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THE INTERNAL STRUCTURE OF RED DWARF STARS

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ABSTRACT

Models are computed for red dwarf stars, with the hydrogen convection zone taken into account. The calculations show that in red dwarfs this outer convection zone extends inward for a distance of about 30 per cent of the radius of the star. The central temperatures of these models are found to be much lower than the central temperatures of models computed under the assumption of radiative equilibrium throughout the star. Thus the earlier difficulty of an overproduction of energy by the proton-proton reaction does not appear in these models.

The observed luminosities, masses, and radii of red dwarf stars such as Castor C and α Cen B can be understood in terms of these models and lead to normal abundances of the elements. For very late-type dwarfs, such as Kr 60 A, other physical effects appear to become important, and the models of the present paper probably are no longer applicable.

I. INTRODUCTION

The problem of the internal structure of red dwarf stars has attracted considerable attention in recent years. The first work on this problem[1, 2, 3] led to the conclusion that the proton-proton reaction, if it is an allowed nuclear transition, is the main mechanism of energy production in these stars. It is now clear that the proton-proton reaction is allowed,[4, 5] but models incorporating this effect and computed under the usual assumption of radiative equilibrium always give, with any reasonable chemical composition, a computed luminosity much higher than the observed luminosity.[4, 6]

A possible reason for this discrepancy has been suggested by Strömgren,[7] who has pointed out that in the red dwarfs the hydrogen convection zone may be expected to extend down deep into the interior. The reason for this is that the lower temperatures and higher densities in these stars favor the convective transport of energy. Models computed by Biermann[8] show that the central temperature is smaller in a star with a large outer convection zone than in a star with no outer convection zone. The reason for this is that in a convective region the temperature gradient is lower than the radiative temperature gradient, so an extensive outer convection zone will reduce the central temperature and hence also the luminosity. These considerations show that a star with a deep hydrogen convection zone may have a quite different structure from a model computed under the assumption of radiative equilibrium throughout, and that the differences in structure are in the right direction to explain the difficulty quoted above. Detailed calculations of models with the effects of convection taken into account are necessary to see whether the red dwarf stars can be understood in these terms. Biermann[8] has dis-

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1 H. A. Bethe and C. L. Critchfield, *Phys. Rev.*, **54**, 248, 1938.

2 R. E. Williamson and G. F. D. Duff, *M.N.*, **109**, 55, 1949.

3 J. B. Oke, *J.R.A.S. Canada*, **44**, 135, 1950.

4 L. H. Aller, *Ap. J.*, **111**, 173, 1950.

5 E. E. Salpeter, *Phys. Rev.*, **88**, 547, 1952.

6 L. H. Aller, J. W. Chamberlain, E. M. Lewis, W. C. Liller, J. K. McDonald, W. H. Potter, and N. E. Weber, *Ap. J.*, **115**, 328, 1952.

7 *A.J.*, **57**, 65, 1952.

8 *A.N.*, **264**, 361, 1938, and references given there.

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cussed stellar models with large hydrogen convection zones, and Öpik[9] has computed several series of models with outer convective regions. However, it was felt that a series of models specifically computed with the opacity, energy production, and chemical abundances to be expected in red dwarf stars was needed, and the present investigation was undertaken along these lines.

Now it is well known that in computing the structure of a star in radiative equilibrium it is a very good approximation to take the boundary conditions at the outer surface as given by $T = 0$ and $P = 0$. Actually, the correct physical condition is that the temperature has a specified value at some given pressure in the atmosphere, but changes in this temperature have no appreciable effect on the internal structure[10] so long as the outer layers are in radiative equilibrium. However, as Biermann[11] has shown, for stars in convective equilibrium in the envelope, the conditions in the outermost region do play a role in determining the internal structure. The reason is that in convective equilibrium there is an adiabatic relation between temperature and pressure similar to the monatomic gas relation,

$$P = KT^{\gamma/(\gamma-1)} , \tag{1}$$

and at any rate the temperature and pressure go to zero together. Thus for stars in convective equilibrium, one cannot simply take $T = 0$ and $P = 0$ as the boundary conditions at the surface of the star, because this would leave K undetermined, while, in fact, K is determined by the atmosphere.[12] Therefore, in the next section a rough model is computed of the outermost layers of a typical red dwarf star, for the purpose of obtaining an estimate of the approximate value of K.

II. THE ATMOSPHERE AND HYDROGEN IONIZATION ZONE

The atmosphere of a late-type dwarf is in radiative equilibrium at the surface, but at some depth the radiative temperature gradient becomes unstable against convection. We wish to discuss the structure of this hydrogen convection zone in later-type dwarfs in order to find the proper boundary condition for the interior. Only relatively crude models of the outer layers of the stars are necessary for our purposes.

We take the average of the two components of Castor C as a typical red dwarf with well-determined mass, radius, and luminosity. Starting from the surface, a model atmosphere[13] is integrated inward to the point at which convective instability sets in. This calculation gives the values of the pressure and temperature at the upper edge of the hydrogen convection zone. From this point an approximate model of the hydrogen convection zone is integrated inward to the region in which the ionization of hydrogen is essentially complete. The approximations made in this model convection zone are then tested and found satisfactory. Finally, the approximation used in the remainder of this paper for describing this zone is discussed. The details of these computations are as follows.

The model atmosphere is computed in the gray-body approximation, taking the opacity as due to H^- only and using a constant opacity per H^- ion. Such a model atmosphere which neglects the effects of the many absorption lines and the frequency variation of κ_ν is quite inaccurate but can be used to find the orders of magnitude of the pressure and temperature at which convection sets in. The equations describing this model atmosphere are:

$$\frac{dP}{d\tau} = \frac{g}{\kappa}, \tag{2}$$

[9] *Pub. Obs. Tartu*, Vol. **30**, No. 4, 1938.

[10] H. N. Russell, *M.N.*, **98**, 199, 1938.

[11] *A.N.*, **257**, 269, 1935.

[12] *Ibid.*; T. G. Cowling, *M.N.*, **98**, 528, 1938.

[13] B. Strömgren, *Festschrift für Elis Strömgren* (Copenhagen: E. Munksgaard, 1940), p. 218.

