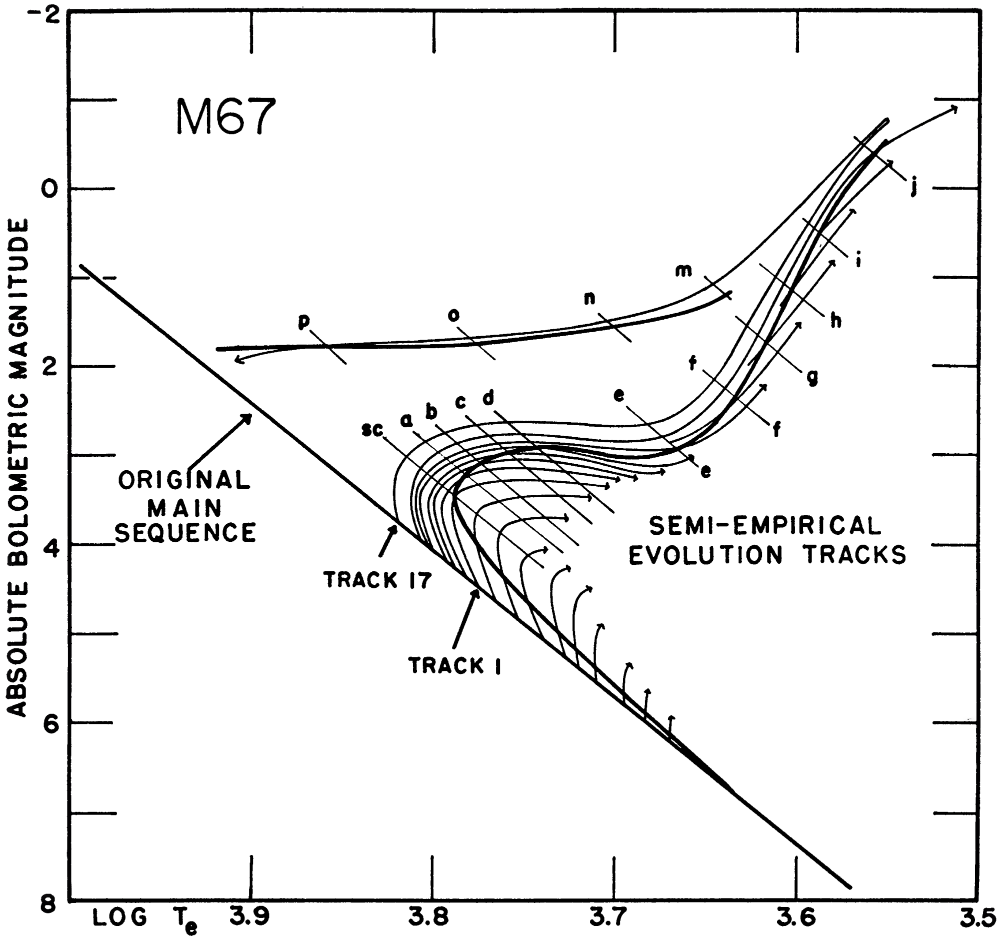

FIG. 2.—Semiempirical tracks of evolution for stars in M67. The various mapping points are shown as lines labeled *SC*, *a*, ... , *p*. The observed C-M diagram for M67 transformed to the M_{bol}, log T_e plane is shown as the heavy line cutting across the evolutionary tracks.

TABLE 2

MAPPING DATA FOR M67

M_{bol} on Original Main Sequence	M_{bol} on M67	Point on Fig 2	M_{bol} on Original Main Sequence	M_{bol} on M67	Point on Fig 2
+7 0	+7 000		+4 2	+3.15	*b*
+6 8	+6 797		+4 1.	+3 00	*c*
+6 6	+6 575		+4 08	+2 90	*d*
+6 4	+6 352		+4 06	+2 98	*e*
+6 2	+6 12		+4 04	+2 35	*f*
+6 0	+5 89		+4 02	+1 75	*g*
+5 8	+5 66		+4 00	+1 17	*h*
+5 6	+5 41		+3 98	+0 55	*i*
+5 4	+5 16		+3 96	−0 34	*j*
+5 2	+4 90		+3 94	+0 42	*k*
+5 0	+4 62		+3 92	+0 60	*l*
+4 8	+4 30		+3 90	+1 20	*m*
+4 6	+3 94		+3 88	+1 55	*n*
+4 4	+3 46	*SC*	+3 86	+1 74	*o*
+4 3	+3 26	*a*	+3 84	+1 78	*p*

