1951ApJ...114..407S

CONTINUOUS EMISSION FROM PLANETARY NEBULAE

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Received June 15, 1951

ABSTRACT

Simultaneous emission of two photons by an H atom in the metastable 2s level is considered as a source of a continuum in planetary nebulae. Detailed calculations show that the probability of two-photon emission is 8.23 $\sec^{-1}$ and that the intensity of radiation emitted by this process increases markedly with increasing frequency. About 32 per cent of electron captures lead directly to the 2s level and then to two-photon emission, since collisional de-excitation proves unimportant. Transitions from the 2p to the 2s level, induced by collisions with free electrons, convert a part of the remaining Lα quanta into this continuous radiation, conversion occurring after a quantum of Lα radiation has been scattered about 10^{10} times, on the average, in a region of ionized H. A neutral hydrogen region may surround the H II region. In such an H I envelope, about 10^{13} scatterings are required for conversion, but the density of neutral H is so much greater that most conversion by collision probably takes place there. The fraction of Lα so converted may vary over a wide range, depending on the physical conditions.

The theory is used to predict the total emission from an ionized hydrogen gas. The Balmer jump is reduced, and the decrement of the continua shortward of the Paschen and Balmer limits is also reduced. A bluish continuum is to be expected in the region from λ 6000 to λ 3646. Our analysis applied to the ultraviolet observations now available results in a reduction of electron temperatures by about 25 per cent. The available wide-slit observations indicate the possibility that a bluish "visual continuum" may exist.

The continuous spectrum of planetary nebulae in the visual region was first measured by Page,[1] and has been analyzed more recently by Aller and Minkowski[2] and by Page.[3] These observations show that longward of the Balmer continuum there exists continuous radiation, with appreciable strength and with a nearly constant intensity per unit wavelength interval between λ 3600 and λ 4800.

Attempts to explain this continuous spectrum have been, so far, uniformly unsuccessful. Previous suggestions have been summarized by Greenstein and Page,[4] who show that emission of radiation in the formation of H^- ions cannot explain the data. The present paper investigates a different source of continuous radiation, based on the simultaneous emission of two quanta from a hydrogen atom in the metastable 2s level. An appreciable fraction of electrons captured by protons will reach the 2s level on their way down to the ground state. In addition, Lα radiation may be converted into two photons of this visual continuum. The absorption of Lα radiation will excite an H atom to the 2p level; and, during the brief interval before the Lα quantum is re-emitted, a collision with a free electron may induce a transition from the 2p to the 2s level. All one-photon transitions from the 2s down to the 1s level are forbidden, and, unless a collision with another free electron induces a transition back to the 2p level, the excited electron will jump down to the 1s level, emitting two photons. Evidently, the sum of the energies of these two photons equals the energy of the Lα photon.

While the probability that an Lα photon will be converted into two photons is relatively small, a single Lα photon is scattered an enormous number of times before it can

[1] *M.N.*, **96**, 604, 1936.

[2] Unpublished.

[3] *Ap. J.*, **96**, 78, 1942.

[4] *Ap. J.*, **114**, 106, 1951.

escape from the planetary nebula. Moreover, the Zanstra process converts most of the stellar energy beyond the Lyman limit into Lα photons. Thus it seems possible that some of the energy emitted by the central star in a planetary nebula can be converted into continuous radiation by two-photon emission.

When this work was nearly complete, it was found that this process had already been considered by Minkowski and Aller,[5] who rejected it because the predicted color distribution was too blue, and, more recently, by A. Y. Kipper;[6] no details of Kipper's work are apparently available in this country.