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$$\kappa = \frac{1 - x_H}{m_H} P_e a_\tau (H^-) \tag{3}$$

$$= (1 - x_H) P_e 8.87 \times 10^{-3} \theta^{5/2} 10^{0.75\theta} ,$$

$$T^4 = \tfrac{3}{4} T_e^4 \rho(\tau) . \tag{4}$$

Here the number in equation (3) is a mean absorption coefficient of H^- taken from the table given by Münch;[14] $\rho(\tau)$ in equation (4) is the solution of the gray-body problem and is tabulated by Mark;[15] and the other symbols have their usual meanings. In addition to these equations, the relation between P_e and P and T is needed. Since the tables of Strömgren[16] do not cover the range of temperatures relevant to red dwarfs, new tables were constructed for this purpose, using the relative abundances among the metals as given in Kuiper's new compilation.[17] The mass, radius, luminosity, and effective temperature of the average component of Castor C used throughout this investigation are listed in Table 1; they were taken from Kuiper's[18] list and differ only slightly from those

TABLE 1

PHYSICAL PARAMETERS OF TYPICAL RED DWARF STARS

Star	Spectrum	log $L/L_\odot$	log $M/M_\odot$	log $R/R_\odot$	T_e
α Cen B......	K1	−0.44	−0.05	−0.08	4860
Castor C av...	M0	−1.20	− .22	− .20	3600
Kr 60 A.......	M4+	−1.74	−0.58	−0.29	2940

of the newer determination by Kron,[19] which was not available when this work was begun.

Model atmospheres were computed for two assumed values of A, the ratio of number of hydrogen atoms to number of metal atoms. Condensed results of these integrations are presented in Table 2. The condition for stability against convection is

$$(n+1)_{\text{rad}} = \left(\frac{d \log P}{d \log T}\right)_{\text{rad}} \geq \left(\frac{d \log P}{d \log T}\right)_{\text{ad}} = (n+1)_{\text{ad}} , \tag{5}$$

the subscripts "rad" and "ad" denoting radiative and adiabatic, respectively. Hydrogen is essentially not ionized, even at the highest temperature in Table 2; so $(n+1)_{\text{ad}} = 2.5$, and it may be seen that instability sets in at approximately $\tau = 1.5$ and at a temperature and pressure that do not vary rapidly with the metal abundance.

From this point inward the star is unstable against convection, and we shall assume that it is, in fact, in convective equilibrium. Furthermore, we shall assume that the temperature gradient is the adiabatic gradient. These two assumptions may be regarded as giving a first approximation to the structure of the convective region; after computing the structure in this approximation, the assumptions will be checked, and, as will be seen below, it turns out that they are satisfactory.

[14] *Ap. J.*, **106**, 217, 1947.

[15] *Phys. Rev.*, **72**, 558, 1947.

[16] *Pub. mind. Med. Kobenhavens Obs.*, No. 138, 1944.

[17] G. P. Kuiper (ed.), *The Atmospheres of the Earth and Planets* (rev. ed.; Chicago: University of Chicago Press, 1952), chap. xii.

[18] *Ap. J.*, **88**, 472, 1938.

[19] *Ap. J.*, **115**, 301, 1952.

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The pressure gradient in the convective region is given by the usual equation,

$$\frac{dP}{dr} = -\frac{GM_r \rho}{r^2}, \tag{6}$$

while the temperature variation is given by the adiabatic relation. The computation has been carried out in the way suggested by Unsöld,[20] using directly the condition that the entropy per unit mass be constant to compute the adiabatic P, T relationship. The

TABLE 2

MODEL STELLAR ATMOSPHERES FOR CASTOR C AV

τ	LOG T	LOG $A = 3.8$		LOG $A = 4.2$	
		log P	(d log P)/(d log T)	log P	(d log P)/(d log T)
0.00	3.466	....	....	....	....
0.05	3.480	4.60	....	4.79	....
0.10	3.491	4.77	....	4.95	....
0.2	3.508	4.94	....	5.11	....
0.3	3.521	5.03	....	5.20	....
0.4	3.533	5.10	....	5.27	....
0.5	3.543	5.15	....	5.32	....
0.6	3.553	5.19	....	5.36	....
0.8	3.569	5.26	....	5.43	....
1.0	3.583	5.30	3.3	5.47	3.2
1.2	3.595	5.34	2.9	5.51	2.8
1.4	3.606	5.37	2.7	5.54	2.6
1.6	3.616	5.40	2.4	5.57	2.4
1.8	3.625	5.42	2.3	5.59	2.2

model convection zone was computed for a composition of pure hydrogen. This neglects the presence of other less abundant elements, in particular, helium; but, as Biermann[8] has pointed out, these have only minor effects. We thus have

$$\frac{S}{k} = 2 \ln \frac{x_H}{1 - x_H} + (1 + x_H)\left(\frac{\chi}{kT} + \frac{5}{2}\right) = \text{Const.}, \tag{7}$$

$$\frac{x_H^2}{1 - x_H} N = \frac{(2\pi m kT)^{3/2}}{h^3} e^{-\chi/kT}, \tag{8}$$

$$P = N(1 + x_H) kT \tag{9}$$

$$\rho = NH, \tag{10}$$

together with equation (6), where S = entropy per hydrogen nucleus, apart from an additive constant, N = number of hydrogen nuclei per unit volume, χ = ionization potential of hydrogen, and the other symbols have their usual meanings. The departure of M_r from its surface value M is negligible.

As starting values we have taken from Table 2 log T = 3.60 and log P = 5.56 as the temperature and pressure at the top of the convection zone. The integration is then carried out by solving equation (7) for x_H as a function of T with S/k = 9.3224, the entropy per nucleus fixed by the starting values of T and P at the top of the convective

[20] *Zs. f. Ap.*, **25**, 11, 1948.

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zone. Equations (8), (9), and (10) then give P and ρ as functions of T; and, finally, equation (6) is integrated to give r as a function of T. The results of this integration of the structure of the model convection zone are presented in condensed form in Table 3.

Next, the accuracy of this first approximation for the structure of the convection zone may be checked. To make this check, the departure of the temperature gradient from the adiabatic gradient may be computed by using the equations given by Biermann[8] and by Öpik[21] for the convective transport of energy. Taking the mean free path of the turbulent elements as given by the scale height, we find that at the level log $T = 4.00$ the true temperature gradient differs from the adiabatic temperature gradient by only 4 per cent

TABLE 3

MODEL HYDROGEN CONVECTION ZONE FOR CASTOR C AV

log T	log x_H	log P	$-$log ρ	$(d \log P)/(d \log T)$	r/R
3.6.....	−7.124	5.560	5.960	2.500	1.000
3.7......	−5.354	5.810	5.810	2.502	1.000
3.8......	−3.949	6.061	5.658	2.538	1.000
3.9......	−2.839	6.325	5.495	2.817	1.000
4.0......	−1.990	6.650	5.274	3.842	1.000
4.1......	−1.391	7.111	4.926	5.337	0.998
4.2......	−0.998	7.691	4.470	6.084	0.997
4.3......	−0.737	8.299	3.994	5.962	0.995
4.4......	−0.555	8.870	3.556	5.444	0.993
4.5......	−0.423	9.384	3.174	4.836	0.991
4.6......	−0.324	9.839	2.849	4.268	0.988
4.7......	−0.250	10.241	2.572	3.791	0.984
4.8......	−0.195	10.601	2.333	3.415	0.981
4.9......	−0.155	10.928	2.123	3.136	0.976
5.0......	−0.125	11.231	1.932	2.937	0.970
5.1......	−0.104	11.517	1.755	2.800	0.963
5.2......	−0.088	11.792	1.587	2.707	0.955
5.3......	−0.077	12.059	1.425	2.645	0.945
5.4......	−0.069	12.322	1.266	2.603	0.932
5.5......	−0.063	12.580	1.110	2.574	0.916
5.6......	−0.058	12.837	0.956	2.554	0.898
5.7......	−0.054	13.091	0.803	2.540	0.876
5.8......	−0.052	13.345	0.651	2.530	0.849
5.9......	−0.050	13.598	0.499	2.522	0.820
6.0......	−0.048	13.850	0.348	2.518	....