TABLE 3

M_{bol} FOR SELECTED POINTS ALONG 17 EVOLUTIONARY TRACKS FOR STARS IN M67

Track No.	Main Seq.	S-C	M_a	M_b	M_c	M_d	M_e	M_f	M_g	M_h	M_i	M_j	M_k	M_l	M_m	M_n	M_o	M_p
1.	4.4	3 46																
2.	4.3	3.36	3.26															
3	4.2	3.26	3.19	3.15														
4.	4.1	3.16	3 09	3 06	3 00													
5.	4.08	3.14	3 06	3 04	2.98	2.90												
6.	4.06	3.12	3 04	3 02	2.96	2.88	2.98											
7	4.04	3 10	3 02	3 00	2.94	2.87	2.96	2.35										
8.	4.02	3 08	3 00	2.98	2.92	2.85	2.94	2.34	1.75									
9	4.00	3 06	2.98	2.95	2.90	2.83	2.92	2.32	1.74	1.17								
10.	3.98	3 04	2.96	2.93	2.88	2.81	2.90	2.30	1 72	1.16	0.55							
11.	3.96	3 02	2.95	2.91	2.86	2.79	2.88	2.28	1.70	1 14	.53	−0.34						
12.	3.94	3 00	2.93	2.89	2.84	2.77	2.86	2.26	1.68	1.12	.51	− .36	0 42					
13	3.92	2.98	2.90	2.87	2.82	2.75	2.84	2.24	1.66	1 10	49	− .38	40	0.60				
14.	3.90	2.96	2.88	2.85	2.80	2.73	2.82	2.22	1 64	1.08	47	− 40	.38	.58	1.20			
15.	3.88	2.94	2.86	2.83	2.78	2.71	2.80	2.20	1.62	1 06	45	− 42	.36	.56	1.18	1.55		
16.	3.86	2.92	2.84	2.81	2.75	2.68	2.77	2.18	1.60	1.04	43	− 45	.34	.54	1.16	1.53	1.74	
17	3.84	2.90	2.82	2.79	2.73	2.66	2.75	2.15	1.58	1.02	0 41	−0 47	0.32	0.52	1.14	1.51	1.72	1.78

the increase in luminosity is 0.94 mag. from the main sequence to the S-C limit. These two numbers have been used previously for the normalization condition of equation (1).

The next step is to obtain the extension of the tracks beyond the S-C limit, which requires use of the luminosity functions. Counts in the two functions $\phi(M_v)$ and $f(M_v)\psi(M_v)$ show that there were as many stars between $M_{\text{bol}} = 4.4$ and $M_{\text{bol}} = 4.3$ along the original main sequence as there are now from points *SC* to *a* (Fig. 2) along the observed C-M diagram. Consequently, stars now at point *a* originated at $M_{\text{bol}} = +4.3$ on the main sequence. Similarly, there were equal numbers of stars between $M_{\text{bol}} = +4.3$ and $M_{\text{bol}} = 4.2$ on the main sequence as now exist between points *a* and *b* on the M67 subgiant sequence. Consequently, the end point of the evolutionary track for stars originally at $M_{\text{bol}} = +4.2$ is at point *b*, on the observed sequence. By this stepwise counting procedure, the end points of the various tracks of evolution are located on the observed C-M diagram. These are listed for M67 in Table 2, where the identification letters *a–p* are shown in Figure 2. After the end points have been located, the complete tracks are found by constructing a parallel line (this assumes homologous models) to

TABLE 4*

CO-ORDINATES OF EVOLUTIONARY TRACK NO. 17 IN M67

Point on Fig 2	M_{bol}	log T_e	Point on Fig 2	M_{bol}	log T_e
Main seq	+3 84	3 814	*h*	+1 02	3 612
SC	+2 90	3 820	*i*	+0 41	3 596
a	+2 82	3 810	*j*	−0 47	3 567
b	+2 79	3 798	*k*	+0 32	3 602
c	+2 73	3 778	*l*	+0 52	3 612
d	+2 66	3 754	*m*	+1 14	3 646
e	+2 75	3 686	*n*	+1 51	3 705
f	+2 15	3 646	*o*	+1 72	3 780
g	+1 58	3 622	*p*	+1 78	3 858

* To obtain co-ordinates of tracks *1–16*, use this table, Table 3, and the relation $\Delta \log T_e = -0\ 060\ \Delta M_{\text{bol}}$

the track found in the immediately preceding step and extending this line to meet the C-M diagram at the appropriate mapping point.

