STUDIES IN THE EQUILIBRIUM OF GLOBULAR CLUSTERS (II)

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Summary

Times of relaxation are calculated for radial and circular motion in an "isothermal gas sphere." The times of relaxation increase outwards with the radius. A model is developed in which relaxation is complete only up to a limited distance from the centre. Calculations with this model give projected densities which agree much better with observation than do those from the simple isothermal case.

In a previous paper, *M.N.* **114**, 191, 1954 (Paper I), attention was given to the distribution of density in globular clusters, with special reference to (1) the fact that the "isothermal gas sphere" is not a finite object and (2) that it is only a correct solution of the differential equation concerned (Poisson's equation) if the stars all have the same mass.

The first point has been dealt with by many authors by introducing an arbitrary cut-off in the solution. In Paper I this appeared as an arbitrary upper limit to the stellar velocities. The second point was examined by introducing a spread of stellar masses—a mass function. In the present paper we attempt to enquire more closely into the nature of the cut-off, in other words to suggest a physical justification for one. Throughout the present paper we deal only with the case where the stars all have the same mass: firstly, to avoid complication, and secondly to avoid producing results which only have meaning in relation to an arbitrary mass function. (All stellar masses equal is also an arbitrary mass function, but it is the simplest.) Secondly, observations of globular clusters continue very obstinately to give no support to the theoretical expectation that massive stars are concentrated towards the centres of clusters.* Perhaps this only means that the mass luminosity law for nearby stars is quite inapplicable to stars in clusters, but it does suggest some caution in assigning a mass function and a mass luminosity law to cluster stars, both derived from observations of "main sequence" stars.

1. By "globular cluster" we mean an assembly of stars possessing spherical symmetry. The relation between the gravitational potential ϕ and the radius r is then Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi\Gamma\rho. \tag{1.1}$$

If the cluster is in complete equilibrium, Liouville's theorem can be applied to it, and hence so can Jeans' theorem. The only satisfactory† solution for the

* But the galactic cluster M37 as analysed by von Zeipel shows a concentration of luminous stars towards the centre, and so do the Hyades, according to van Bueren.

† i.e. stable against collisions.

velocity distribution is Maxwell's distribution, and if $\nu_m(v, r)\,dm\,dv$ is the number of stars per unit volume at r with velocities between v and $v+dv$ and masses between m and $m+dm$,

$$\nu_m(v, r) = N_0 f(m) \frac{4}{\sqrt{\pi}} (\beta m)^{3/2} . \exp\{2\beta m(\phi - \phi_0)\} . v^2 . \exp\{-\beta m v^2\} \qquad (1.2)$$

where N_0 is the number of stars per unit volume at the centre of the cluster, $f(m)$ the mass function at the centre of the cluster, and β is a constant.*

Then if $\nu_m(r)\,dm$ is the number of stars at r with masses between m and $m+dm$, integrated over all velocities,

$$\nu_m(r) = N_0 f(m) \exp\{2\beta m(\phi - \phi_0)\}$$

leading to an expression for the density, $\rho(r) = \int m\nu_m(r)\,dm$, which depends on the central mass function $f(m)$. However, throughout this paper we confine ourselves to the case where all the masses are the same, and then

$$\nu_m(r) = N_0 \exp\{2\beta m(\phi - \phi_0)\}.$$

We write $\beta m = j^2$ and $2j^2(\phi - \phi_0) = -\psi$. Then

$$\rho = \rho_0\, e^{-\psi}. \qquad (1.3)$$

Further we introduce a dimensionless radius z defined by

$$z = r \times (8\pi\Gamma\rho_0 j^2)^{1/2} \qquad (1.4)$$

and then equation (1.1) becomes

$$\frac{d}{dz}\left(z^2 \frac{d\psi}{dz}\right) = z^2 e^{-\psi}. \qquad (1.5)$$

Equation (1.5) is the well known equation of the isothermal gas sphere and its solution has been tabulated.

In passing we may note that if in (1.4) $z = r/l$, where l is a length, then if ρ_0 is in solar masses per cubic parsec and j in (km/sec)$^{-1}$,

$$l = 3{\cdot}05(\rho_0 j^2)^{-1/2} \text{ parsecs.}$$

2. We now turn to the calculation of the time of relaxation in the cluster. It was remarked in Paper I that "once a high velocity star has got a velocity anywhere near the velocity of escape it spends almost all its time well away from the centre of the cluster and experiences very few collisions relative to the number experienced by a low velocity star which spends its time near the centre." In this section we follow up this remark.

