OBSERVATIONAL APPROACH TO EVOLUTION. II. A COMPUTED LUMINOSITY FUNCTION FOR K0–K2 STARS FROM $M_v = +5$ TO $M_v = -4.5$

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ABSTRACT

Any successful theory of stellar evolution must explain the apparent sequence of luminosity class III giant stars in the H-R diagram. This problem is similar to an explanation for the double-peaked luminosity function for K stars with a first maximum at the main sequence and a second maximum at an absolute magnitude corresponding with stars of luminosity class III. The basic equation is given for the luminosity function $\phi_0(\text{Sp}, M_v)$. It is based upon evolutionary considerations in which giant stars are assumed to have once been main-sequence objects, which have moved from this sequence as hydrogen was converted into helium. Empirical tracks of evolution, which are needed to evaluate the equation, are obtained from a composite color-magnitude diagram of ten open clusters. The equation is solved numerically for K0–K2 stars from $M_v = +5$ to $M_v = -4.5$ and is shown to lead in a natural way to a double-peaked function. The predicted luminosity function agrees well with observational data. The evolutionary explanation is also consistent with the kinematics of giant stars as compared with main-sequence stars.

I. INTRODUCTION

The H-R diagram for stars in the general field shows a well-defined and relatively narrow giant sequence for stars of luminosity class III. Any successful theory of stellar evolution must explain this sequence. The problem may be stated as follows: From the time that Hertzsprung (1905) first made a separation of giant and dwarf stars, it has been evident that the luminosity function, $\phi(\text{Sp}, M_v)$, for stars of a given spectral class later than G0 is double-peaked, reaching a first maximum at the main sequence and reaching a second and final maximum at absolute magnitudes corresponding to stars of luminosity class III. An explanation of this phenomenon is required. This paper is concerned with a semitheoretical determination of $\phi(\text{Sp}, M_v)$ from recent evolutionary ideas which require that the giant stars were once main-sequence objects which moved away from this sequence as hydrogen was converted into helium.

We shall restrict the discussion to the K0–K2 stars from $M_v = +5$ to $M_v = -4.5$. The evolutionary tracks which are followed by stars in this magnitude interval are obtained empirically from the systematics of the color-magnitude diagrams for galactic clusters. Ultimately, of course, these tracks must be explained by evolving models such as those computed by Schwarzschild and his school or, more recently, by Hoyle and Hazelgrove (1957); but such models do not now cover the mass range needed in the present context (from $\mathfrak{M} = \mathfrak{M}_\odot$ to $\mathfrak{M} = 15\,\mathfrak{M}_\odot$).

The next section presents the observational data from which the evolutionary tracks are derived. Section III gives the theory of the $\phi(\text{Sp}, M_v)$ function. Section IV contains the numerical evaluation of the theoretical function, together with a comparison with observational data.

II. COMPOSITE COLOR-MAGNITUDE DIAGRAMS FOR ELEVEN STAR CLUSTERS

In a recent paper (Johnson and Sandage 1955, Fig. 6) the data on color-magnitude (C-M) diagrams for a number of clusters were summarized. More recently, results for several additional clusters have become available, and the ideas of how to obtain correct photometric distances have changed. A rediscussion of the problem is in order.

A composite C-M diagram for ten galactic clusters and one globular cluster is shown in Figure 1. The principal difference between this diagram and the one previously given is the use of new distance moduli for nine of the ten open clusters. These have been obtained by a method which accounts for the effect of stellar evolution on the main sequence. Johnson and his collaborators have emphasized in a series of papers (Johnson and Hiltner 1956; Johnson and Sandage 1956; Johnson, Sandage, and Wahlquist 1956) that distance moduli, determined by the photometric method, depend upon which "standard" main sequence is used to fit the unknown clusters. The main sequence of the nearby stars is not a good one to use because these stars are in various and unknown stages of evolution. Distances obtained by using this sequence will, in general, be too large if an unevolved sequence, such as the faint stars in a galactic cluster, is compared with it. There have been various attempts to determine a standard "age-zero" main sequence (Johnson and Hiltner 1956, Table 1), but these depend upon the validity of the theory of evolving stellar models. Differential corrections, obtained from the theory, have usually