The detailed co-ordinates for seventeen tracks constructed in this manner are given in Tables 3 and 4. Table 3 lists the bolometric magnitude for the junction of each of the seventeen tracks with the homology invariant lines marked *SC*, *a*, *b*, . . . , *p*. Table 4 gives the M_{bol} and log T_e for track *17*. The log T_e for tracks *1–16* may be computed from the log T_e of track *17* by using $\Delta \log T_e = -0.060\ \Delta M_{\text{bol}}$, where ΔM_{bol} between each track and No. *17* is obtained from Table 3. This is the equation for the slope of the homology line.

The time for stars to travel along the various segments of the seventeen tracks may now be computed if we assume that the observed C-M diagram is the locus of constant time connecting all the evolutionary paths. The age of the cluster is determined by the time it takes stars to evolve from the main sequence to the Schönberg-Chandrasekhar limit (*SC* in Fig. 2) along track *1*. At line *SC*, track *1* intersects the C-M diagram where the cluster main sequence ceases to exist. This time is given by

$$T = 0.007\, \mathfrak{M}\, c^2 \int_{Xq=0}^{Xq=0\ 07} \frac{X\, d q_i}{L_i}, \tag{2}$$

where $\mathfrak{M}$ is the mass of stars which evolve along track *1* (i.e., the mass corresponding to $M_{\text{bol}} = +4.4$); X is the fractional hydrogen abundance; dq_i the fraction of the mass burned at the ith stage of the process toward the S-C limit; and L_i the corresponding luminosity. With $\mathfrak{M} = 1.2\ \mathfrak{M}_\odot$, with the details of the $dq = f(L)$ relation given by the S-C theory summarized in the appendix, and with the S-C limit taken as $M_{\text{bol}} = +3.46$ for M67, equation (2) gives $T = 5.1 \times 10^9$ years as the age of this cluster. This age is to be considered with caution because it depends rather critically on the details of the evolution to the main-sequence break point. Although, as we mentioned earlier, the various theoretical models which have been computed in recent years all give nearly the same C-M diagrams, they do not give nearly so good agreement on the time scales. The differences range to about a factor of 2. Although we shall adopt 5.1×10^9 years as the age of both M67 and M3, it must be emphasized that this particular value results from only one of the possible theoretical models. (The models of Hoyle and Schwarzschild 1955, for example, give about 6×10^9 years for this time.) Furthermore, the equality of ages for M3 and M67 is an assumption made merely for convenience of comparison. The distance moduli for these two clusters are not yet known accurately enough to decide the question of an age difference.

The times along the segments of each track beyond the S-C limit are found in the following way. It is assumed that stars take 5.1×10^9 years to travel from the main sequence to SC along track *1*. The time will be shorter for stars along track *2* because track *2* has the higher luminosity. If we assume homologous evolution and neglect the slight mass increase for stars along track *2* compared with *1*, the time along track *2* from the main sequence to SC will be $(5.1 \times 10^9)(\langle L_1 \rangle / \langle L_2 \rangle)$, where $\langle L_1 \rangle$ and $\langle L_2 \rangle$ are the mean luminosities along these intervals of the two tracks. Since data from Table 3 show that $\langle L_1 \rangle / \langle L_2 \rangle = 0.9124$ from the main sequence to SC, the time along track *2* to SC is 4.653×10^9 years. Now the total time for stars to reach the C-M diagram at point a along track *2* is 5.1×10^9 years. It therefore follows that the time for stars to go from SC to a on track *2* is 0.447×10^9 years. We may continue the process to track *3*. The ratio $(5.1 \times 10^9)(\langle L_1 \rangle / \langle L_3 \rangle)$ gives 4.243×10^9 years for stars along track *3* to reach SC from the main sequence. Similarly, the time from SC to a is $(0.447)(\langle L_{2,\, SC\text{-}a} \rangle / \langle L_{3,\, SC\text{-}a} \rangle) \times 10^9$ years $= 0.413 \times 10^9$ years. Therefore, from a to b along track *3* takes $(5.100 - 4.243 - 0.413) \times 10^9 = 0.444 \times 10^9$ years. Times along all the various segments of the seventeen evolutionary tracks have been obtained in this manner and are given in Table 5 for M67.