A formula, number 2.355, given by Chandrasekhar in his *Principles of Stellar Dynamics* asserts that $\Sigma\Delta E^2$ the sum of the squares of the exchanges of energy experienced in time dt by a star of mass m_2 and velocity v_2 moving through a field of N stars per unit volume, all of mass m_1, and having a Maxwellian distribution of velocities with a parameter j, is given by

$$\Sigma\Delta E^2 = 8\pi N\Gamma^2 m_1^2 m_2^2 v_2 G(x_0) \ln(q v_2^2)\,dt$$

where

$$G(x_0) = (2x^2)^{-1}\left\{\operatorname{erf} x - \frac{2}{\sqrt{\pi}} x \exp(-x^2)\right\}$$

$$x_0 = j v_2$$

$$q = \left(\frac{6}{\pi N}\right)^{1/3} . \{\Gamma(m_1 + m_2)\}^{-1}.$$

* cf. Paper I, section 4.

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Here $m_1 = m_2 = m$. If we take* as our definition of the time of relaxation $dt = T$ when $\Sigma \Delta E^2 = \overline{E^2}/e$, where $\bar{E} = 3m/4j^2$, then

$$T^{-1} = \tfrac{128}{9}\pi e N j^4 \Gamma^2 m^2 v_2 G(jv_2) \ln(qv_2^2). \qquad (2.2)$$

To calculate this for a star moving in a circular orbit is simple, as v_2 is constant throughout the history of the star, but for any other motion it is necessary to follow the history of the velocity of the star as it describes its orbit.

Consider first a star which has the circular velocity at some point in the cluster. Since $g\,dr = -d\phi$ we have $2j^2 gl\,dz = d\psi$, where $l = (8\pi\Gamma\rho_0 j^2)^{-1/2}$. But $v_c^2 = gr$ so that

$$2j^2 v_c^2 = 2j^2 glz = z\frac{d\psi}{dz}.$$

In any motion $v^2 - 2\phi = \text{constant} = 2E/m$ where E is the energy in the motion: hence $j^2v^2 + \psi = \text{constant} = 2j^2E/m$. Hence if an object moves radially with the speed of the circular velocity at a point where $\psi = \psi_1$ the apocentre of the movement occurs where $\psi = \psi_2$ such that

$$\psi_2 = \psi_1 + \tfrac{1}{2}\left(z\frac{d\psi}{dz}\right)_1 \qquad (2.3)$$

and from tabulated solutions of equation (1.5) giving ψ as a function of z, the value of ψ_2 can always be found for any ψ_1.

The circular velocity at ψ_1 is given by

$$jv_c = (\psi_2 - \psi_1)^{1/2} = \left\{\tfrac{1}{2}\left(z\frac{d\psi}{dz}\right)_1\right\}^{1/2}. \qquad (2.4)$$

Substitution in equation (2.2) now gives T_c, the time of relaxation for a star moving with the circular velocity. We have in fact

$$T_c^{-1} = \tfrac{128}{9}\pi e\rho j^3 \Gamma^2 m(\psi_2 - \psi_1)^{1/2} G\{(\psi_2 - \psi_1)^{1/2}\} \ln\{qj^{-2}(\psi_2 - \psi_1)\} \qquad (2.5)$$

where

$$q = \left(\frac{6m}{\pi}\right)^{1/3} (2\Gamma m)^{-1} \{\rho_0 \exp(-\psi_1)\}^{-1/3}.$$

Then

$$T_c^{-1} = \tfrac{16}{9}\, e\frac{\Gamma mj}{l^2} \exp(-\psi)(\psi_2 - \psi)^{1/2} G\{(\psi_2 - \psi)^{1/2}\} \ln\{qj^{-2}\{\psi_2 - \psi)\}$$

using $l^{-2} = 8\pi\rho_0\Gamma j^2$.