I. PROBABILITY OF TWO-PHOTON EMISSION

The general theory of two-photon processes has been given by M. Goppert Meyer,[7] and a detailed application of the theory in the case of the 2s—1s transition in H has been given by Breit and Teller.[8] Since Breit and Teller's numerical computations were approximate and did not consider at all the change of intensity with frequency, more detailed computations are required. Let the frequencies of the two photons emitted be $y\nu_{12}$ and $(1-y)\nu_{12}$, where ν_{12} is the frequency of an Lα photon; evidently

$$\nu_{12} = \tfrac{3}{4}\, cR\,, \tag{1}$$

where c is the velocity of light and R is the Rydberg constant for H. Let $A(y)dy$ be the probability that a photon is emitted with a frequency in the range $\nu_{12}dy$. From equation (6.2) in the paper by Breit and Teller, we have

$$A(y) = \frac{9\alpha^6 cR}{2^{10}}\,\psi(y)\,, \tag{2}$$

where α is the fine-structure constant $2\pi e^2/hc$, and

$$\psi(y) = y^3(1-y)^3 \left| \sum_{m=2}^{\infty} R^{1s}_{mp} R^{2s}_{mp} \left(\frac{3}{1+3y-4/m^2} + \frac{3}{4-3y-4/m^2} \right) + \int_0^{\infty} C_{1s} C_{2s}\, dx \left(\frac{3}{1+3y+4x^2} + \frac{3}{4-3y+4x^2} \right) \right|^2 . \tag{3}$$

The quantities R^{ns}_{mp} and C_{ns} are radial quantum integrals defined by Breit and Teller.

Values of $\psi(y)$ have been computed in detail, with R^{ns}_{mp} taken from the tabulation by H. Bethe,[9] and C_{ns} from the paper by M. Stobbe;[10] since Stobbe's tables were not sufficiently complete, new values of these functions were computed from his formulae. The resultant values of $\psi(y)$ are given in Table 1. Since $\psi(1-y)$ equals $\psi(y)$, no values are given for y greater than 0.5. The emissivity j_ν per unit frequency interval is proportional to $h\nu A(y)$, and therefore varies as $y\psi(y)$; values of the relative emissivities are given in the last column.

The familiar Einstein coefficient $A_{2s,\,1s}$ for the two-photon transition is given by

$$A_{2s,\,1s} = \frac{1}{2}\int_0^1 A(y)\,dy = \frac{9\alpha^6 cR}{2^{11}} \int_0^1 \psi(y)\,dy\,. \tag{4}$$

The factor of $\tfrac{1}{2}$ is required, since there are two photons, and each pair is counted twice. Numerical integration yields the result

$$\int_0^1 \psi(y)\,dy = 3.770\,. \tag{5}$$

[5] Informal communication.

[6] *A.J.U.S.S.R.*, **27**, 321, 1950.

[7] *Ann. d. Phys.*, **9**, 273, 1931.

[8] *Ap. J.*, **91**, 215, 1940.

[9] *Handb. d. Phys.* (Berlin: J. Springer, 1933), **24–1**, 442.

[10] *Ann. d. Phys.*, **7**, 661, 1930.

Inserting numerical values into equation (4), we find

$$A_{2s,\,1s} = 8.227 \text{ sec}^{-1} , \tag{6}$$

a value close to the upper limit found by Breit and Teller; an increase in α above the value used by Breit and Teller is partly responsible for this relatively high value of $A_{2s,\,1s}$.

II. EXCITATION OF 2s LEVEL

We consider, now, the processes by which an electron can reach the 2s state under conditions prevailing in planetary nebulae. The first and simplest process is that in which an electron reaches the 2s state by electron capture, either by direct capture in this state or by capture in a higher state, with subsequent cascading downward to the 2s state.

We first compute the probability $X_{r,\,n}$ that an electron, on recombination with a proton in the level of total quantum number n, passes through the 2s state on its way down. Let $\Gamma_{n'l',\,nl}$ be the number of electrons jumping from the level $n'l'$ to the level nl per second per cubic centimeter; jumps down from and up to the free state will be repre-