It is clear that these times are directly connected with the observed numbers of stars at each point on the M67 subgiant and giant sequences. If there are very few stars in any given segment of the evolved sequences, then the magnitude interval along the original main sequence from which these stars came was very small. And this means that the evolution is fast through the segment in question. Similarly, in regions where the observed number of stars is large, the time a star spends in these regions is long. This one-to-one correspondence between the evolutionary times and the number of stars along the sequences is shown by comparing the entries of Table 5 with the observed C-M diagram of M67 (Johnson and Sandage 1955, Fig. 3). Table 5 shows that there is an abrupt change in the computed times at point c. It is here that the number of stars per color interval becomes smaller in the observed diagram.

The preceding paragraph points up a major difficulty with this method of finding the evolutionary tracks by the use of the luminosity functions. There must be sufficient numbers of stars along the evolved sequences of any cluster in question to be statistically significant, because it is these numbers which determine the location of the mapping points a–p and the times along the segments. Although M67 is among the richest galactic clusters in the sky, the number of stars along its sequences is not quite sufficient to give precision results. For this reason the detailed results in Tables 2, 3, 4, and 5 should be taken with caution. The situation is somewhat better in M3 because of the larger total population.

TABLE 5

TIMES ALONG THE VARIOUS EVOLUTIONARY TRACKS FOR M 67 (TIMES ARE IN 10^9 YEARS)

Track No.	*MC–SC*	*SC–a*	*a–b*	*b–c*	*c–d*	*d–e*	*e–f*	*f–g*	*g–h*	*h–i*	*i–j*	*j–k*	*k–l*	*l–m*	*m–n*	*n–o*	*o–p*
1.	5.1000																
2.	4.6533	0.4467															
3	4.2429	4133	0 4438														
4.	3.8695	.3769	4067	0.4469													
5	3 7975	.3684	.3975	4387	0.0979												
6.	3 7281	.3616	.3902	4307	0961	0 0933											
7	3 6612	.3550	.3830	4248	0948	0921	0.0891										
8.	3.5941	.3485	.3761	4169	0931	0904	0880	0 0929									
9	3.5294	.3421	.3675	4075	0914	0887	0864	0917	0.0953								
10.	3 4647	.3359	.3608	4000	0897	0871	0848	0900	0941	0.0929							
11.	3 4000	.3314	.3558	.3927	0881	0855	0832	0884	0924	0912	0 0913						
12.	3.3377	.3253	.3493	.3855	0865	0839	0817	0868	0907	0895	0896	0.0935					
13	3.2776	.3178	.3414	.3784	0849	0824	0802	0851	0891	0879	0880	0918	0.0954				
14.	3.2177	.3120	.3351	.3714	0833	0809	0787	0836	.0875	0863	0864	0901	0937	0.0933			
15	3 1598	.3063	.3290	.3647	0818	0794	0773	0821	0859	0847	0848	0885	0920	0916	0.0921		
16.	3 1022	.3007	.3230	.3563	0796	0773	0756	0806	0843	0832	0827	.0864	0903	0849	0904	0.1025	
17	3 0448	0.2952	0.3171	0.3498	0 0782	0.0759	0.0738	0 0789	0 0828	0 0817	0 0812	0 0849	0 0887	0 0833	0.0887	0.1006	0.0944

III. SEMIEMPIRICAL TRACKS FOR M3

The intersection points of the various evolutionary tracks for stars in M3 with the observed C-M diagram are given in Table 6. These data were obtained from counts in the observed luminosity function (Sandage 1954*b*) and in an original luminosity function constructed to satisfy equation (1). At the time the computations were made (Sandage 1954*a*), the Salpeter function was not available, nor had the more recent dis-

TABLE 6

MAPPING DATA FOR M3

M_{bol} on Original Main Sequence	M_{bol} on Observed M3 Sequence	Point on Fig 3	M_{bol} on Original Main Sequence	M_{bol} on Observed M3 Sequence	Point on Fig 3
+7 0	+7 0		+4 6	+3 94	
+6 8	+6 797		+4 4	+3 46	*SC*
+6 6	+6 575		+4 3	+3 14	*a*
+6 4	+6 352		+4 2	+2 64	*b*
+6 2	+6 12		+4 1.	+1 83	*c*
+6 0	+5 89		+4 05	+1 03	*d*
+5 8	+5 66		+4 03	+0 34	*e*
+5 6	+5 41	.	+4 02	+0 06	*f*
+5 4	+5 16		+4 01	−0 46	*g*
+5 2	+4 90		+4 00	−1 19	*h*
+5 0	+4 62		+3 99	−2 27	*i*
+4 8	+4 30		+3 98	−4 00	*j*