On the other hand if a star is moving radially so that its apocentre occurs where $\psi = \psi_2$, at any other point $j^2v^2 = \psi_2 - \psi$ so that the time dt required to pass through an element of dimensionless radius dz is

$$dt = \frac{l\,dz}{v} = \frac{jl\,dz}{(\psi_2 - \psi_1)^{1/2}}. \qquad (2.6)$$

Hence the time to complete an oscillation is

$$4\mathrm{T} = 4jl \int_{\psi=0}^{\psi=\psi_2} (\psi_2 - \psi)^{1/2} \left(\frac{d\psi}{dz}\right)^{-1} d\psi \qquad (2.7)$$

and T can be found from the tabular solution. Now by (2.1) and (2.6) the sum of the squares of the energy exchanges experienced in moving through dz is

$$\Sigma\Delta E^2 = 8\pi N_0 \exp(-\psi)\Gamma^2 m^4 l\, G\{(\psi_2 - \psi)^{1/2}\} \ln\{qj^{-2}(\psi_2 - \psi)\}\, dz$$

*i.e. e^{-1} of the quantity defined by Chandrasekhar as $\bar{T}_E$.

Hence in performing a quarter oscillation (moving from $\psi=0$ to $\psi=\psi_2$) in time T,

$$\frac{\Sigma\Delta E^2}{\bar{E}^2}=\frac{128\pi}{9}N_0\,\Gamma^2m^2j^4l\int_{\psi=0}^{\psi=\psi_2}\exp(-\psi)\,G\,\{(\psi_2-\psi)^{1/2}\}\ln\{qj^{-2}(\psi_2-\psi)\}\frac{dz}{d\psi}\,d\psi.$$

Accordingly $\Sigma\Delta E^2/\bar{E}^2=1/e$ in time. $T_r=n\mathrm{T}$ if

$$1=ne\frac{128\pi}{9}\rho_0j^3\,\Gamma^2mjl\int_0^{\psi_2}\exp(-\psi)G\{(\psi_2-\psi)^{1/2}\}\ln\{qj^{-2}(\psi_2-\psi)\}\frac{dz}{d\psi}\,d\psi.$$

Hence T_r, the time of relaxation for radial oscillation is given by

$$T_r^{-1}=\frac{16e}{9}\frac{\Gamma mj}{l^2}\frac{\int_0^{\psi_2}\exp(-\psi)\,G\,\{(\psi_2-\psi)^{1/2}\}\ln\{qj^{-2}(\psi_2-\psi_1)\}(dz/d\psi)\,d\psi}{\int_0^{\psi_2}(\psi_2-\psi_1)^{-1}(dz/d\psi)\,d\psi}. \quad (2.8)$$

From formulae (2.2) and (2.8) we can calculate the times of relaxation for the extreme cases of circular and radial motion for any total energy $E=\frac{1}{2}mj^{-2}\psi_2$. Both times of relaxation are inversely proportional to j/l^2, characteristic of the particular cluster, and also inversely proportional to m, the common mass of the individual stars.

Numerical results are given in Table 1. For the purposes of this table we have taken $m=$ the solar mass, $j^{-1}=1$ km/sec, and $l=1$ parsec.

TABLE I

Times of relaxation for circular and radial motions, etc., in the isothermal gas sphere

$\xi=\ln z$ (where z is dimensionless radius)	2	3	4	5
log τ years	7·11	7·60	8·05	8·48
log T_c years	8·86	9·88	10·78	11·55
log T_r years	8·37	8·96	9·50	10·01
log T_c/T_r	0·49	0·92	1·28	1·54
log ρ/ρ_0	$\bar{2}$·71	$\bar{3}$·64	$\bar{4}$·76	$\bar{5}$·95
ψ_1	2·97	5·42	7·47	9·32
ψ_2	4·22	6·55	8·42	10·26

In the example shown in Table 1 complete equipartition cannot be established beyond $\xi=3$ or 4 in a life of 10^{10} years : and if the life of the cluster is substantially less than that the full quota of circular velocities can hardly extend much beyond $\xi=3$ (or $z=e^3=20$). The effect of this defect on the distribution of density in the cluster is discussed in the next section.

3. We are led to the idea that at some considerable time after the formation of a cluster, relaxation is substantially complete at and near the centre, but that in any finite time after formation there are some great distances from the centre at which there has not yet been time to relax the stellar velocities. If the cluster started out in a restricted volume, it would push out some stars beyond this initial volume, at first on orbits initially present in the distribution, and later on orbits arising as a result of relaxation of velocities at and near the centre.