TABLE 1

RELATIVE PROBABILITIES AND INTENSITIES OF TWO-PHOTON EMISSION

y	λ(A)	Probability $\psi(y)$	Emissivity $y\psi(y)$	y	λ(A)	Probability $\psi(y)$	Emissivity $y\psi(y)$
0.00	....	0	0	0.30	4052	4.546	1.363
.05	24,313	1.725	0.0863	.35	3473	4.711	1.649
.10	12,157	2.783	0.2783	.40	3039	4.824	1.929
.15	8105	3.481	0.5222	.45	2702	4.889	2.200
.20	6078	3.961	0.7922	0.50	2431	4.907	2.454
0.25	4862	4.306	1.077				

sented by $\Gamma_{f,\,nl}$ and $\Gamma_{n'l',\,f}$, respectively. Let $\Gamma^{*}_{n'l',\,nl}$ represent the corresponding quantity in thermodynamic equilibrium with the same density of protons and electrons and at a temperature corresponding to the mean kinetic energy of protons and electrons. Evidently, from the principle of detailed balancing,

$$\Gamma^{*}_{n'l',\,nl} = \Gamma^{*}_{nl,\,n'l'} \,. \tag{7}$$

Also, since the electron velocity distribution is Maxwellian, we have

$$\Gamma^{*}_{f,\,nl} = \Gamma_{f,\,nl} \,. \tag{8}$$

We wish to compute the fraction of electrons, captured in the level of total quantum number n, which are captured in the level of angular momentum l. From equations (7) and (8) we see that this fraction, y_{nl}, is given by

$$\nu_{nl} = \frac{\Gamma_{f,\,nl}}{\sum_{l} \Gamma_{f,\,nl}} = \frac{\Gamma^{*}_{nl,\,f}}{\sum_{l} \Gamma^{*}_{nl,\,f}} \,. \tag{9}$$

To a first approximation the ratio on the right-hand side of equation (9) is given by the ratio of gf values. The use of an integrated f value for transitions to the continuum neglects the shape of the radiation spectrum and the detailed form of $df/d\nu$ for continuous absorption, but should give an adequate first approximation. Equation (9) then becomes

$$y_{nl} = \frac{g_{nl} f_{nl,\,f}}{\sum_l g_{nl} f_{nl,\,f}}. \tag{10}$$

Evidently in the case $n = 2$ we have

$$X_{r,\,2} = y_{20}. \tag{11}$$

For captures of electrons in levels of higher n, we must consider subsequent transitions. We may write, in general,

$$X_{r,\,n} = \frac{\sum_l z_{nl} y_{nl}}{\sum_l y_{nl}}, \tag{12}$$

where z_{nl} is the fraction of electrons captured in the level n, l which cascade down to the level 2, 0. For the level $n = 3$ we have

$$z_{3l} = \begin{cases} 0 & \text{for } l = 0 \\ 1 & \text{for } l = 1 \\ 0 & \text{for } l = 2. \end{cases} \tag{13}$$

Transitions down to the ground level ($n = 1$) are neglected, since any photons emitted in this way will be immediately reabsorbed. For higher levels, more complicated results are obtained. The relative probabilities of two competing downward transitions may be determined directly from the transition probabilities given by Bethe.[9] For $n = 4$, for example, we have

$$z_{4l} = \begin{cases} 0.42 & \text{for } l = 0 \\ 0.74 & \text{for } l = 1 \\ 0.26 & \text{for } l = 2 \\ 0.00 & \text{for } l = 3. \end{cases} \tag{14}$$

If we now substitute these results for z_{nl} in equation (12) and use in equation (10) the value of $f_{nl,\,f}$ given by Bethe, we find

$$X_{r,\,n} = \begin{cases} 0.38 & \text{for } n = 2 \\ 0.45 & \text{for } n = 3 \\ 0.38 & \text{for } n = 4. \end{cases} \tag{15}$$

For greater values of n, the value of $X_{r,\,n}$ decreases gradually. However, the number of electron captures on the nth level varies as $1/n^3$ for binding energies less than the kinetic energy of the free electron. In planetary nebulae, where the electron temperature corresponds to a mean kinetic energy of about 1 volt, captures on levels about $n = 5$ may therefore be neglected. The fraction of electrons captured which reach the 2s state should be somewhere between 0.30 and 0.35; we shall assume a mean value of 0.32. Collisional de-excitation may be taken into account in the manner discussed below, with the result that, if T is 10,000°, X_r, the weighted mean of $X_{r,\,n}$, becomes

$$X_r = \frac{0.32}{1 + 8.2 \times 10^{-6} n_e}. \tag{16}$$

A second mechanism for reaching the 2s state is radiative excitation of the 2p state, followed by a collision with a free electron, inducing a transition to the 2s state. The low probability for this collisional process is offset by the very high number of times that a quantum of Lα radiation will be absorbed and re-emitted before it leaves the nebula. On the assumption that all excitation is by radiative absorption of Lα quanta, we now wish to compute the ratio between two-quantum jumps from 2s to 1s and one-quantum jumps from 2p to 1s. This ratio, which we denote by ζ, gives the probability that a quantum of Lα radiation will be converted into two photons when it is absorbed by an H atom.