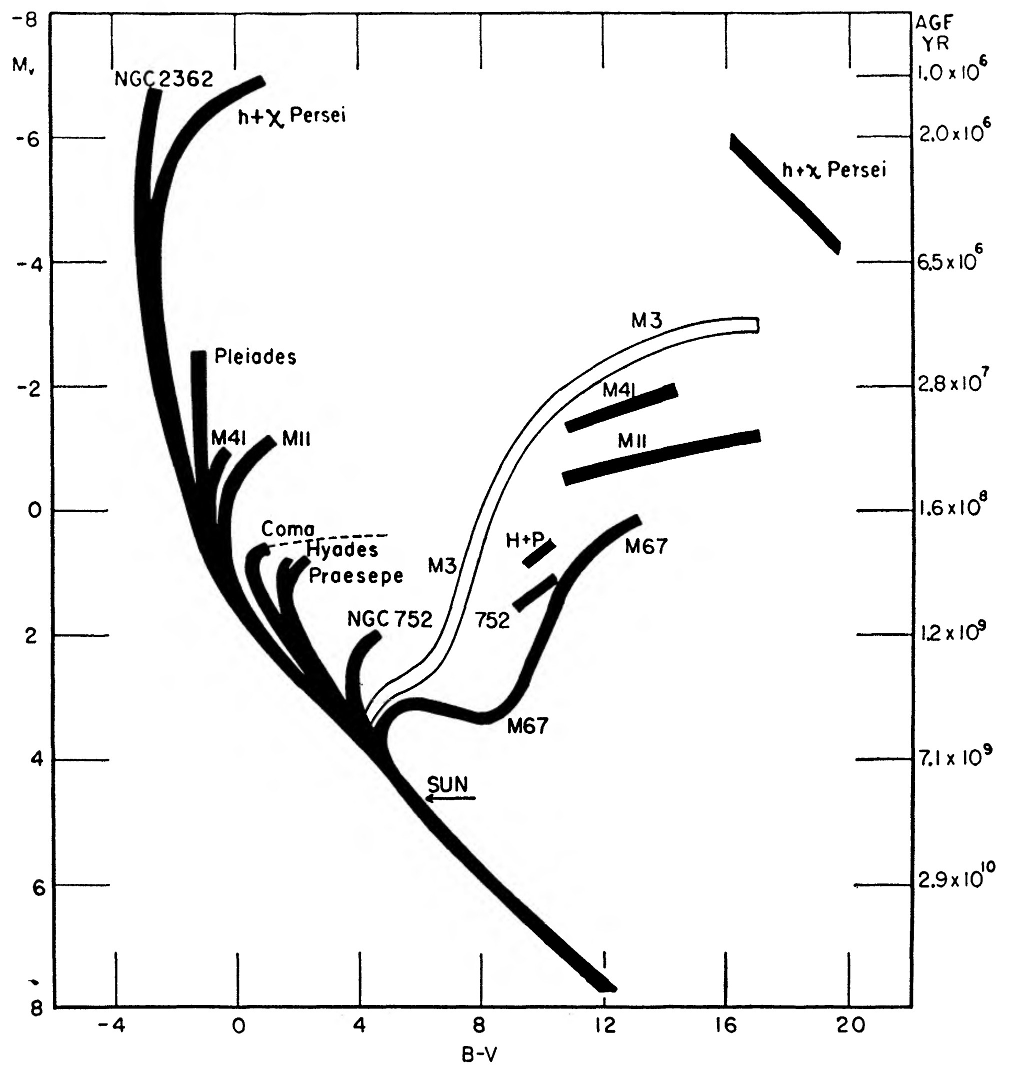

FIG. 1.—The composite color-magnitude diagram for ten galactic and one globular cluster. The age of each cluster whose main-sequence termination point is at a given M_v is shown on the right. See text for details.

been applied to the observed sequences in certain open clusters so as to find the position of the unevolved age-zero main sequence. However, against this procedure, it can be argued that the theory of stellar models is not yet well enough known to give a final answer. A purely empirical method for obtaining the age-zero main sequence is desired. Such a method has already been used (Johnson and Sandage 1956) in comparing the main sequence of M3 with an unevolved sequence. It consists of using the bits of age-zero main sequences available in certain galactic clusters. It is well known that the effects of evolution are apparent in the C-M diagram to a point 3 mag. below termination of the main sequence. Fainter than this, evolution can be neglected. Photometry in many galactic clusters has been carried considerably fainter than the required 3 mag. below the break point, and these parts of the diagrams are important for this problem.