of its value. This justifies the approximation used above. This same calculation also shows that at this level the flux transported by radiation is less than one one-hundredth of that transported by convection, and it also gives $v \simeq 1.4$ km/sec for the mean velocity of turbulence.

Finally, it may be seen from Table 3 that at log $T = 6.0$ hydrogen is nearly completely ionized and that $n + 1$ has very nearly reached the monatomic gas value, 2.5, so that from this point inward the adiabatic relation (1) holds with $\gamma/(\gamma - 1) = n + 1 = 2.5$. Furthermore, the constant K in this equation may be determined from the last values of P and T in Table 3, log $P = 13.85$, log $T = 6.00$, which give log $K = -1.15$.

III. THE CONVECTIVE ENVELOPE

In the previous section it was seen how a model convection zone can be integrated inward through the region of ionization of hydrogen. This process can be continued further inward as far as desired; but, once the region in which the ionization is nearly

[21] *M.N.*, **110**, 559, 1950.

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complete has been reached, a simpler procedure is available. For in the interior regions hydrogen is practically completely ionized, and the adiabatic relation is that of a monatomic gas, i.e., $n + 1 = 2.5$. We then have, with the substitutions

$$P = p \frac{GM^2}{4\pi R^4}, \qquad T = t \frac{\mu H}{k} \frac{GM}{R}, \qquad M_r = qM, \qquad r = xR, \tag{11}$$

the equations

$$p = Et^{2.5}, \tag{12}$$

$$\frac{dp}{dx} = -\frac{pq}{tx^2}, \tag{13}$$

$$\frac{dq}{dx} = \frac{px^2}{t}. \tag{14}$$

The value of the constant E is fixed by comparing equation (12) with equation (1), with $\gamma/(\gamma - 1) = 2.5$, which gives

$$E = 4\pi K \left(\frac{\mu H}{k}\right)^{2.5} G^{1.5} M^{0.5} R^{1.5}. \tag{15}$$

For the model convection zone computed in the previous section for Castor C av, the value of this quantity is $E = 13.6$.

However, this value for Castor C av is only approximate, since it depends upon the opacity in the atmosphere and upon the detailed theory of turbulence, neither of which is accurately known. It appears that the theory of stellar interiors is sufficiently advanced that we can find the conditions in the interior to a good approximation; we must allow for our uncertainty of the structure of the outermost parts of the star by permitting the model of the envelope to vary, using the model computed in the previous section only as a guide. Furthermore, other red dwarfs will differ from Castor C av in their E values. For these reasons we shall not restrict ourselves to constructing one model with the computed E for Castor C av, but rather shall compute a series of models corresponding to a series of values of E which bracket the value for Castor C. We shall finally fix the value of E by the condition that the over-all model of a star fits the observational data, but we expect that the value of E will agree with that of the computed model convection zone within its uncertainties, which means within a factor of about 2 or 3.

For computational purposes we shall take the monatomic gas region as extending to the surface, that is, we shall take as boundary conditions

$$q = 1, \qquad p = 0, \qquad \text{at} \qquad x = 1. \tag{16}$$

This means that we shall make an error of a few per cent in the computed radius of the star; for, once the region of ionization of hydrogen is reached, the temperature gradient is somewhat less steep than that given by $n + 1 = 2.5$. This is illustrated in Figure 1, which shows the variation of t with x for the model convection zone tabulated in Table 3 and for the solution with $E = 13.6$ and the boundary conditions of equation (16). The scales of x are shifted by 0.025, to make the two curves coincide for high temperatures, and therefore the computed radius of the simplified model is incorrect by 2.5 per cent.

Thus, to describe the structure of the outer convective zone, integrations are needed of equations (12)–(14) with boundary conditions (16). A series of 24 such solutions was computed by numerical integration,[22] covering the range[23] $E = 8$ to $E = 46$. These integrations were performed on the electronic computer of the Institute for Advanced

[22] To be published in a forthcoming *Princeton U. Obs. Contr.*

[23] The value of E for the solution regular at the center, which is tabulated in *Brit. Assoc. Adv. Sci., Math. Tables*, Vol. 2 (1932), may be shown to be $E = 45.48$ in the notation used in the present paper.

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Study.[24] The form in which the equations were used in the machine integration was suggested by Dr. von Neumann, the basic coding was laid out by Dr. Goldstine, while Mrs. Selberg performed the final coding and supervised the actual computations. In addition to the variables x, p, t, and q, the homology invariants,

$$U = \frac{d \log M_r}{d \log r} = \frac{p x^3}{q t}, \qquad V = -\frac{d \log P}{d \log r} = \frac{q}{t x}, \tag{17}$$

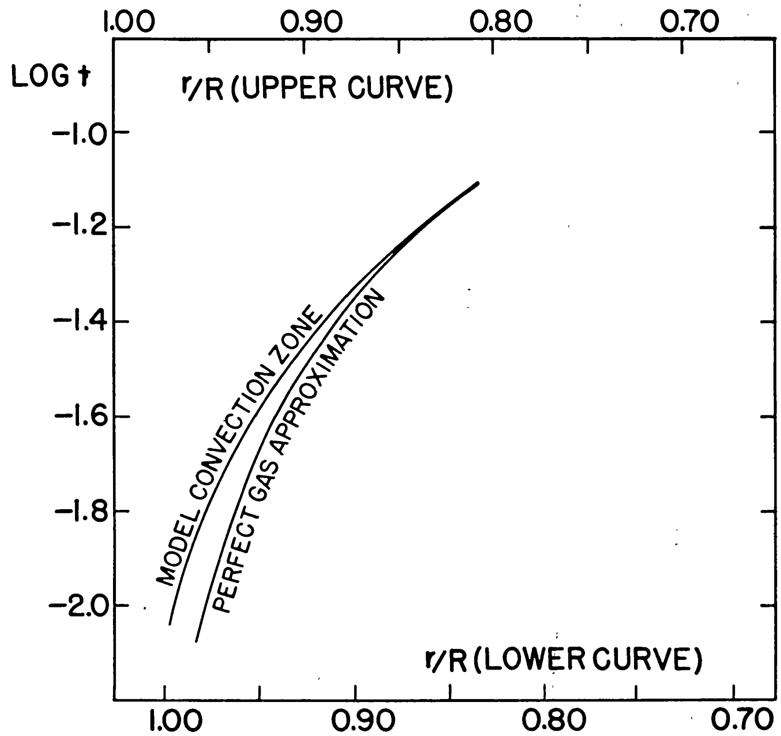

FIG. 1.—Temperature as a function of depth in the model convection zone and in the monatomic gas approximation.

were also computed for each point of each solution. A few of these numerical solutions are plotted in the U, V plane in Figure 2.