This ratio clearly depends on the probability of collisionally induced transitions, which has been considered in detail by Breit and Teller.[8] Their computations may readily be modified to include the Lamb shift[11] of the 2s level relative to the $2p_{1/2}$ level; this shift has only a small effect, since the energy of the transition is in any case very small compared to the energy of the incident electron.[12]

With obvious modifications, the equations S' and S'' by Breit and Teller may be combined to yield for the transition probability $C_{2s,\,2p}$

$$C_{2s,\,2p}=\frac{6n_e h^2}{\pi v m^2}\left\{\ln\frac{m v^2}{|E_{2s}-E_{2p_{1/2}}|}+2\ln\frac{m v^2}{|E_{2s}-E_{2p_{3/2}}|}\right\}, \tag{17}$$

where n_e is the electron density per cubic centimeter; m is the electron mass; and v is the velocity of the free electron in centimeters per second. In wave numbers,

$$|E_{2s}-E_{2p_{1/2}}|=0.035\text{ cm}^{-1}, \tag{18}$$

$$|E_{2s}-E_{2p_{3/2}}|=0.365\text{ cm}^{-1}. \tag{19}$$

Substituting numerical values and replacing $1/v$ by the harmonic mean at temperature T, we have

$$C_{2s,\,2p}=6.21\times10^{-4}\frac{n_e}{T^{1/2}}\ln(5.7T)\left[1+\frac{0.78}{\ln(5.7T)}\right]\text{sec}^{-1}. \tag{20}$$

The second term in brackets on the right-hand side of this equation is small compared to unity and will be neglected here. Collisional transitions induced by collisions with neutral H atoms may be neglected if n_H does not exceed $10^3 n_e$; the cross-section for such encounters will be several orders of magnitude less than the 10^{-12} cm^2 predicted for electron collisions.

The relative populations of the 2s and 2p levels are readily computed from the condition that statistical equilibrium exists, i.e., that the number of electrons jumping out of each level equals the number jumping in. We neglect stimulated two-quantum emissions. The quantity ζ equals the ratio of populations in the 2s and 2p levels, multiplied by the ratio $A_{2s,\,1s}/A_{2p,\,1s}$. After some analysis we obtain

$$\zeta=\frac{g_{2s}C_{2s,\,2p}}{g_{2p}A_{2p,\,1s}}\frac{1}{1+C_{2s,\,2p}/A_{2s,\,1s}}. \tag{21}$$

The ratio g_{2s}/g_{2p} is $\frac{1}{3}$, and, if we insert numerical values from equation (20), taking the value of $A_{2s,\,1s}$ from equation (6), we have

$$\zeta=\frac{3.31\times10^{-13}n_e\ln 5.7T}{T^{1/2}+7.5\times10^{-5}n_e\ln 5.7T}. \tag{22}$$

[11] *Phys. Rev.*, **72**, 241, 1947.

[12] This situation is in marked contrast to that prevailing in a constant electrical field, where the Lamb shift decreases the radiative transition probability from 2s to 1s by a factor of about 1000, according to G. Luders, *Zs. f. Naturforsch.*, **5a**, 608, 1950.

When T is 10,000° K, a standard value for most planetary nebulae, equation (22) yields

$$\zeta = \frac{3.62 \times 10^{-14} n_e}{1 + 8.2 \times 10^{-6} n_e}. \tag{23}$$

For values of n_e between 10^3 and 10^4 per cubic centimeter—typical values for most planetaries—ζ is about 10^{-10}. Of about 10^{10} Lα quanta absorbed by an H atom, one will give rise to a two-quantum jump. If a region of neutral H surrounds the planetary, n_e in such a region will be less by a factor of 10^3, and ζ will equal about 10^{-13}.