TABLE 7

M_{bol} FOR SELECTED POINTS ALONG 11 EVOLUTIONARY TRACKS FOR STARS IN M3

Track No	Main Seq	*SC*	M_a	M_b	M_c	M_d	M_e	M_f	M_g	M_h	M_i	M_j
1	4 4	3 46										
2.	4 3	3 36	3 14									
3	4 2	3 26	3 04	2 65								
4	4 1	3 16	2 94	2 55	1 83							
5	4 05	3 11	2 89	2 50	1 78	1 03						
6	4 03	3 09	2 87	2 48	1 76	1 01	0 34					
7	4 02	3 08	2 86	2 47	1 75	1 00	33	0 06				
8	4 01	3 07	2 85	2 46	1 74	0 99	32	05	−0 46			
9	4 00	3 06	2 84	2 45	1 73	0 98	31	04	− 47	−1 19		
10	3 99	3 05	2 83	2 44	1 72	0 97	30	03	− 48	−1 20	−2 27	
11 .	3 98	3 04	2 82	2 42	1 71	0 96	0 29	0 02	−0 49	−1 21	−2 28	−4 00

cussions of the M3 luminosity function (Sandage 1957*a*, Paper I) and the C-M diagram (Johnson and Sandage 1956) been made. These refinements of the basic data, although important for other purposes, have a nearly negligible effect on the computed evolutionary tracks. Consequently, it was felt that a recomputation of the M3 tracks using these more modern data was unnecessary.

Eleven evolutionary tracks were constructed from the S-C limit to the top of the red giant branch. The M_{bol} at the intersections of each of the tracks with the homology lines SC, a, . . . , j are given in Table 7. These intersection points are shown in Figure 3, where the complete tracks are drawn. The co-ordinate data for track *11* are given in Table 8. The log T_e and M_{bol} from this table can be used as starting values to compute the complete co-ordinate data for tracks *1–10* by the use of the relation $\Delta \log T_e = -0.060\ \Delta M_{bol}$, where ΔM_{bol} may be obtained from the data of Table 7. Finally, the times taken for stars to evolve along the various segments are given in Table 9. Comparison of the M3 and M67 data is made by noting that Tables 6–9 for M3 correspond with Tables 2–5 for M67.