IV. THE CENTRAL RADIATIVE REGION

Red dwarf stars have been shown[25] to be in radiative equilibrium at their centers; they do not have convective cores. Hence the equations of radiative equilibrium may be integrated from the center outward to the point at which they become convectively unstable according to criterion (5). From this point outward the structure is convective and is described by the equations in the preceding section.

We assume that the energy production is completely due to the proton-proton reac-

[24] This portion of the work was sponsored by the Office of Naval Research under Contract No. N-7-ONR-388, Task Order I, and by the Ordnance Corps of the United States Army under Contract No. DA-36-034-ORD-1023-RD.

[25] P. Naur and D. E. Osterbrock, *Ap. J.*, 117, 306, 1953.

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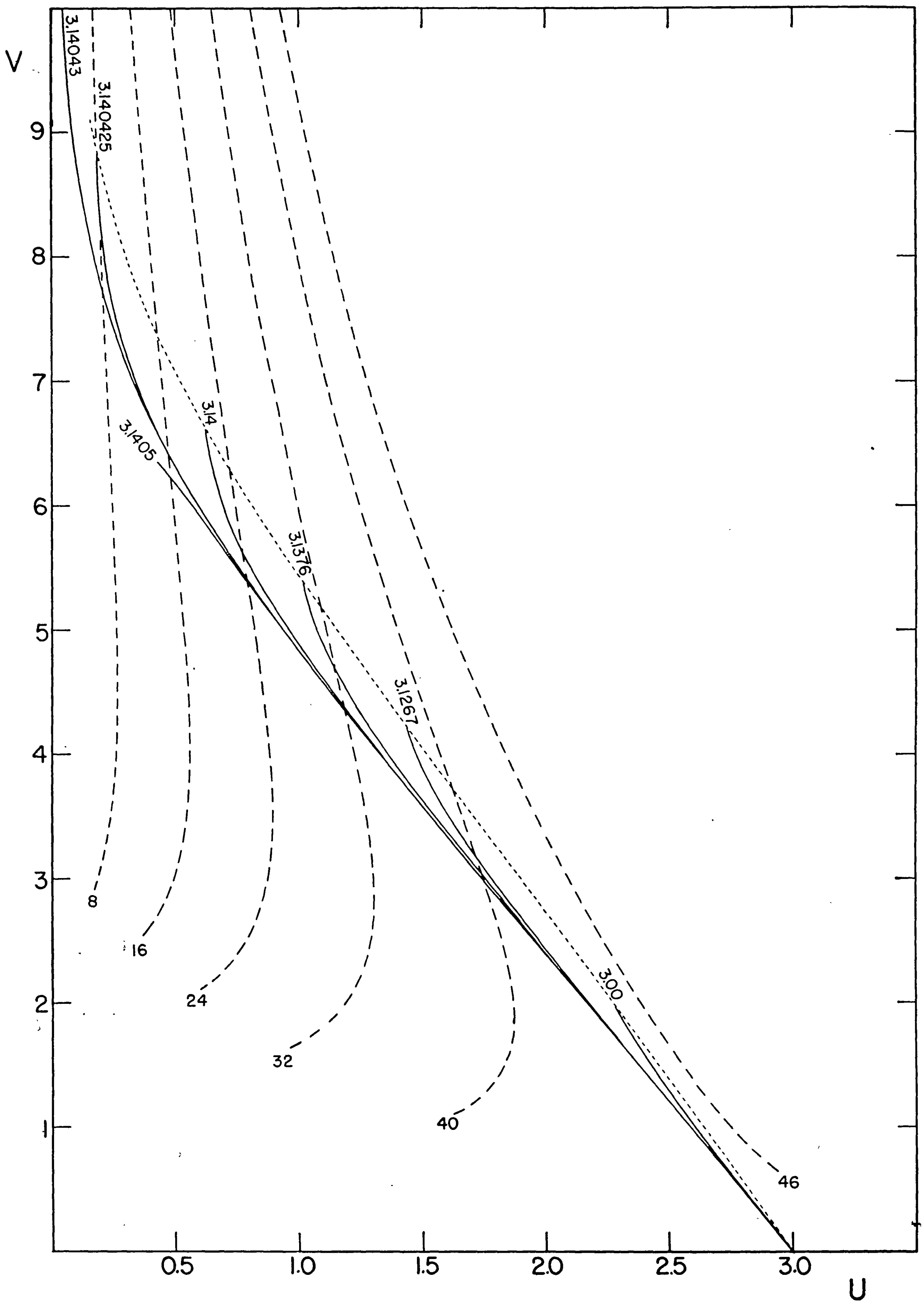

FIG. 2.—Representations of models in terms of the homology invariants U and V. Full lines are radiative-core solutions, each labeled with the value of $(n+1)_c$; long dashed lines are convective envelopes, each labeled with the value of E. Each model is constructed from a radiative-core solution, together with an interpolated convective-envelope solution. The short dashed line is the locus along which the radiative solutions become convective, that is, along which $(n+1) = 2.5$.