III. MEAN FREE PATH OF Lα QUANTUM

We have seen that some 30–35 per cent of the electrons captured by a proton produce two-quantum emission. The others each produce, among other things, a quantum of Lα radiation; and, if the nebula is sufficiently thick optically, these quanta will also be converted into the visual two-photon continuum. Here we consider the mean free path of an Lα quantum, on the average, before it is converted into two quanta by this process.

The situation is idealized by the assumption that, at some distance from the central star, the density of neutral H is constant in space. With this assumption, the necessity for solving the diffusion equation is eliminated; and the simplified analysis of Brownian motion may be applied. Let s_ν be the mean free path of a photon before absorption and let the absorption coefficient of neutral H for this photon be a_ν. If n_H is the density of neutral H, then we have

$$s_\nu = \frac{1}{n_H a_\nu}. \tag{24}$$

If the probability of conversion into two-quantum radiation is ζ per absorption, then the photon will travel $1/\zeta$ mean free paths, on the average, before it is so converted. The directions of successive paths will be uncorrelated, and the mean square distance l_ν^2 will increase proportionally to the number of paths traveled; hence l_ν^2 will equal s_ν^2 multiplied by $1/\zeta$.

To obtain a realistic picture, we must take into account the Doppler change of ν in successive paths, depending on the thermal motion of the absorbing atom and on the angles between the absorbed and emitted photon. This type of noncoherent scattering has been considered by Henyey[13] and applied to planetary nebulae by Zanstra.[14] The detailed correlation of frequencies between the absorbed and subsequently re-emitted photon will be ignored in this first approximation, and only the statistical distribution of frequencies will be considered; this distribution may be assumed to follow the Maxwellian distributions of H-atom velocities. The mean square distance traveled must be averaged over this distribution of frequencies, and we have the basic equation

$$l^2 = \frac{1}{\zeta n_H^2} \int_0^\infty \frac{\phi(\nu)\, d\nu}{a_\nu^2}, \tag{25}$$

where $\phi(\nu) d\nu$ is the fraction of atoms emitting quanta in the frequency range $d\nu$. From the usual Doppler formula and the Maxwellian velocity distribution we have

$$\phi(\nu) = \frac{1}{\pi^{1/2} b} e^{-(\Delta\nu/b)^2}, \tag{26}$$

where $\Delta\nu$ is the difference in frequency from the undisplaced frequency and

$$b^2 = \frac{2kT\nu^2}{m_H c^2}. \tag{27}$$

[13] *Proc. Nat. Acad. Sci.*, **26**, 50, 1940.
[14] *B.A.N.*, Vol. **11**, No. 401, 1949.

If a Doppler profile of the line-absorption coefficient were assumed and a_ν therefore varied as $\phi(\nu)$ for all $\Delta\nu$, the integral in equation (25) would diverge; physically the Lα radiation would leak out of the nebula in the far wings of the profile. Actually, only the center of the line profile is given by the usual Doppler formula, and for large $\Delta\nu$ the resonance wings dominate. To an adequate approximation we may write

$$a_\nu = \frac{\pi e^2 f}{m c} \times \begin{cases} \phi(\nu) & (\Delta\nu \leqslant b) \quad \text{(28a)} \\ \phi(\nu) + \dfrac{\gamma}{\pi(\Delta\nu)^2} & (\Delta\nu > b), \quad \text{(28b)} \end{cases}$$

where e and m are the electronic charge and mass and γ is the damping constant, numerically equal to $A_{2p,\,1s}/4\pi$.