We now investigate a model of a cluster in which relaxation is complete inside a certain radius R, but stars only occur outside this radius if they travel in orbits part of which lie within R (or touch R). The stars outside R are thrown out, as it were, by the equilibrium at and inside R.

Consider an orbit such that the velocity is v_1 and the direction makes an angle α_1 with the radius vector at R. Then if the corresponding quantities are v_2 and α_2 at a distance xR, from the centre of the cluster (where $x>1$), we have, since the force is always central,

$$v_1 \sin \alpha_1 = x v_2 \sin \alpha_2.$$

Again,

$$j^2 (v_1{}^2 - v_2{}^2) = \psi_2 - \psi_1.$$

Accordingly if V_1, V_2 are the velocities in the orbit which touches both $r=R$ and $r=xR$ (i.e. such that $\sin \alpha_1 = \sin \alpha_2 = 1$),

$$V_1 = x V_2$$

$$j^2 V_1{}^2 = (\psi_2 - \psi_1)\, x^2/(x^2-1), \quad j^2 V^2{}_2 = (\psi_2 - \psi_1)/(x^2-1).$$

Three cases arise, as follows,

Case I. If $j^2 v_1{}^2 < \psi_2 - \psi_1$, the orbit cannot get to xR for any value of α_1.

Case II. If $\psi_2 - \psi_1 < j^2 v_1{}^2 < j^2 V_1{}^2$, the orbit intersects $r=xR$ for some values of α_1 (see Fig. 1).

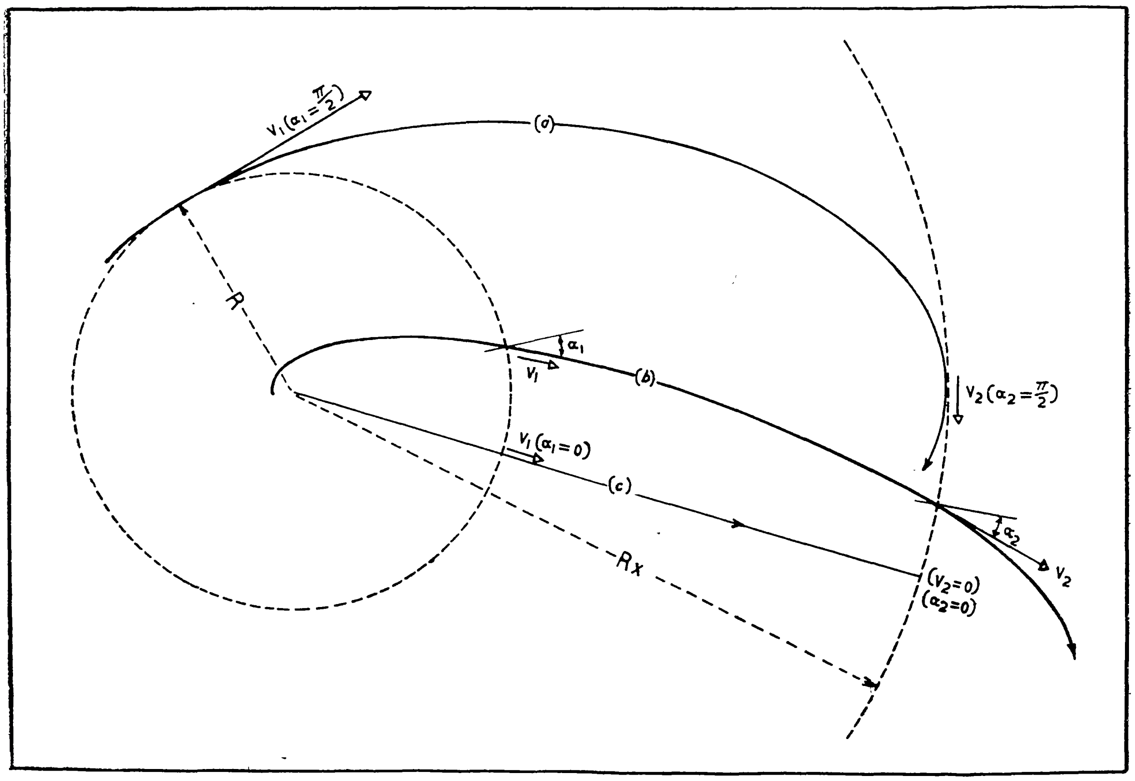

FIG. 1.—*Examples of orbits intersecting* $r=xR$.
(a) $\alpha_1 = \alpha_2 = \pi/2$.
(b) $0 < \alpha_1 < \pi/2$.
(c) $\alpha_1 = \alpha_2 = 0$.