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tion, which may be represented over the temperature range in which we are interested by the interpolation formula

$$\epsilon = \epsilon_0 \rho T^{4.5} \tag{18}$$

with

$$\epsilon_0 = \frac{0.0903}{(10^7)^{4.5}} X^2 . \qquad (19)^{26}$$

The opacity is assumed to be due to photoionization of metals, present in the Russell mixture proportions, as given in the tables of Morse.[27] In his paper the opacity is written

$$\kappa = \frac{4.34 \times 10^{25} Z}{T^{3.5}} \rho (1+X) \left(\frac{g}{t}\right), \tag{20}$$

and the guillotine factor (t/g) is tabulated. For the red dwarf models we shall approximate the guillotine factor by the interpolation formula

$$\log\left(\frac{t}{g}\right) = 0.38 + 0.5 \log \rho (1+X), \tag{21}$$

which gives

$$\kappa = \kappa_0 \rho^{0.5} T^{-3.5} \tag{22}$$

with

$$\kappa_0 = 1.81 \times 10^{25} Z (1+X)^{0.5} . \tag{23}$$

Under these assumptions the equations describing the structure in the radiative region can be reduced as follows:

Dimensionless variables:
Equations (11) and

$$L_r = fL . \tag{24}$$

Basic equations:
Equations (13), (14), and

$$\frac{dt}{dx} = -C \frac{p^{1.5} f}{t^8 x^2}, \tag{25}$$

$$\frac{df}{dx} = D p^2 t^{2.5} x^2, \tag{26}$$

with

$$C = \frac{3 \kappa_0}{4ac} \left(\frac{1}{4\pi}\right)^{2.5} \left(\frac{k}{\mu H G}\right)^{7.5} \frac{LR^2}{M^6}, \tag{27}$$

$$D = \frac{\epsilon_0}{4\pi} \left(\frac{\mu H G}{k}\right)^{4.5} \frac{M^{6.5}}{LR^{7.5}}. \tag{28}$$

Logarithmic variables:

$$\lambda = \log \frac{p}{p_0}, \qquad \psi = \log \frac{q}{q_0},$$
$$\tau = \log \frac{t}{t_0}, \qquad \phi = \log \frac{f}{f_0}, \qquad y = \log \frac{x}{x_0}. \tag{29}$$

[26] The numerical value comes from the value of the energy production by the proton-proton reaction given by Salpeter (*op. cit.*).

[27] *Ap. J.*, **92**, 27, 1940.

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with

$$\frac{q_0}{t_0 x_0}=1\,, \qquad \frac{p_0 x_0^3}{t_0 q_0}=1\,,$$
$$C\,\frac{p_0^{1.5} f_0}{t_0^{9.0} x_0}=1\,, \qquad D\,\frac{p_0^2 t_0^{2.5} x_0^3}{f_0}=1\,. \tag{30}$$

Logarithmic equations:

$$\log\left(-\frac{d\lambda}{dy}\right)=\psi-\tau-y\,,$$
$$\log\left(+\frac{d\psi}{dy}\right)=\lambda-\psi-\tau+3y\,,$$
$$\log\left(-\frac{d\tau}{dy}\right)=1.5\lambda-9.0\tau+\phi-y\,,$$
$$\log\left(+\frac{d\phi}{dy}\right)=2\lambda+2.5\tau-\phi+3y\,. \tag{31}$$

Conditions at the center:

$$f=q=0\,, \qquad \tau=0\,, \qquad \text{at} \qquad x=0\,. \tag{32}$$

Homology invariants:

Equations (17) and

$$U=\frac{d\psi}{dy}\,, \qquad V=-\frac{d\lambda}{dy}\,,$$
$$W=\frac{d\log L_r}{d\log r}=\frac{d\phi}{dy}\,,$$
$$n+1=\frac{d\log P}{d\log T}=\frac{d\lambda/dy}{d\tau/dy}\,. \tag{33}$$

It may be noted that we have chosen the five constants p_0, q_0, t_0, f_0, and x_0 to fulfil the four requirements (30). This leaves one constant still at our disposal, which we may therefore choose at will without restricting the physical solution. We have fixed this constant by setting t_0 equal to the value of t at $x = 0$, a procedure which gives the boundary condition (32), $\tau = 0$ at $x = 0$. This leaves only one arbitrary choice to be made, the value of λ at $x = 0$, which we write λ_c; and the radiative interiors thus form a one-parameter series of models which may be classified by the value of λ_c or, alternatively, by the value of the polytropic index at $x = 0$,

$$\log(n+1)_c=-2.5\lambda_c\,. \tag{34}$$

This relation follows from equation (33), noting that $q/f = dq/df$ at $x = 0$ and using conditions (31).

Starting values for the integrations were found by power-series expansions valid near $x = 0$. Eleven numerical solutions[22] of these equations were integrated outward from the center to the point at which $n + 1$ fell to 2.5. From this point outward a convective envelope must be fitted to construct a complete stellar model.

Several of these solutions are plotted in the U, V plane in Figure 2, together with the solution radiative to the surface, discussed in the next section, and also a solution that asymptotically becomes isothermal, with $n + 1$ increasing steadily outward.

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V. CONSTRUCTION OF COMPLETE MODELS

A complete model consists of a central radiative region, described by the integrations of Section IV, together with a convective envelope, described by the integrations of Section III. At the interface between these regions, x, p, q, t, and f and therefore also U, V, W, and $n+1$ must be continuous. A series of models has been constructed using these conditions by fitting an interpolated convective exterior to each computed radiative interior. The fitting procedure is as follows:

From an integration of a radiative interior, read off values U_1, V_1, and W_1 for the point at which $n+1=2.5$, and the values of y_1, λ_1, τ_1, ψ_1, and ϕ_1 at this point. Interpolate in the U, V plane between the convective exterior solutions to find the solution which passes through the point $U=U_1$ and $V=V_1$, and from this convective solution find the values x_1, p_1, t_1, and q_1 at this point. Then, solving the equations

$$y_1=\log\frac{x_1}{x_0}, \qquad \lambda_1=\log\frac{p_1}{p_0},$$
$$\tau_1=\log\frac{t_1}{t_0}, \qquad \psi_1=\log\frac{q_1}{q_0}, \tag{35}$$

gives the four quantities x_0, p_0, t_0, and q_0. In addition, in the convective exterior, by integrating equation (26), with the boundary condition that at the outer surface of the star $L_r=L$, or in our notation $f=1$ at $x=1$, we find

$$f_1=1-D\int_{x_1}^{1} p^2 t^{2.5} x^2 dx\,. \tag{36}$$

This integration can be carried out for each convective solution and interpolated between solutions. Equation (36), together with the relation

$$\phi_1=\log\frac{f_1}{f_0} \tag{37}$$

and the last of equations (30), may be solved for the three unknowns D, f_1, and f_0. The procedure used was to assume a rough value of D, compute the corresponding f_1 by equation (36), then the corresponding f_0 by equation (37), and, finally, a new value of D by equation (30). This iteration method converges very rapidly to the correct values of D, f_1, and f_0.

It may be noted that we have been able to carry out the fitting for f directly, without using the homology invariant W defined by equation (32), as a consequence of the fact that the structure of the convection zone does not depend on the distribution of energy sources in it.

Next the value of C may be computed from the third of conditions (30), and, finally, with p_0 and t_0 known, the central values p_c and t_c can be computed from the values λ_c and τ_c ($=0$). The mathematical characteristics of the models computed in this way are listed in Table 4.