If equation (28) is substituted in equation (25), we find that the integral comes mostly from values of the integrand for which the Doppler wings and the resonance wings are about equal. If we let w be the value of $(\Delta\nu/b)^2$ at which these two contributions to a_ν are equal, then from equations (26) and (28) we see that w satisfies the equation

$$w e^{-w} = \frac{\gamma}{\pi^{1/2} b}. \quad (29)$$

In the present instance γ is several orders of magnitude less than b, and w is moderately large. If we define a new variable, u, by the relation

$$u = \left(\frac{\Delta\nu}{b}\right)^2 - w, \quad (30)$$

then the integral in equation (25) becomes

$$\int_0^\infty \frac{\phi(\nu)\, d\nu}{a_\nu^2} = \frac{m^2 c^2 b^2 e^w}{\pi^{3/2} e^4 f^2 w^{1/2}} \int_{-w}^{\infty} \frac{e^u (1 + u/w)^{3/2} du}{(1 + e^u + u/w)^2}. \quad (31)$$

In deriving equation (31) we have used equation (28b) for all values of $\Delta\nu$; this diminishes somewhat the value of the integral for small $\Delta\nu$, but this region of $\Delta\nu$ contributes a negligible amount to the integrand in any case. When w is infinitely great, the integral in equation (31) equals unity. For $w = 10$, the integral differs from unity by about 10 per cent, a difference which we shall neglect. We have, combining equations (24) and (31),

$$l^2 = \frac{m^2 c^2 b^2 e^w}{\pi^{3/2} e^4 f^2 n_H^2 w^{1/2} \zeta}. \quad (32)$$

For Lα radiation, f is 0.416, and γ is 4.97×10^7 sec^{-1}. If we set T equal to 10,000° K, b is 1.06×10^{11}. The corresponding value of w found from equation (29) is 10.60. Substituting numerical values in equation (32) and making use of equation (23) for ζ, we obtain

$$l = 2.4 \times 10^3 \frac{(1 + 8.2 \times 10^{-6} n_e)^{1/2}}{n_e^{1/2} n_H} \text{ parsecs}. \quad (33)$$

If l found from equation (33) is small compared to the radius R of the planetary nebula, then all the Lα radiation will be converted into two-photon emission. If, on the other hand, l much exceeds R, the previous analysis is not strictly applicable. In this case the quanta will escape from the nebula after they have traveled a distance R, on the average, from the point of origin. Since the number of scatterings required to travel a distance R

varies as R^2, the number of scatterings experienced by a photon before escape will be less than the number $1/\zeta$ required for conversion into two-photon radiation by the fraction $(R/l)^2$, where l is the mean free path computed from equation (33) on the assumption that R is effectively infinite. If we denote by X_c the fraction of Lα radiation so converted by collisions, we have, approximately,

$$X_c = \left(\frac{R}{l}\right)^2 . \tag{34}$$

It should be noted that, if the H atoms are distributed in a filamentary system, with regions of high density embedded in regions of lower density, the mean value of $1/n_H^2$ will be increased, and l will exceed the value given in equation (33).

The fraction X_c of Lα radiation converted into two-photon emission by collisions can be evaluated from a comparison between the size of the planetary and the free path, l, given by equation (33). Observations give only the average n_e in the main body of the nebula; n_H is variable, amounting to about $10^{-3}n_e$ in the main body and increasing at the outer boundary as n_e decreases. The estimate of the mean value of $n_e^{1/2}n_H$ must be based on a definite model for the nebula. Page and Greenstein[15] identify the visible portion of the nebula with the ionized hydrogen region (Strömgren sphere) surrounding the central star. They find that the observed radii R of planetary nebulae agree with those predicted on the basis of Strömgren's theory,[16] using the observed n_e, T_e, and the T_s, R_s of the exciting star. We adopt the model used by Strömgren, that of a homogeneous sphere of pure hydrogen of density n, surrounding a star which radiates as a black body.

First, we consider the fraction of Lα radiation converted within the H II region; we denote this fraction by $X_{c\text{II}}$. Equation (33) is valid only if n_e and n_H are constant. However, for approximate results we may use this equation for actual nebulae, introducing the following mean value of $n_e^{1/2}n_H$:

$$\overline{n_e^{1/2}n_H} = \frac{n^{3/2}}{R}\int_0^R x^{1/2}(1-x)\,dr , \tag{35}$$

where x, the fraction of H ionized, is given by the Strömgren theory.[16] We will assume that R equals s_0; since n_e is less than 10^5, we will drop the correction term in the numerator of equation (33). Since R is much smaller than l, we have, from equation (34),

$$X_{c\text{II}}^{1/2} = \frac{R}{l} = \frac{n^{3/2}}{2.4\times 10^3}\int_0^R x^{1/2}(1-x)\,dr . \tag{36}$$