The following approach resulted from a discussion between Harold Johnson and the author. The Hyades, whose geometrical parallax is well determined by van Bueren (1952), is chosen as the zero-point cluster. Van Bueren's modulus of 3.03 is adopted. Both Eggen (1950*a*) and Johnson and Knuckles (1955) have carried the C-M diagram of the Hyades to an apparent visual magnitude of about $V = 10.5$. Because the main sequence terminates near $V = 3.5$, there is about 4 mag. of unevolved main sequence from $V = 6.5$ to $V = 10.5$ in the Hyades data. These apparent magnitudes correspond to absolute

TABLE 1

STANDARD AGE-ZERO MAIN SEQUENCE

M_v	$(B-V)_0$	M_v	$(B-V)_0$	M_v	$(B-V)_0$	M_v	$(B-V)_0$
−2 0	−0 24	+0 5	−0 11	+3 0	+0 28	+5 5	+0 72
−1.5	− .22	+1 0	− 06	+3.5	+ 39	+6.0	+0 82
−1 0	− 19	+1.5	00	+4 0	+ 47	+6 5	+0 93
−0 5	− 16	+2 0	+ 08	+4 5	+ 55	+7 0	+1 04
0 0	−0 14	+2 5	+0.17	+5 0	+0 63	+7.5	+1 16

visual magnitudes of $M_v = 3.5$ to $M_v = 7.5$. Data for the Pleiades (Eggen 1950*b*; Johnson and Morgan 1953) also extend to apparent visual magnitude $V = 10.5$. There is nearly a 1.5-mag. overlap in absolute magnitude between the Pleiades and the faint unevolved end of the Hyades sequence, and a comparison of the clusters in this interval gives the apparent modulus of the Pleiades as $m-M = 5.55$ when $E_y = 0.035$ is used (Johnson and Morgan 1953). The Pleiades main sequence is now used to extend the empirical age-zero sequence to about $M_v = 0.5$, at which point evolutionary effects begin to enter. Finally, NGC 2362 (Johnson and Morgan 1953) may be fitted to the Pleiades in the unevolved region from $M_v = +0.5$ to $M_v = +1.5$. The NGC 2362 sequence may be used to $M_v \approx -2.0$ before evolutionary effects begin to appear. In this manner an unevolved main sequence has been constructed from $M_v = +7.5$ to $M_v = -2.0$. The assumption has been made that any differences in chemical composition which may exist between the clusters do not cause differences in the positions of the main sequences for unevolved stars.

Table 1 gives this standard unevolved main sequence where the zero point of the absolute magnitudes depends solely on van Bueren's modulus for the Hyades. It is desirable to have an independent check of this zero point. The thirteen stars which comprise the nucleus of the Ursa Major stream provide such a check. Both Miczaika (1954) and Eggen (1955) have published colors and magnitudes for these stars, together with values for the cluster parallax. The color–absolute-magnitude diagram plotted from these data shows a sequence which agrees to within ± 0.15 mag. with that of Table 1 fainter than $M_v = +2$. There is no systematic difference. This check appears to be satisfactory. The C-M diagram for the Ursa Major nucleus begins to deviate from the values in Table 1 brighter than $M_v = +2$. This is, of course, expected because these stars have started to evolve. One additional check can be made by using the C-M diagram of the nearby stars. Comparison of the diagram defined by Table 1 of this paper with that of Table 8 of Johnson and Morgan's compilation (1953) shows that the nearby stars deviate from Table 1 brighter than $M_v = +4.5$, as expected from evolutionary theory. Between $M_v = +4.5$ and $M_v = +7.5$ there is good statistical agreement between the two C-M diagrams, if β Com, μ Cass, Gmb 1830, and τ Cet are omitted. These four stars average about 0.5 mag. fainter than Table 1. At least one of them, Gmb 1830, is a well-known subdwarf. The other three may be somewhat similar. This comparison between Table 1 and the nearby stars seems satisfactory and tends to confirm the correctness of the foregoing procedure.