Although attention has been confined to models that would fit red dwarf stars, for purposes of comparison one additional model has been constructed which is in radiative equilibrium to the surface. To construct this model, a radiative solution[22] was integrated from the outer boundary inward according to equations (13), (14), (25), and (26) with the boundary conditions $q=f=1$ and $t=p=0$ at $x=1$. The parameters C and D were varied to fulfil the conditions of fitting this radiative envelope to a radiative interior. The mathematical characteristics of this model are also listed in Table 4 as model XII of that table. In addition, Table 5 presents a comparison of this model with

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TABLE 4

MATHEMATICAL PROPERTIES OF RED DWARF MODELS

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
$(n+1)_c$	3.00	3.10	3.1133	3.1267	3.134	3.1376	3.14	3.1403	3.14036	3.14041	3.140425	3.14043
E	44.65	41.71	40.39	37.67	34.29	30.65	21.43	16.82	14.69	11.05	8.02	....
$\log x_0$	−0.885	−0.928	−0.941	−0.966	−0.993	−1.020	−1.090	−1.129	−1.149	−1.187	−1.225	−1.537
$\log p_0$	+1.209	+1.296	+1.328	+1.393	+1.471	+1.556	+1.794	+1.936	+2.009	+2.153	+2.299	+3.541
$\log q_0$	−1.166	−1.208	−1.218	−1.235	−1.250	−1.262	−1.283	−1.291	−1.293	−1.297	−1.300	−1.303
$\log f_0$	−0.054	−0.082	−0.087	−0.093	−0.096	−0.098	−0.099	−0.100	−0.100	−0.100	−0.100	−0.100
U_1	2.260	1.786	1.653	1.432	1.214	1.017	0.620	0.455	0.385	0.272	0.184	....
V_1	2.037	3.302	3.655	4.247	4.845	5.403	6.650	7.280	7.592	8.205	8.860	....
W_1	0.681	0.199	0.133	0.063	0.027	0.010	0.000	0.000	0.000	0.000	0.000	....
x_1	0.412	0.506	0.527	0.558	0.586	0.610	0.655	0.676	0.686	0.705	0.723	....
q_1	0.309	0.508	0.558	0.636	0.707	0.766	0.873	0.913	0.930	0.953	0.971	....
$\log p_1$	+0.567	+0.329	+0.262	+0.148	+0.025	−0.097	−0.415	−0.603	−0.705	−0.913	−1.144	....
$\log t_1$	−0.433	−0.516	−0.538	−0.571	−0.604	−0.633	−0.698	−0.732	−0.749	−0.783	−0.819	....
f_1	0.889	0.980	0.987	0.996	0.998	0.999	1.000	1.000	1.000	1.000	1.000	....
$\log p_c$	+1.018	+1.099	+1.131	+1.195	+1.272	+1.358	+1.595	+1.737	+1.810	+1.954	+2.100	+3.342
$\log t_c$	−0.281	−0.280	−0.277	−0.269	−0.257	−0.242	−0.193	−0.161	−0.144	−0.110	−0.075	+0.234
$\log C$	−5.172	−5.308	−5.340	−5.385	−5.418	−5.435	−5.420	−5.385	−5.361	−5.309	−5.248	−4.645
$\log D$	+0.886	+0.810	+0.773	+0.691	+0.583	+0.456	+0.066	−0.180	−0.310	−0.570	−0.836	−3.156

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the model computed by Williamson and Duff,[28] using the same opacity law ($\alpha = 0.5$ in their notation) but with a convective core, that is, with a very high temperature dependence of the energy generation. This comparison gives some insight into the effects arising from the concentration of energy sources toward the center of the star.

To summarize, a series of models has been computed, each consisting of a radiative core and a convective envelope. These models form a one-parameter series, each member of which may be labeled by the value of $(n + 1)_c$. All the other quantities are determined from the fitting conditions, and so, for instance, one may alternatively use E as the single parameter which describes a model. Thus the value of E determines the model and in turn determines the other mathematical properties of the model, including x_1, the radius of the interface, and C and D.

VI. PHYSICAL RESULTS

We shall now use the models described in the previous sections to determine the physical conditions in red dwarf stars. Let us confine our attention to a particular star, say Castor C av, for which the observations give the values of M, R, and L. Suppose, now, that we know the correct model which describes this star, that is, we know the value of

TABLE 5

COMPARISON OF RADIATIVE MODELS WITH $\kappa \propto \rho^{0.5} T^{-3.5}$

	$\epsilon \propto \rho T^{4.5}$ No Convective Core	$\epsilon \propto \rho T^{n}$ n Large with Convective Core
log C	−4.645	−4.744
log p_c	+3.3422	+3.1950
log t_c	+0.2337	+0.2273

E, having determined it from the structure of the atmosphere. Then the values of C and D are determined. With these values, equations (27) and (28) give two conditions on the chemical composition. Now the composition is determined by three parameters, X, Y, and Z, but there is one relation between them, namely, $X + Y + Z = 1$, so we may regard two of these parameters—say, X and Z—as the two independent parameters which describe the composition. Thus the above two conditions just determine the two parameters of the composition. This is essentially the procedure used in finding the composition of, say, the sun from its internal structure.[29]

In the present case, however, the correct model is not known because the value of E found in Section III cannot be regarded as an accurate determination. We therefore do not have a single model but a one-parameter series of models, classified by the values of E. We must determine not only the composition but also the model, so we have three quantities, E, X, and Z, to be determined by only two conditions, equations (27) and (28). It is apparent that we cannot determine a unique solution but can rather determine a series of possible sets of solutions. The procedure followed was to assume a value of hydrogen content X and then solve for the other two unknowns, E and Z. The solutions were carried out graphically, as shown in Figure 3, in which the solid curve is a plot of C against D for the series of models. Each dot represents a computed model; the curve has been sketched through these dots. For a given star, such as Castor C av, we choose a

[28] *M.N.*, **109**, 46, 1949.

[29] M. Schwarzschild, *Ap. J.*, **104**, 203, 1946; I. Epstein, *Ap. J.*, **114**, 428, 1951.