The integrand is known as a function of r from Strömgren's differential equation (12) for $1-x$ less than 1 and from his approximation formula (17) for the region r about equal to s_0. A scale parameter, α, exists which measures both the thickness of the layer in which hydrogen becomes neutral and the fraction of hydrogen neutral near the star. The combination αs_0 is independent of the properties of the star:

$$\alpha s_0 = \alpha R = \frac{1}{6.3n} . \tag{37}$$

The unit of length is 1 parsec for R; n is the total number of hydrogen atoms and ions per cubic centimeter. The fraction of neutral hydrogen in the inner part of the sphere is given by

$$1-x = \frac{\alpha}{1-(r/R)^3}\left(\frac{r}{R}\right)^2 , \qquad (1-x) < 1 . \tag{38}$$

Thus n_H depends mainly on α. The mean $(1-x)x^{1/2}$ has been obtained by numerical

[15] *Ap. J.*, **114**, 98, 1951. [16] *Ap. J.*, **89**, 526, 1939.

integration; it is 2.7α for $\alpha = 10^{-2}$ and 3.8 for $\alpha = 10^{-3}$. At $r/R = 0.7$ the value of $(1 - x)x^{1/2}$ from equation (38) is 0.75α. The mean ionization determined by integration is seriously influenced by the large value of $1 - x$ in the rim of the ionized region but is not changed in order of magnitude. The maximum value of $(1 - x)x^{1/2}$ is 0.38, but this value is encountered in a shell only $0.007R$ thick. The order of magnitude of the mean free path should be correct, in spite of the large variation of $1 - x$.

Combining equations (36) and (37) and the results of the integrations, we obtain

$$X_{c\,\text{II}} = 3.2 \times 10^{-8} n\,, \qquad \text{for} \qquad \alpha = 10^{-2}\,, \tag{39a}$$

$$X_{c\,\text{II}} = 6.4 \times 10^{-8} n\,, \qquad \text{for} \qquad \alpha = 10^{-3}\,. \tag{39b}$$

The value of α can be computed from observational data. For a low-temperature and density ($n_e = 750$) nebula, NGC 40, the compilation by Page and Greenstein results in $\alpha = 1.5 \times 10^{-3}$; for NGC 7009, $n_e = 6800$ and $\alpha = 0.7 \times 10^{-3}$. Thus expression (39*b*) is sufficiently good; $X_{c\,\text{II}}$ is 4.1×10^{-5} for NGC 40 and 5.0×10^{-4} for NGC 7009. Certain nebulae with apparently large n_e, such as NGC 6790 and IC 4997, would have $X_{c\,\text{II}}$ equal to 1.5×10^{-3} and 4.6×10^{-3}, respectively. Thus the known planetaries convert an insignificant fraction of Lα radiation into two-photon emission by means of collisions within the H II region.

Page and Greenstein[15] pointed out that, if a planetary nebula has been expanding for a sufficiently long period, the visible disk or shell will be only the ionized core of the gas. The apparent boundary of the H emission is then the edge of the Strömgren sphere, and the neutral gas may extend far beyond. Without necessarily adopting the hypothesis of continuous ejection of matter from the star, we may still assume that the gas is expanding radially outward to large distances from the visible boundary at a uniform velocity. Then the density $n(r)$ varies as $1/r^2$, and the total number of atoms per square centimeter outside the visible boundary is $Rn(R)$. Thus in approximate results we can replace this H I region by a uniform layer of thickness R. In such a region the free electrons come primarily from C and Mg, which will be largely ionized. The electron density n_e may be set equal to $n_H/2000$, and we have

$$\overline{n_e^{1/2} n_H} = 2.2 \times 10^{-2} n^{3/2}\,. \tag{40}$$

We let $X_{c\,\text{I}}$ be the fraction of Lα radiation converted into two-photon emission in the H I shell outside the planetary. Then, by use of equations (33) and (34), we have

$$X_{c\,\text{I}}^{1/2} = \frac{R}{l} = 0.9 \times 10^{-5} n^{3/2} R\,. \tag{41}$$