Case III. If $j^2 v_1{}^2 > j^2 V_1{}^2$, the orbit intersects $r=xR$ for all values of α_1.

These cases arise from

$$\sin \alpha_2 = \sin \alpha_1 \, . \, \frac{v_1}{x v_2} = \sin \alpha_1 \, . \, v_1 \{x^2 v_1{}^2 - (x^2-1)\, V_1{}^2\}^{-1/2}.$$

Hence, $\sin \alpha_2 > 1$, which is impossible, unless $\alpha_1 < A$, where

$$\sin^2 A = x^2 - (x^2-1)\, V_1{}^2/v_1{}^2.$$

When $\quad j^2v_1^2=\psi_2-\psi_1=\{(x^2-1)/x^2\}.j^2V_1^2, \qquad \sin^2 A=0.$

When $\quad v_1^2=V_1^2, \qquad \sin^2 A=1.$

Accordingly in Case II, $0<\sin^2 A<1$, and A can always be found. All (v_1, α_1) stars for which $0<\alpha<A$ get to xR. In Case III, all values of α_1 give a possible value of α_2, (but not all values of α_2 are possible.)

These rules enable us to calculate the number of stars at xR from the number at R. If ν_1 is the number of stars per unit volume at R, the number with velocities between v_1 and v_1+dv_1, moving in directions between α_1 and $\alpha_1+d\alpha_1$ is

$$\nu_1 \frac{4}{\sqrt{\pi}} j^3 v_1^2 \exp(-j^2v_1^2)\, dv_1 \,|\sin\alpha_1|\, d\alpha_1.$$

The number moving inwards and outwards over the sphere of radius R in unit time is

$$\nu_1 \frac{4}{\sqrt{\pi}} j^3 v_1^3 \exp(-j^2v_1^2)\, dv_1 \,|\sin\alpha_1|\cos\alpha_1\, d\alpha_1 \,.\, 4\pi R^2$$

integrated over all values of α_1 from 0 to $\pi/2$ and v_1 from 0 to ∞. The number crossing the sphere at xR is the same integrated over all permissible values of α_1 and v_1. Each class (v_1, α_1) contributes to ν_2, the number per unit volume at xR, inversely as $v_2\cos\alpha_2$, so that

$$\nu_2=\nu_1\,.\,\frac{4}{\sqrt{\pi}}\int_{\alpha_1=0}^{\alpha=A}\int_{j^2v_1^2=\psi-\psi_1}^{v_1=\infty}\frac{v_1^3\exp(-j^2v_1^2)\,dv_1\,.\sin\alpha_1\cos\alpha_1\,d\alpha_1}{v_2\cos\alpha_2}\,.\,\frac{R^2}{xR^2}.$$

Since $v_1 \sin\alpha_1=xv_2\sin\alpha_2$, we have $xv_2\cos\alpha_2=(x^2v_2^2-v_1^2\sin_2\alpha_1)^{1/2}$ and, carrying out the α integral, we get

$$\frac{\nu_2}{\nu_1}=\frac{4}{\sqrt{\pi}}j^3\int_{j^2v_1^2=\psi_2-\psi_1}^{j^2v_2^2=\infty} v_1\exp(-j^2v_1^2)\,dv_1\left\{v_2-\left(v_2^2-\frac{v_1^2\sin^2 A}{x^2}\right)^{1/2}\right\}.$$