Table 2 lists the revised apparent moduli and the assumed reddening for the eleven clusters shown in Figure 1. These moduli were obtained by fitting the C-M diagrams to the sequence of Table 1. The photometric data have been taken from the following sources: NGC 2362 (Johnson and Morgan 1953); h + χ Persei (Johnson and Morgan 1955; Johnson and Hiltner 1956); Pleiades (Johnson and Morgan 1953); M41 (Cox 1954); M11 (Johnson, Sandage, and Wahlquist 1956); Coma Berenices and the Hyades

TABLE 2

REVISED MODULI AND REDDENING FOR ELEVEN CLUSTERS

Cluster	$m-M$ Apparent	E_y	Cluster	$m-M$ Apparent	E_y
NGC 2362.	11 20	0 116	Hyades	3 03	0 00
h+χ Persei	13 50	.567	Praesepe.	5.90	.00
Pleiades	5.55	.035	NGC 752	7.8	.024
M41	9 20	.06	M67	9 50	.060
M11.	12 40	42	M3	15 68	0 00
Coma Ber.	4 45	0.00			

(Johnson and Knuckles 1955); Praesepe (Johnson 1952); NGC 752 (Johnson 1953); M67 (Johnson and Sandage 1955); M3 (Johnson and Sandage 1956).

Most of the features of the composite diagram of Figure 1 are now well known and are discussed elsewhere (Johnson and Sandage 1955). One point, previously mentioned by Reddish but not proved by his preliminary data (1954), concerns the systematic widening of the Hertzsprung gap toward brighter magnitudes. The gap is wedge-shaped, going to zero near M67 and expanding to $\Delta(B-V) = 1.5$ mag. and $M_v = -6$. All clusters studied so far confirm this general feature.

The other new datum shown along the right-hand ordinate of Figure 1 is the age of each cluster. This is determined from the absolute magnitude of the main-sequence termination point in the usual way by assuming homologous evolution to a 12 per cent limit. The age of M3 is taken to be 5.1×10^9 years, and all other times are then given by

$$T = 1.10 \times 10^{10} \frac{\mathfrak{M}}{L} \text{ years}, \tag{1}$$

wheren $\mathfrak{M}$ and L are in solar units. The bolometric magnitudes and masses of Table 1 in Paper I are used. These times are only approximate, but they are of the right order.

One final point should be mentioned. The main-sequence termination point for M67 is fainter than that of M3 in the new normalization of Figure 1. The modulus for M3 is based upon the usual assumption that $\bar{M}_v = 0.00$ for the RR Lyrae stars. To make the sequence of M3 and M67 terminate at the same absolute magnitude requires that $\bar{M}_v =$

+0.4 for the cluster variables. These considerations show how extremely uncertain the ages of globular clusters really are. They are tied directly to the absolute magnitude of the cluster variables, which is presently not well known.

III. THEORY FOR $\phi(\mathrm{Sp}, M_v)$

Our problem is to compute the luminosity function of K0–K2 stars from absolute visual magnitude +5 to −4.5. Because it is assumed that main-sequence stars are progenitors of the giants, it is clear that $\phi(\mathrm{Sp}, M_v)$ for the giants will be related to the main-sequence luminosity function. This relation is as follows: Let $M_{v,\,g}$ be the absolute visual magnitude of a giant star and $M_{v,\,d}$ be the absolute visual magnitude of the same star when it was on the main sequence. Let $\phi_d(M_v)$ be the presently observed luminosity function along the main sequence and $\phi_g(\mathrm{Sp}, M_v)$ be the luminosity function of the giants. Finally, let $R(\mathrm{Sp}, M_{v,\,g})$ be the ratio of the time a star spends in the giant state at the spectral type Sp and magnitude $M_{v,\,g}$ to the time spent on the main sequence. If the rate of formation of stars is uniform in time, then the ratio of the number of stars in the giant state at spectral type Sp and in a magnitude interval $dM_{v,\,g}$ to the number along the main sequence in the corresponding internal $dM_{v,\,d}$ must equal R. That is to say,

$$\frac{\phi_g(\mathrm{Sp}, M_{v,\,g})\, dM_{v,\,g}}{\phi_d(M_{v,\,d})\, dM_{v,\,d}} = R(\mathrm{Sp}, M_{v,\,g}), \tag{2}$$

where $dM_{v,\,g}$ and $dM_{v,\,d}$ are related by the evolutionary tracks (i.e., all giants in $dM_{v,\,g}$ have come from $dM_{v,\,d}$).