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value of X—say $X = 0.90$—and compute as a function of Z the values of C and D for this hydrogen content by equations (27) and (28). These values of C and D are indicated by the dashed line; the point where this dashed line intersects the solid curve gives the model (value of E along the solid curve) and the value of Z (along the dashed line) corresponding to this value of X. Then the values x_1, t_1, p_1, q_1, t_c, p_c, etc., can be determined for this model by interpolation with E as argument, and finally the values of the physical quantities, such as T_1, the temperature at the interface, and ρ_c and T_c, the central temperature, and the density may be found. This whole process may be repeated

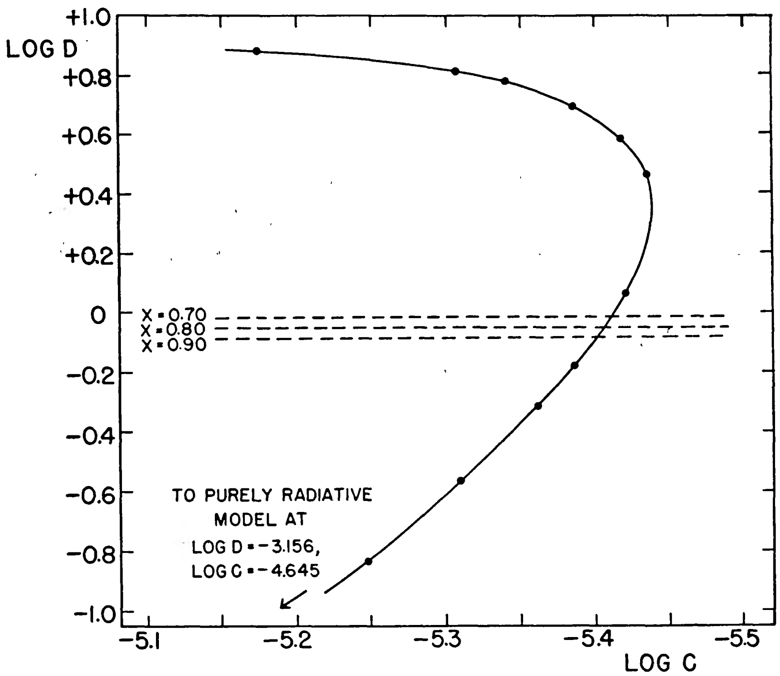

FIG. 3.—The relation between the parameters C and D for the models of this paper (*solid line*). Each dot represents a computed model. Dashed curves are possible values of C and D for Castor C av, for different assumed hydrogen abundances.

for any assumed value of X. We thus determine a table of possible compositions of Castor C av, together with the possible central temperatures, depths of convection zone, and so on, corresponding to each possible composition. This information is summarized in Table 6.

This same procedure may be applied to any red dwarf star for which observed values of M, R, and L are available. We have taken as two typical examples the stars α Cen B and Kr 60 A, whose spectral types are K1 and M4+, respectively. The values of mass, luminosity, and spectral type listed in Table 1 for these two stars are from an unpublished table of the best-determined stellar masses, compiled by Dr. K. Aa. Strand and Dr. R. G. Hall at Yerkes Observatory.[30] The effective temperatures are from Kuiper's tabulation,[14] and the radii then follow immediately. The possible abundances determined for α Cen B are listed in Table 6 also, and it may be seen that they agree well with those determined for Castor C av. There are a few other stars in the list of Strand and Hall intermediate in spectral type between α Cen B and Castor C av, and they likewise may be fitted with reasonable abundances. It must be noted that no claim is made for the detailed accuracy of these derived abundances, because the approximations used in con-

[30] Private communication. I am indebted to Drs. Strand and Hall for giving me permission to quote this table.

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structing the models introduce some errors. As shown in the next section, however, these errors are small.

The star Kr 60 A, which has the latest spectral type, M4+, of all stars considered here, cannot be fitted to one of these models with any reasonable abundances; it would require a very low hydrogen content. We shall return to a discussion of this star in the next section.

We have seen, however, that red dwarfs of spectral type M0 or earlier can be fitted by the models of this paper. In deriving possible sets of abundances, we have not assumed values of X smaller than 0.70 because we expect the abundances to be similar to those that occur in the sun, the atmospheres of early-type stars, planetary nebulae, etc.[31]

TABLE 6

POSSIBLE ABUNDANCES OF ELEMENTS AND CORRESPONDING PHYSICAL QUANTITIES

X	Y	Z	E	x_1	q_1	T_1	T_c	ρ_c
	Castor C av							
0.90.......	0.091	0.009	18.7	0.669	0.900	2.2×10^6	7.8×10^6	81 gm/cm³
.80.......	.184	.016	19.4	.666	.894	2.4×10^6	8.3×10^6	79
.70.......	0.271	0.029	19.9	0.663	0.888	2.6×10^6	8.9×10^6	76
	α Cen B							
.90.......	0.087	0.013	10.2	0.710	0.959	2.1×10^6	10.3×10^6	90
.80.......	.178	.022	10.6	.707	.956	2.3×10^6	11.0×10^6	87
0.70.......	0.261	0.039	10.9	0.705	0.954	2.5×10^6	11.8×10^6	85

It should be noted not only that the fitting can be carried out but also that the resulting value of Z is small. This means that red dwarf stars can be understood in terms of normal abundances, consisting mainly of hydrogen and with helium much more abundant than the metals. Furthermore, as shown by Table 6, over the range of values of X considered likely, that is, $X \geqslant 0.70$, the values of central temperature, depth of the convection zone, and temperature at the bottom of the convection zone do not change much with the assumed value of the hydrogen content.

VII. CHECKS OF ASSUMPTIONS

Finally, we shall turn our attention to a check of the assumptions made in computing the structure of the red dwarf models. First of all, we consider the parameter E, which describes the outer convection zone. From the model convection zone for Castor C av, the value of this quantity is $E \simeq 13.6$, while the determination from the interior of the same star gives E in the range of 18.7 to 19.9. These values therefore agree within the uncertainty of the determination of the atmospheric value of E. It appears that the determination from the interior structure is relatively firm, so that future work should attempt to explain the structure of the atmosphere, using the value of E derived from the interior. As we have not computed a model convection zone for α Cen B, we cannot apply the same check to this star.

Similarly, we shall apply our other checks to the Castor C av model, and we shall use as a definite case the model with $X = 0.80$ to compute the run of temperature and den-

[31] See, e.g., *Trans. I.A.U.*, **7**, Part IV, 1950.

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sity. A condensed table of the physical variables in this model is presented in Table 7. The interface between the radiative center and the convective exterior is at $r/R = 0.67$.

With these values of ρ and T, the guillotine factors may be read off from Morse's tables[27] and compared with the interpolation formula (21). Only the values of the guil-

TABLE 7

PHYSICAL VARIABLES FOR CASTOR C AV MODEL
($X = 0.80$)

r/R	M_r/M	L_r/L	log T	log ρ
0.00	0.000	0.000	6.919	+1.896
0.10	0.020	0.161	6.896	+1.846
0.20	0.134	0.642	6.835	+1.697
0.30	0.334	0.918	6.752	+1.463
0.40	0.546	0.988	6.662	+1.166
0.50	0.717	0.998	6.572	+0.840
0.60	0.837	1.000	6.477	+0.512
0.70	0.915	1.000	6.326	+0.220
0.80	0.969	1.000	6.101	−0.118
0.90	0.997	1.000	5.753	−0.640
1.00	1.000	1.000	....	....