When $\psi_2-\psi_1<j^2v_1^2<j^2V_1^2$ we have $\sin^2 A=x^2V_2^2/V_1^2$ and when $j^2v_1^2>j^2V_1^2$ we have $\sin^2 A=1$. Accordingly

$$\begin{aligned}\frac{\nu_2}{\nu_1}&=\frac{4}{\sqrt{\pi}}j^3\int_{j^2v_1^2=\psi_2-\psi_1}^{j^2v_1^2=j^2v_1^2} v_1v_2\exp(-j^2v_1^2)\,dv_1\\&\quad+\frac{4}{\sqrt{\pi}}j^3\int_{j^2v_1^2=j^2V_1^2}^{\infty} v_1\exp(-j^2v_1^2)\,dv_1\left\{v_2-\left(v_1^2-\frac{v_1^2}{x^2}\right)^{1/2}\right\}\\&=\frac{4}{\sqrt{\pi}}j^3\int_{j^2v_1^2=\psi_2-\psi_1}^{\infty} v_1v_2\exp(-j^2v_1^2)\,dv_1\\&\quad-\frac{4}{\sqrt{\pi}}j^3\int_{j^2{}_1V^2}^{\infty} v_1\left(v_2^2-\frac{v_1^2}{x^2}\right)^{1/2}\exp(-j^2v_1^2)\,dv_1\end{aligned}$$

and remembering that $j^2v_2^2=j^2v_1^2-(\psi_2-\psi_1)$, $v_1dv_2=v_1\,dv_1$,

$$\frac{\nu_2}{\nu_1}=\exp\{-(\psi_2-\psi_1)\}\left\{1-\frac{4}{\sqrt{\pi}}\int_{y_1}^{\infty}\tfrac{1}{2}\left(y-\frac{j^2v_1^2}{x^2}\right)^{1/2}\exp(-y)\,dy\right\}$$

in which $y=j^2v_2^2$ and $y_1=j^2V_1^2-(\psi_2-\psi_1)=(\psi_2-\psi_1)/(x^2-1)$.

Again

$$y-\frac{j^2v^2}{x^2}=\frac{x^2-1}{x^2}(y-y_1),$$

so that

$$\frac{\nu_2}{\nu_1}=\exp\{-(\psi_2-\psi_1)\}\left[1-\left(\frac{x^2-1}{x^2}\right)^{1/2}\exp\left(-\frac{\psi_2-\psi_1}{x^2-1}\right)\right]. \qquad (3.1)$$

The differential equation to be solved is

$$\frac{d}{dz}\left(z^2\frac{d\psi}{dz}\right)=z^2\exp(-\psi)$$

from $\quad z=0$ to $z=R/l$ (say $z=z_1$) subject to $\psi=0$ and $\dot{\psi}=0$ at $z=0$. $\quad$ (3.2)

$$\frac{d}{dz}\left(z^2\frac{d\psi}{dz}\right)=z^2\exp(-\psi)\left[1-\left(\frac{z^2-z_1^2}{z_1^2}\right)^{1/2}\exp-\left\{\frac{z_1^2(\psi-\psi_1)}{z^2-z_1^2}\right\}\right]$$

for all $z>z_1$, subject to ψ and $\dot{\psi}$ being continuous at $z=z_1$. ψ_1 is the value of ψ at $z=z_1$.

A solution of equations (3.2) was made, with $z_1=30$, and carried from $z=30$ to $z=100$. It is shown in Table 2, with some comparisons, including projections. It was found that if we denote ρ/ρ_0 by η, then ηz^3 was approximately constant (there is a small term linear with z) from $z=60$ to $z=200$, and in making the projection it was supposed that this relation held good outwards for such a distance as could sensibly affect the projection. Notice that this gives a projected density roughly proportional to r^{-2}, as compared with r^{-1} for the isothermal case.