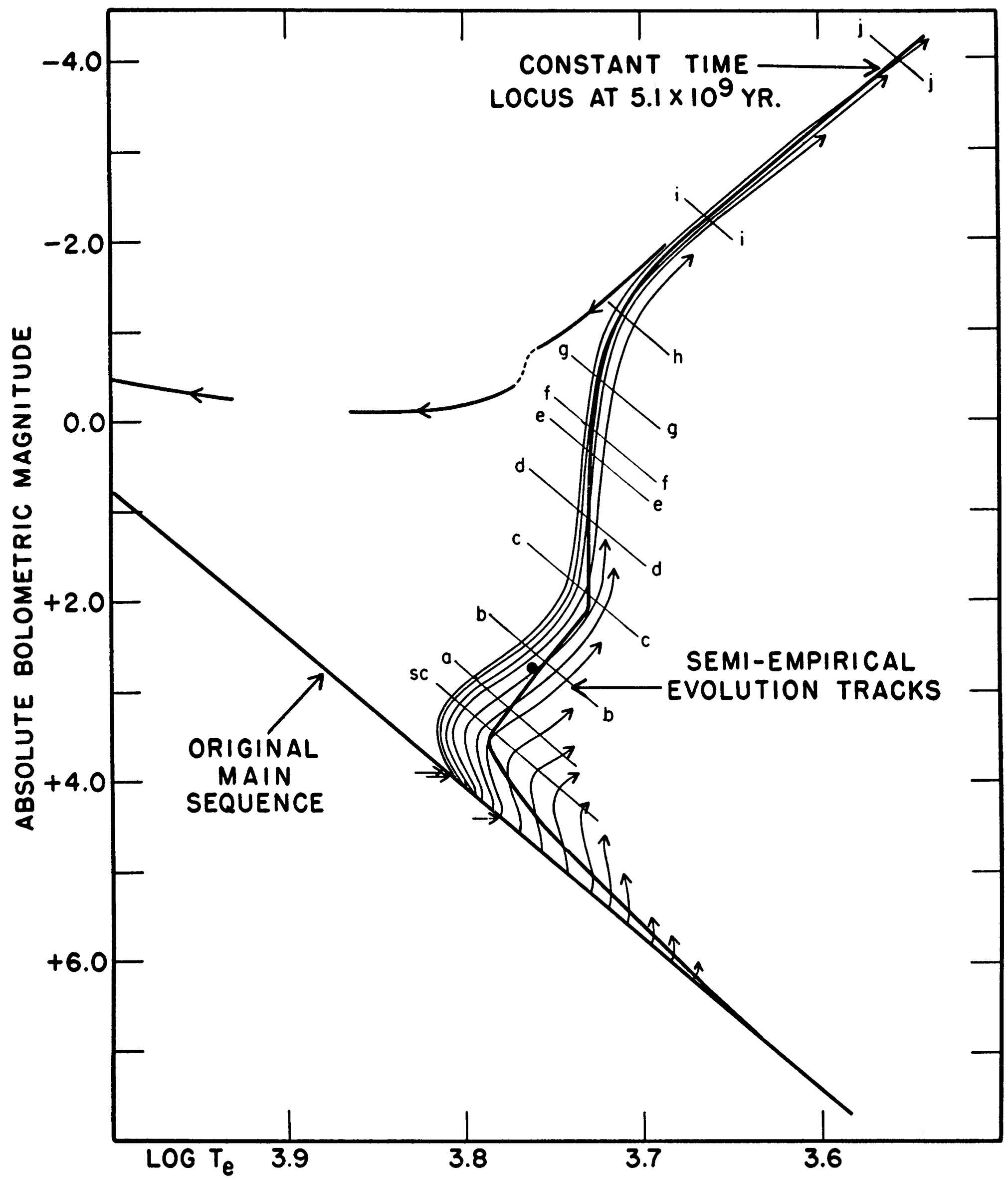

FIG. 3.—Semiempirical tracks of evolution for stars in M3

IV. DISCUSSION

a) Amount of Fuel Burned

It is of interest to compute the fraction of the star's mass which has been exhausted of hydrogen at each stage of the evolution of both clusters. The amount of hydrogen consumed while the star radiates L_i ergs/sec for a time Δt_i is

$$X\Delta q_i = 0.00955 \left(\frac{L_i}{\mathfrak{M}}\right) \Delta t_i , \tag{3}$$

TABLE 8*

CO-ORDINATES OF EVOLUTIONARY TRACK NO. 11 IN M3

Point on Fig 2	M_{bol}	log T_e	Point on Fig 2	M_{bol}	log T_e
Main seq.	+3 98	3 805	*e*	+0 29	3 737
SC	+3 04	3 808	*f*	+0 02	3 731
a	+2 82	3 796	*g*	−0 49	3 727
b	+2 43	3 769	*h*	−1 21	3 713
c	+1 71	3 741	*i*	−2 28	3 665
d	+0 96	3 735	*j*	−4 00	3 555

* To obtain co-ordinates of tracks *1–10* use this table, Table 7, and the relation Δ log T_e = −0 060 ΔM_{bol}

TABLE 9

TIMES ALONG THE VARIOUS EVOLUTIONARY TRACKS FOR M3 (TIMES ARE IN 10^9 YEARS)

Track No	*MS–SC*	*SC–a*	*a–b*	*b–c*	*c–d*	*d–e*	*e–f*	*f–g*	*g–h*	*h–i*	*i–j*
1	5 1										
2	4 653	0 447									
3	4 243	408	0 449								
4	3 869	372	410	0 450							
5	3 696	355	.391	430	0 228						
6	3 627	348	384	422	224	0 095					
7	3.594	345	380	.418	222	094	0 046				
8	3 561	342	377	414	.220	093	046	0 047			
9	3 529	339	373	410	218	092	045	047	0 046		
10	3 496	336	370	407	216	092	045	046	045	0 048	
11	3 465	0 333	0 367	0 403	0 214	0 091	0 044	0 046	0 045	0 048	0 046