For an extended interval $\Delta M_{v,\,g}$ in the giant region (say 0.5 mag.), equation (2) becomes

$$\phi_g(\mathrm{Sp}, M_{v,\,g})\, \Delta M_{v,\,g} = \int_{M_{d,\,u}}^{M_{d,\,l}} \phi_d(M_{v,\,d}) R(\mathrm{Sp}, M_{v,\,d})\, dM_{v,\,d}, \tag{3}$$

where the limits, $M_{d,\,u}$ and $M_{d,\,l}$ are the upper and lower bounds of the magnitude interval on the main sequence corresponding to $\Delta M_{v,\,g}$ in the giant region. Equation (3) is the fundamental equation of the problem. It is clear that equation (3) may also be expressed in terms of Salpeter's initial luminosity function $\psi(M_{v,\,d})$ defined by equation (1) of Paper I. With the approximation to the mass-luminosity relation given by $L = \mathrm{Const.}\ \mathfrak{M}^{3.5}$, equation (3) becomes

$$\phi_g(\mathrm{Sp}, M_{v,\,g})\, \Delta M_{v,\,g} = \int_{M_{d,\,u}}^{M_{d,\,l}} \psi_d(M_{v,\,d}) R(\mathrm{Sp}, M_{v,\,d})\, 10^{2/7 M_v} dM_{v,\,d}. \tag{4}$$

Any of equations (2), (3), and (4) can be used to determine ϕ_g. Numerical application with equation (2) is made by replacing dM by ΔM. The resulting equation is an approximation to equation (3) valid if ϕ_d does not vary greatly over the interval $\Delta M_{v,\,d}$. The only choice in the use of equation (3) or equation (4) is in the desirability of using either ϕ_d or ψ_d. In the subsequent applications equation (4) has been used, with ψ_d taken from Table 2 of Paper I.

IV. APPLICATION

To evaluate equation (4), we must first relate $M_{d,\,l}$ and $M_{d,\,u}$ with $\Delta M_{v,\,g}$ through the evolutionary tracks. By a process similar to the semiempirical approach employed earlier for the evolution of the M3 stars (Sandage 1954), we can use Figure 1 to approximate the tracks of evolution. This has been done in Figure 2, where the tracks are shown as dotted lines and the region of the K0–K2 stars is shown as a cut in the C-M diagram. The tracks were drawn freehand but were determined by the position of the giants in each of the clusters. Near the beginning of the tracks the relation was preservep between the tracks and the observed sequences as given by the theory of Schönberg and Chandrasekhar (1942), shown in Figure 3. This figure has been reproduced from a previous paper (Sandage 1954).

The open circles on the main sequence in Figure 2 give the starting points for the individual evolutionary tracks which intersect the K1 line at M_v values of -4.5, -3.5, -2.5, -1.5, -0.5, $+0.5$, $+1.0$, and $+2.0$. Additional tracks were also constructed but not shown. The adopted data for the mapping from the main sequence into the giant region is given in Table 3 for intervals of $\Delta M_g = 0.5$ mag. along the K0–K2 cut. This table defines the limits of integration of equation (4).

All factors in equation (4) are now known except R(Sp, M). The value for R will eventually be determined from the theory, since it is the ratio of the time spent by a