TABLE 2

Solutions and projections of isothermal and modified isothermal

z	ψ		$\log\eta$		log projected intensity	
	Iso.	mod.	Iso.	mod.	isothermal	modification
0	0·00		0·00		1·52	1·75
2	0·56		$\bar{1}$·76		1·36	1·59
5	2·04		$\bar{1}$·11		0·96	1·18
10	3·74		$\bar{2}$·38		0·54	0·75
20	5·41		$\bar{3}$·65		0·18	0·34
30	6·29	6·29	$\bar{3}$·27	$\bar{3}$·27	0·00	0·00
40	6·87	6·86	$\bar{3}$·02	$\bar{4}$·85	$\bar{1}$·88	$\bar{1}$·75
50	7·31	7·27	$\bar{4}$·83	$\bar{4}$·57	$\bar{1}$·79	$\bar{1}$·58
60	7·66	7·57	$\bar{4}$·67	$\bar{4}$·35	$\bar{1}$·70	$\bar{1}$·45
70	7·96	7·80	$\bar{4}$·55	$\bar{4}$·16	$\bar{1}$·65	$\bar{1}$·34
80	8·20	7·99	$\bar{4}$·46	$\bar{4}$·00	$\bar{1}$·60	$\bar{1}$·24
90	8·41	8·14	$\bar{4}$·35	$\bar{5}$·87	$\bar{1}$·55	$\bar{1}$·16
100	8·61	8·28	$\bar{4}$·26	$\bar{5}$·74	$\bar{1}$·52	$\bar{1}$·08

To obtain a good comparison of these results with Gascoigne's observed curves,* the zero of the log of projected light was chosen at

$z=30$ for isothermal and modified isothermal solutions

$r=3.5$ minutes for 47 Tuc.

and $\quad r=7{\cdot}23$ minutes for ω Cen.

These four points were plotted as one in Figure 2, thus determining the scale of the abcissae.

* *M.N.*, **116** (in press).

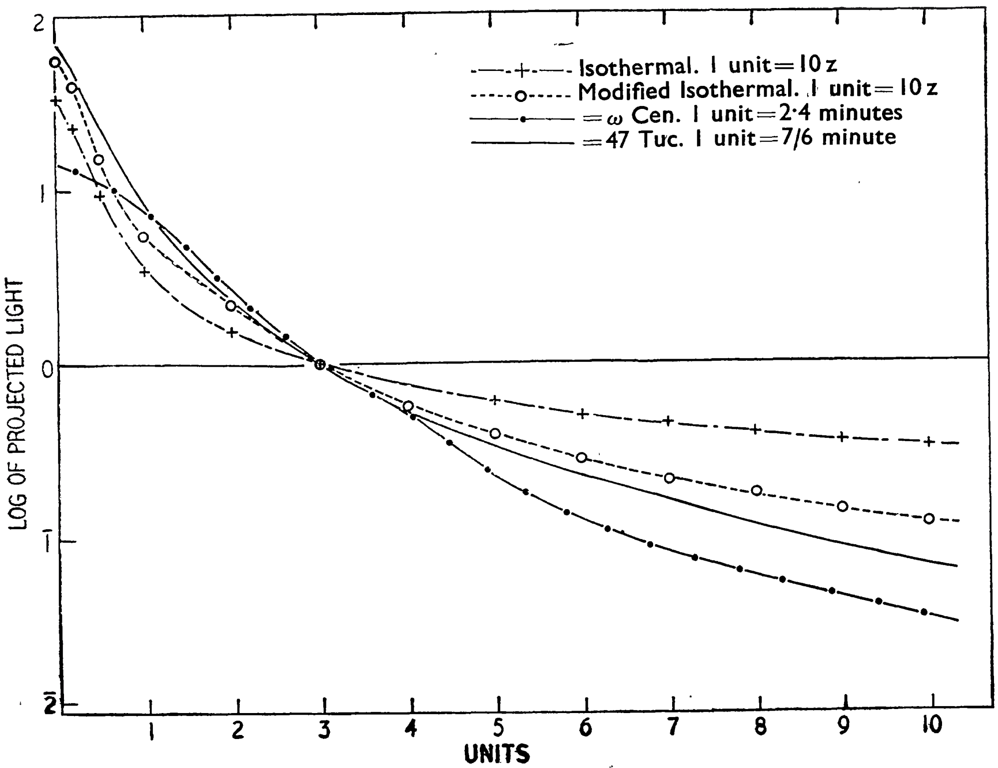

FIG. 2.—*Comparison of theoretical models with observations.*

The modification certainly agrees better with observation than does the unmodified isothermal solution, but much more work is needed, both theoretical and observational, to decide these outstanding questions :

What is the mass function in clusters ?

What is the mass-luminosity law ?

How far out in radius do clusters extend and what is the nature of the cut-off ?

Is equipartition of energy confined to the centres, or even absent ?

Commonwealth Observatory,
Mount Stromlo,
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1955 *December.*