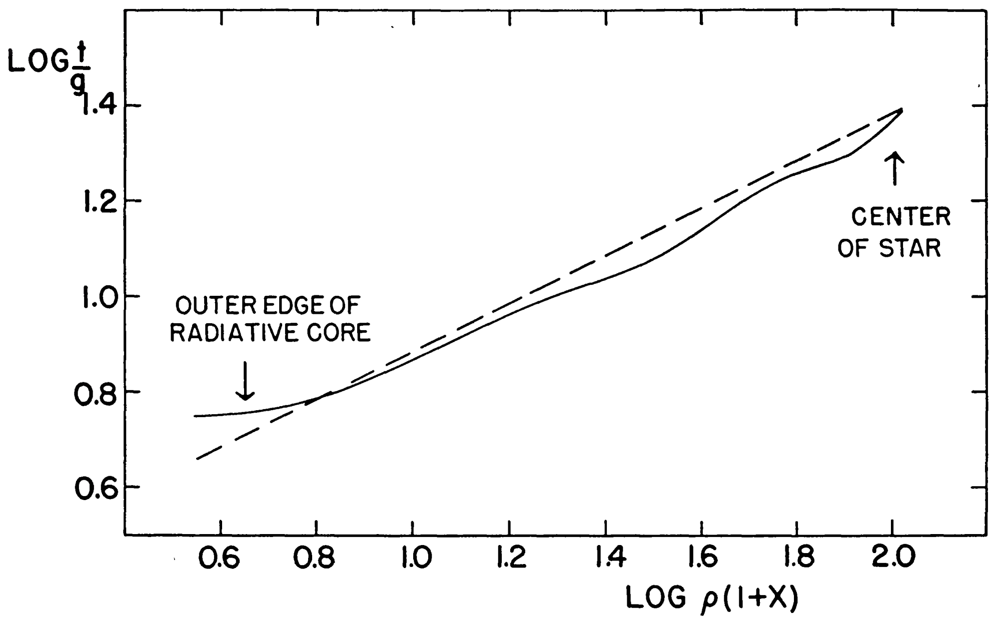

FIG. 4.—Guillotine factor as a function of density. The solid line shows the values determined for Castor C av, while the dashed line represents the interpolation formula (25).

lotine factor in the radiative region up to its boundary are needed, and these are plotted in Figure 4 against log $\rho(1 + X)$. It may be seen that the straight-line approximation of equation (21) is sufficient for our purposes. The opacity at the center is 2.7 cm^2/gm, and the electron-scattering opacity, $0.20\,(1 + X) = 0.4$ cm^2/gm, is small enough to be neglected in comparison with this.

The relative contribution of the free-free transitions[2, 6, 32] in hydrogen and helium to

[32] F. Hoyle, *Trans. I.A.U.*, 7, 481, 1950.

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the opacity must also be estimated. If this opacity is expressed in the form of equation (20), then the free-free opacity in hydrogen and helium corresponds to a guillotine factor[27]

$$\left(\frac{t}{g_{ff}}\right)=\frac{1179\,Z}{g_{ff}}. \tag{38}$$

The appearance of Z in this equation comes from the fact that we are expressing the free-free opacity as proportional to Z by using equation (25), and we must again divide out this dependence in the guillotine factor. The usual procedure is to assume $g_{ff} = 1$, but, according to calculations of Zirin,[33] the effective g factors for free-free transitions at $T \simeq 8 \times 10^6$ and $\rho(1 + X) \simeq 135$ gm/cm³ (the central conditions for the Castor C av model) are 0.30 for hydrogen and 0.35 for helium, the reduction from the value $g_{ff} = 1$ being due to shielding effects. If we take a mean value $g_{ff} = \frac{1}{3}$, we find

$$\log\left(\frac{t}{g_{ff}}\right)=3.55+\log Z\,, \tag{39}$$

which makes this guillotine factor larger than the guillotine factor of the metals opacity, even at $Z = 0.01$. Since the opacity is inversely proportional to the guillotine factor, the conclusion may be drawn that the opacity is determined mainly by the metals and that the interpolation formula (21) is sufficient for our present purposes.

One further check is necessary to test the effect of the thermal conductivity on the flux of energy. In terms of the conductivity, a conductive opacity[34] may be defined which may be compared directly with the radiative opacity. This conductivity may be evaluated by means of the tables of Mestel,[35] and the result is that at the center of the Castor C av model the radiative opacity is about a hundred times more important than the conductive opacity; so the latter may safely be neglected. We thus see that the various approximations used in computing the red dwarf models are justified for dwarfs of spectral type M0 or earlier.

In the case of Kr 60 A, the red dwarf with the smallest well-determined mass, it may be that conduction and incipient degeneracy begin to affect the structure, as suggested by Williamson and Duff[2] and by Aller.[4] It is clear from Section VI that the models of the present paper, which seem to take most of the other physical effects into account, do not fit Kr 60 A with any reasonable abundances of the elements. However, it should also be mentioned that there is some uncertainty in the bolometric correction for Kr 60 A.

VIII. SUMMARY

In this paper models of red dwarf stars have been constructed in which the presence of the outer convection zone has been taken into account. These models show that the red dwarf stars can be understood as stars with normal abundances of the elements, similar to those in other astronomical objects. Values of the helium content Y and the metal content Z for various assumed hydrogen contents X are summarized in Table 6.

The essential feature of red dwarf stars in which they differ from early-type stars is that in the red dwarfs the convection zone extends deep down into the interior of the star. For the two stars considered—α Cen B, spectral type K1, and Castor C av, spectral type M0—the depth of the convection zone is of the order of 30 per cent of the radius of the star.

The various physical quantities listed in Table 6 are relatively well determined in each star, in spite of the uncertainty in the exact hydrogen content. Furthermore, it may be

[33] Private communication. I am very grateful to Dr. Zirin for informing me of the results of his work prior to publication.

[34] T. D. Lee, *Ap. J.*, **111**, 625, 1950.

[35] *Proc. Cambridge Phil. Soc.*, **46**, 331, 1950.

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noted that the temperature at the bottom of the convection zone is in the general range of 2.0–2.5 million degrees in both stars, although their spectral types are quite different. In this connection it may furthermore be noted that Strömgren[36] has found 2.5 million degrees as the temperature at the bottom of the outer convective zone in the sun, a still earlier-type dwarf.

In conclusion, I wish to thank all those who have helped me in the course of this investigation: Professor Bengt Strömgren, who suggested this problem to me and first discussed it with me; Professor Martin Schwarzschild, who gave generously of his time for discussions and advice; Mrs. Reeves, Mr. Skumanich, Mr. Rogerson, and especially Mr. Härm, who did much of the numerical work; and Dr. von Neumann, Dr. Goldstine, and Mrs. Selberg, at the Institute for Advanced Study, who advised and coded and carried out the numerical solutions for the convective exteriors with the electronic computer.

[36] *Mat. Tidsskr.* (Copenhagen), Sec. B, p. 96, 1950.

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