where L and $\mathfrak{M}$ are expressed in solar units and Δt_i is in 10^9 years. Tables 3, 5, 7, and 9 give the data necessary to evaluate $X\Delta q_i$ for each segment of all evolutionary tracks. The fraction of the total hydrogen which has been consumed at each intersection point *SC*, *a*, . . . , *p* is found by summing the individual $X\Delta q_i$'s. The results are given in Table 10. These calculations suggest that 80 per cent of the available hydrogen has been consumed by the time the M3 stars reach point *j*. Comparison of these results for M3 with the preliminary theoretical values of q for the globular-cluster giant models of Hoyle and Schwarzschild (1955) is made in Table 10, where we have assumed $X = 0.9$. Here the theoretical q values are tabulated at our mapping points *SC–j*. The values at these points have been interpolated at the proper M_{bol}, log T_e from Tables 1 and 8 of Hoyle and Schwarzschild's paper. The "observed" and the theoretical q values agree tolerably well to about point h. From h to j our values are higher, which means that our time scale along this segment is longer by a factor of about 3 than that implied by the Hoyle-Schwarzschild (H-S) models. If the H-S time scale is correct from h to j, then there should be fewer stars here by a factor of about 3 than are actually observed. It is difficult to ascertain whether this difference between theory and observation is significant because of the several factors involved in the determination of time scales

TABLE 10

FRACTION OF MASS BURNED AT VARIOUS POINTS ALONG TRACKS IN M67 AND M3

M67		M3			
Point on Fig 2	Xq	Point on Fig 3	Xq	q for $X=0\ 9$	q (Hoyle and Schwarzschild)
SC	0 070	*SC*	0 070	0 078	0 13
a	082	*a*	083	092	15
b	096	*b*	102	113	18
c	112	*c*	137	152	20
d	116	*d*	174	193	25
e	120	*e*	204	227	28
f	125	*f*	226	251	29
g	133	*g*	259	288	31
h	148	*h*	.317	352	33
i	173	*i*	467	519	0 39
j	224	*j*	0 813	0 903	
k	279				
l	313				
m	336				
n	351				
o	364				
p	0 375				

by the method of this paper. These factors are the value of the original luminosity function from $M_{bol} = +4.00$ to $M_{bol} = +3.98$ and the number of stars on the observed giant sequence from $M_{bol} = -1.19$ to $M_{bol} = -4.00$. The numbers are so small that the time interval is not extremely well determined. Furthermore, a small change in the time interval causes a rather appreciable change in the computed q because of the high luminosity involved between points h and j. It is between h and j that our computed q values are least reliable.

The essential agreement of the q values to point h means that the time scale of the H-S models in this interval differs by at most a factor of 2 from that in Table 9. Turning the problem around, this means that the theoretical models are capable of predicting nearly the correct luminosity function along the evolving sequences at least to point h, and this is a very satisfactory feature of the H-S theory. The comparison of our values of q for M67 with those computed from the H-S theory for the "population I" giants cannot be made because the H-S models do not fit the observations.

We have seen that at the final point j on the globular-cluster giant branch the stars have burned out 80 per cent of their hydrogen. One hypothesis which has been proposed for the termination of the giant branch is that stars, upon reaching point j, have exhausted their hydrogen supply. With no hydrogen fuel left, these stars decline in luminosity and move along the horizontal branch toward the main sequence because of increased chemical homogeneity. The identification of the end of the giant sequence with the stage of nearly complete hydrogen exhaustion finds some support in the "observational" q values of M3. However, a different situation appears to exist for M67, where only 37 per cent of the hydrogen has been consumed at j. The M67 observations, as they stand, would therefore seem to require a more subtle reason than hydrogen exhaustion for the termination of the giant sequence. This is in line with the belief of Hoyle and Schwarzschild that the onset of helium-burning terminates the giant branch.

It must be admitted, however, that the observational basis for the rejection of fuel exhaustion as the sole cause for termination is uncertain because the exact end point of the M67 giant branch is unknown, owing to the few stars near $M_v = 0$. It is conceivable that stars in M67 evolve beyond $M_{\text{bol}} = -0.5$, $\log T_e = 3.55$, into the late K and early M types before coming back on a horizontal branch. But, since no late K and M stars are observed, we must require that the rate of evolution be fast in this region compared with the total number of stars in M67. However, for stars to exhaust 90 per cent of their total mass requires that $M_{\text{bol}} = -4.0$ must be reached. (This is in consequence of the fast time scale demanded by the lack of observed stars.) Such a high luminosity appears to be unlikely from consideration of the systematics of C-M diagrams for all clusters (Fig. 1; Sandage 1957*b*, Paper II). It thus seems probable that some agent other than hydrogen exhaustion is responsible for the termination of the giant sequence, such as the H-S hypothesis of helium-burning.