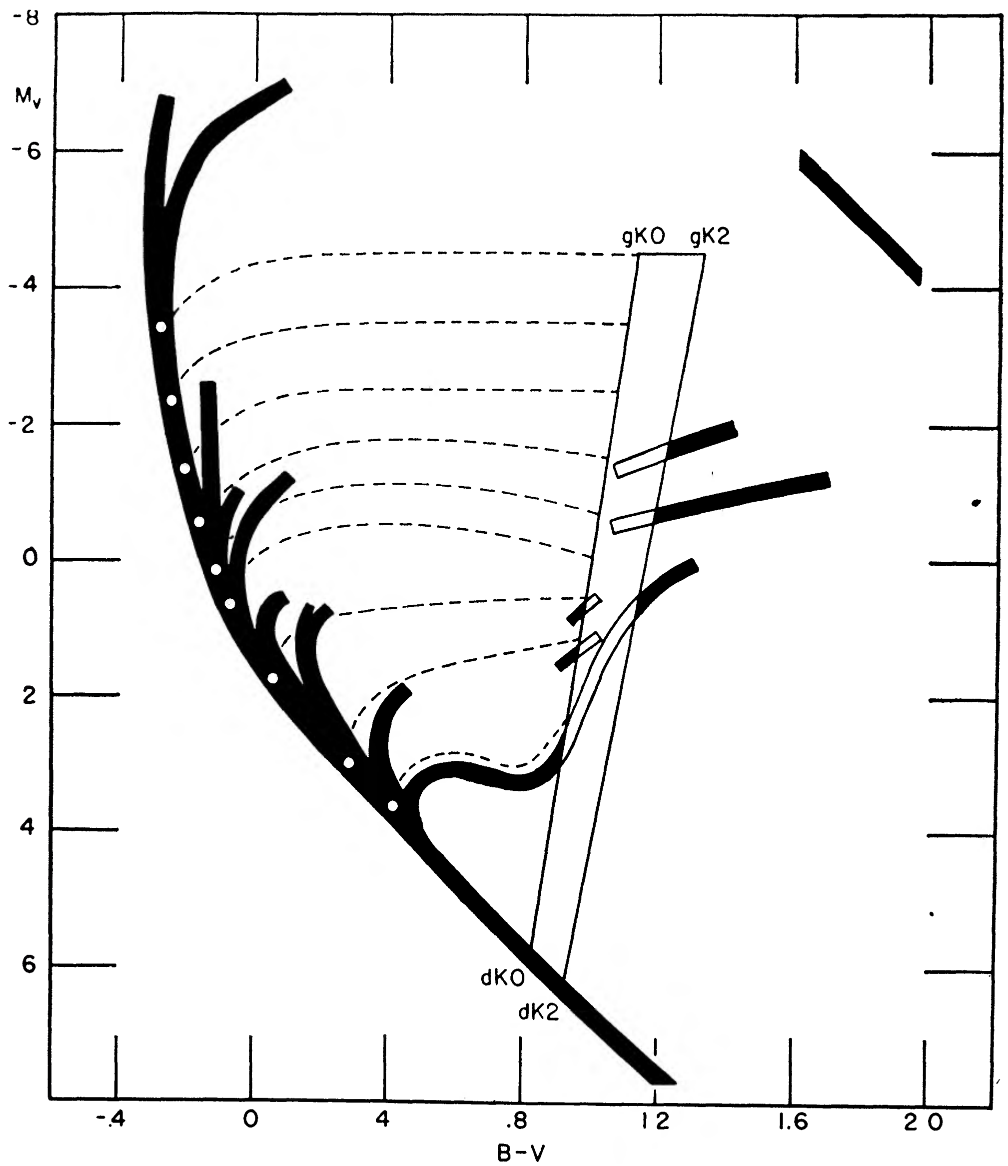

FIG. 2.—The adopted tracks of evolution for stars of different absolute magnitudes. The problem in the text is to determine the number of stars in the K0–K2 cut at each absolute magnitude.

given star in the K0–K2 giant stage to the time spent in its evolution close to the main sequence. A first approximation to $R(M_v)$ may be obtained from the observational data in open clusters by comparing the ratio of giant stars to main-sequence stars in the first-magnitude interval below the termination point. The comparison has been made in a previous paper on M11 (Johnson, Sandage, and Wahlquist 1956), and the indications were that R is approximately independent of absolute magnitude from M41, whose giants occur near $M_v = -2$, to the Hyades, with giants near $M_v = +0.5$. A semi-theoretical argument may also be advanced for assuming R to be independent of M_v. For homologous models, the time spent in any given stage of evolution will vary as $\mathfrak{M}/L$. Since R measures the ratio of the rates of evolution on the main sequence to that in the giant region and since *both* rates may be proportional to L^{-1}, then R will be independent of L. In this first approximation to the problem we shall therefore assume that R is indeed independent of absolute magnitude over the range of M_v considered.