b) *Lifetime of the RR Lyrae Stars*

Table 9 does not contain the evolution times for stars on the horizontal branch in M3 or, in particular, the times in the RR Lyrae domain. These times are easy to estimate, however. There are about six hundred stars along the horizontal branch from $B - V = 0.50$ to $B - V = -0.10$, of which about two hundred are RR Lyrae variables. Extension of the mapping data of Table 6, together with counts at $M_{\text{bol}} = +3.95$ in the Salpeter function given in Paper I, normalized to M3, show that the six hundred horizontal-branch stars came from an interval of 0.05 mag. along the original main sequence from $M_{\text{bol}} = +3.98$ to $M_{\text{bol}} = +3.93$. The data of Tables 6 and 9 show that the time taken for stars to traverse the horizontal branch is therefore 2.3×10^8 years (i.e., the time for stars to move between successive mapping points is 4.6×10^7 years for an interval of 0.01 mag. along the original main sequence). The lifetime of an RR Lyrae star is about one-third this value because the length of the RR Lyrae domain is about one-third the total length of the horizontal branch. Consequently, the time spent in the RR Lyrae phase in M3 is about 8×10^7 years. Presumably, the RR Lyrae star enters the region of instability at $B - V = 0.39$, with a period of oscillation of about 1 day; increases in density as it evolves through the RR Lyrae domain; and leaves the region at $B - V = 0.17$, with a period of about 0.3 day. The rate of change of the period is of the order of 0.7 day per 8×10^7 years or $\Delta t/t = 2.4 \times 10^{-11}$. This amounts to about 0.1 second per 100 years, which is too small a change to be detected with confidence from present observational data.

The period changes which have been observed for selected RR Lyrae stars in M3 (Belserene 1952) average about twenty times larger than the expected evolutionary value. Furthermore, half the changes indicate increasing periods, while half show decreasing periods. On the evolutionary hypothesis, all periods should either be decreasing if the stars evolve from right to left in the C-M diagram or increasing if they evolve from left to right. It would therefore seem reasonable to assume that the observed period changes are due to unknown perturbations not directly connected with evolution. These perturbations may be periodic or abrupt, but in any case they effectively mask the much smaller evolutionary change.

It is a great pleasure to thank Dr. Martin Schwarzschild for the several discussions which led to the present method and for reading and commenting upon the manuscript.

APPENDIX

The details of the Schönberg-Chandrasekhar (1942) tracks of evolution in the M_{bol}, log T_e plane are summarized in this appendix. In Table 11 we adopt the fraction of the mass exhausted of hydrogen as the independent variable. This is called q in our notation and ν by Schönberg and Chandrasekhar (S-C). The limiting value of q was found to be 0.101 by S-C for models where $\mu_c/\mu_e = 2.2$ with Kramers opacity in the radiative envelopes. Sandage and Schwarzschild (1952) found the limiting value to be 0.12 for $\mu_c/\mu_e = 2.0$, with electron scattering as the predominant opacity source near the interface of the core and envelope and Kramers opacity farther out. We have multiplied the S-C value of q by 1.2 to correct for this difference.

The luminosity and the radius vary as $L_0/\sqrt{Q}$ and Q, respectively, in the S-C models, where L_0 and Q are defined in the original paper (1942). Table 2 and Figures 2 and 3 of the 1942 paper give $L_0/\sqrt{Q} = f(q)$ and $Q = g(q)$ required to draw the tracks in the M_{bol}, log T_e plane. The main-sequence model with $q = 0$ is the Cowling model, whose luminosity and radius are denoted by L_c and R_c. The variation of L and R with q immediately gives the variation of the log of the surface temperature from the relation

$$4\Delta \log T_e = \log \frac{L}{L_c} - 2 \log \frac{R}{R_c}.$$

Table 11 shows the results.

TABLE 11

DETAILS OF THE SCHÖNBERG-CHANDRASEKHAR TRACKS

q	L/L_c	Δ mag	R/R_c	$\Delta \log T_e$
0.000	1 000	0 00	1 000	0 000
012	1 038	04	1 022	− 001
024	1 100	10	1 049	000
036	1 170	17	1 076	+ 001
048	1 269	26	1 111	+ 003
060	1 390	36	1 149	+ 006
072	1 525	46	1 193	+ 007
.084	1 682	56	1 251	+ 008
096	1 902	70	1 316	+ 010
108	2 138	83	1 396	+ 010
0.120	2 377	0 94	1 584	+0 019

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