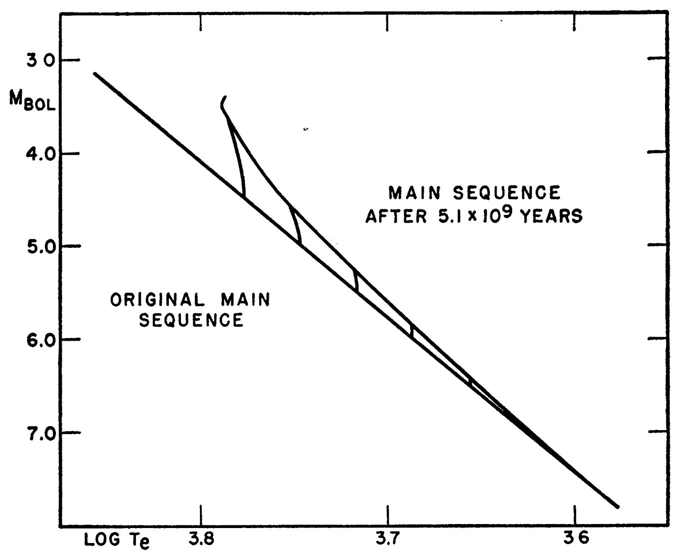

FIG. 3.—The deviation of the main sequence from the original sequence due to evolution

TABLE 3

MAIN-SEQUENCE TO GIANT MAPPING DATA AND RESULTS FOR ϕ_g

$M_{v,\,d}$	(Sp Class)$_d$	Corresponding $M_{v,\,g}$	ϕ_g
−3.35 to −2 85...	B1 V	−4 5 to −4.0	0 2
−2 85 to −2 35 .	B1 V	−4 0 to −3 5	0 5
−2 35 to −1 85.	B2 V	−3.5 to −3 0	0 8
−1 85 to −1.35..	B2 V	−3.0 to −2 5	1.4
−1 35 to −0.90	B3 V	−2.5 to −2 0	2 1
−0.90 to −0.55	B4 V	−2.0 to −1 5	2 6
−0.55 to −0 25.	B5 V	−1.5 to −1 0	2.8
−0.25 to +0.15.	B7 V	−1.0 to −0 5	5 3
+0 15 to +0 65	B8 V	−0.5 to 0.0	10 2
+0 65 to +1.85	A0 V	0.0 to +0 5	55 0
+1.85 to +3 05	A5 V	+0 5 to +1 0	156
+3 05 to +3 75.	F2 V	+1 0 to +1 5	200
+3 75 to +4 05 .	F6 V	+1 5 to +2 0	130
+4 05 to +4 15 .	F7 V	+2 0 to +2 5	51 4
+4.15 to +4 20. . .	F7 V	+2 5 to +3 0	28 0

Equation (4) has been evaluated numerically with the $\psi(M_v)$ function given in Table 2 of Paper I and with the limits of integration given by Table 3 of the present paper. The numerical values for $\phi_g(\mathrm{Sp}, M_{v,g})$ are also given in Table 3, where the numbers are relative and have been normalized to agree approximately with the observational values of $\phi(\mathrm{K0\text{-}K2}, M_v)$ now to be discussed.

The determination of $\phi(\mathrm{Sp}, M_v)$ from observational data is difficult because of the large amount of statistically unbiased data which is required. Oort (1932) and more recently Halliday (1955) give the $\phi(\mathrm{Sp}, M_v)$ function for K stars. It is Halliday's data to which the values in Table 3 have been normalized. Both the computed values (*solid dots*) and Halliday's observed values (*open triangles*) are shown in Figure 4. The agreement is quite good to $M_v = -1$, beyond which Halliday's values decrease considerably more rapidly than do our values in Table 3. This may be due to the effect of the Hertzsprung gap on our calculations. The gap begins to cut into the K0–K2 region near $M_v = -1$, and this will reduce the value of $R(M_v)$ below that which applies for fainter K stars. We have neglected this effect in the computation.

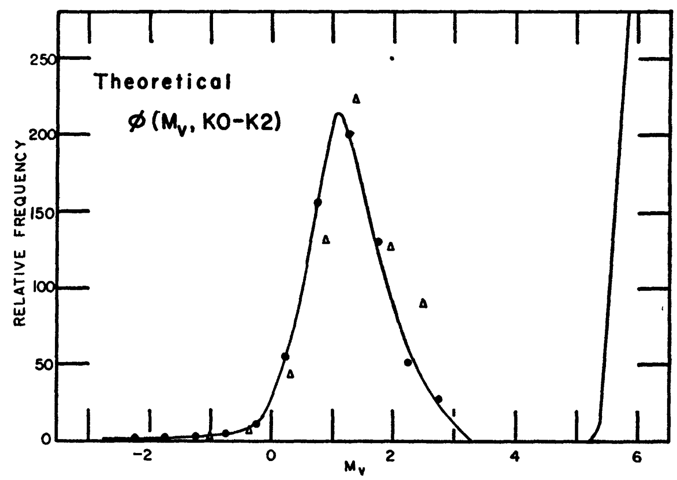

FIG. 4.—The calculated values of $\phi(\mathrm{K0\text{–}K2}, M_v)$ given as closed circles. Halliday's observed function is shown as open triangles. The increase toward the main sequence near $M_v = +6$ is schematic.

The large peak in both the computed and the observed functions near $M_v = +1.0$ shown in Figure 4 represent a section of the apparent sequence of giants. In this absolute-magnitude interval the agreement between the computed values and Halliday's observed values shows that the current ideas of stellar evolution are capable of a quantitative explanation for the giants.

It is instructive to see the physical reason for the concentration of giants near $M_v = +1.0$. It arises from the peculiarities of the evolutionary tracks for stars fainter than $M_v = 0.0$. These tracks move everything into a narrow giant region. Stars like those in the Hyades move on nearly horizontal tracks, while stars similar to those in M67 move on sharply inclined tracks which reach the same place at $M_v \approx +0.5$, $B-V = 1.2$. This process can be labeled the "funnel effect."

The first and second columns of Table 3 give the absolute magnitudes and original spectral classes for stars on the main sequence which are now giants at the M_v given in the third column. The second column was obtained from the $M_v = f$ (Sp. class) relation of Johnson and Morgan (1953). According to Table 3, about 70 per cent of the present K0–K2 giants were originally F2–F7 dwarfs, while most of the remaining 30 per cent were originally A0–A5 main-sequence stars. There are two consequences of this prediction, both of which can, in principle, be checked observationally. One is that there should be a mass spread among the luminosity class III giants, with 70 per cent having masses of about $1.5\mathfrak{M}\odot$ (corresponding to $M_v = 3.5$), while 30 per cent should have masses like the A stars. Masses for the giants are yet too poorly known to test this supposition. The second consequence concerns the space motions of the giants. The kinematic properties of about two out of every three giant stars should be identical with those of F2–F7 dwarfs. Only one out of three should have space motions like the A stars. Table 4 lists the components of the velocity ellipsoid along the principal axes for stars of various spectral classes taken from a compilation by Parenago (1950). It is quite evident that Table 4 is in good agreement with our present expectations. Most of the K0–K2 giants have come from the F dwarfs. Furthermore, the fact that some of the giants have come from the A stars is beautifully confirmed by the special treatment of the data on space motions of main-sequence A stars and K giants made recently by Vyssotsky (1951).

TABLE 4

VELOCITY DISPERSIONS OF GIANT AND DWARF STARS

Spectral Interval	σ_1 (km/sec)	σ_2 (km/sec)	σ_3 (km/sec)	$\epsilon(\sigma)$ (km/sec)
B7– A2	5 5	10 5	16 2	1 4
A3– A8	7 9	9 3	19 1	0 7
A9– F1	9 5	12 8	23 9	0 6
dF2–dF4	11 7	17 0	26 8	0 8
dF5–dF7	16 7	21 4	31 8	0 9
dF8–dG2	22 6	27 5	46 0	2 1
gG9–gK1	15 7	20 5	30 5	0 7
gK2–gK5	17 3	20 5	30 6	0 7
gM	16 3	22 5	31 2	0 7

It may be fair to summarize the present results by saying that current ideas of stellar evolution explain, in a natural way, the observed double-peaked luminosity function for K0–K2 stars. Because the same type of argument can be made for stars of different spectral types (from G0 to M0 stars, for example), it seems clear that the apparent sequence of class III giants is not a sequence at all in the normal definition of the term but is rather an accident which depends upon the fortuitous merging of the evolutionary tracks for all stars which were fainter than $M_v = 0.0$ on the initial main sequence. The general scheme presented appears to be confirmed by kinematical results.

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