The density of a typical planetary may be taken as $n = 5000$ and $R = 0.1$ parsecs; then $X_{c\,\text{I}} = 0.10$, i.e., about one-tenth of Lα is converted into two-photon emission. This is much larger than the $X_{c\,\text{II}}$, which would be near 0.001, for these standard conditions. We can expect $X_{c\,\text{I}}$ to vary from zero to about $\frac{1}{4}$, depending on physical conditions in this outer envelope. If we take into account the probable fall of temperature in H I regions and the consequent increase of ζ and decrease of b, the value of $X_{c\,\text{I}}$ is somewhat increased. If we let $X/(1 - X)$ be the ratio of the energy of two-photon radiation to Lα radiation, evidently X may vary from about 0.32 to nearly 1 in planetaries of differing density. A more refined theory of radiative transfer, taking into account the large spatial variations of n_H through the nebula, both in H I and H II regions, would be required for a more accurate and detailed calculation of X_c.

IV. RELATIVE INTENSITY OF TWO-PHOTON EMISSION

The previous sections give information on the fraction of electron captures that produce two-photon emission, and they also yield information on the distribution of this emission with wave length. It remains to compare the intensity of this continuous radiation with other features of the hydrogen emission spectrum. We shall here make comparison with the Balmer and Paschen emission continua.

Let j_{2q} be the emissivity per gram in the two-quantum continuum per unit frequency from the planetary nebula, and let j_P and j_B be the corresponding quantities for the Paschen and Balmer continua. The variation of j_{2q} with frequency is given by

$$j_{2q}(\nu) = n_{2q} C_{2q} y \psi(y) , \tag{42}$$

where $\psi(y)$ is defined above, n_{2q} is the total number of quanta emitted in two-photon transitions per gram per second, and C_{2q} is a constant to be determined. Similarly, for j_P and j_B we have

$$j_P(\nu) = n_P C_P e^{-h\nu/kT} \qquad (\nu > \nu_P) , \tag{43}$$

$$j_B(\nu) = n_B C_B e^{-h\nu/kT} \qquad (\nu > \nu_B) , \tag{44}$$

where ν_P and ν_B are the frequencies of the Paschen and Balmer limits, respectively. The functional form of equations (43) and (44) is evident from Kirchhoff's law, since the absorption coefficient for photoionization varies approximately as $(1 - e^{-h\nu/kT})/\nu^3$, when stimulated emission is taken into account.

The constant C_{2q} may be determined from the condition that

$$4\pi\rho \int_0^\infty j_{2q} \frac{d\nu}{h\nu} = n_{2q} , \tag{45}$$

with similar equations for j_P and j_B; in this way we obtain

$$C_{2q} = \frac{h}{4\pi\rho \int_0^\infty \psi(y)\, dy} = \frac{h}{4\pi\rho \times 3.770} , \tag{46}$$

$$C_P = \frac{h}{4\pi\rho Ei(h\nu_P/kT)} , \tag{47}$$

$$C_B = \frac{h}{4\pi\rho Ei(h\nu_B/kT)} , \tag{48}$$

where $Ei(x)$ is the familiar exponential integral.

Lastly, we must determine the ratio of n_{2q}, the number of electrons jumping from 2s to 1s, to n_P, the number of electrons captured on the level $n = 3$. Let n_c be the total number of electrons captured on all levels, excluding the lowest. Then, evidently,

$$\frac{n_{2q}}{n_c} = 2X . \tag{49}$$

The number of electrons captured on the level of total quantum number m may be written, approximately,[17]

$$n_m = \frac{B\beta^2}{m^3} e^{\beta/m^2} Ei \frac{\beta}{m^2} , \tag{50}$$

where β is a constant for all m, and

$$B = \frac{2 A n_i n_e}{(\pi L)^{1/2}} = \left(\frac{2^{11} k}{3^3 \pi m^5}\right)^{1/2} \frac{h e^2}{c^3} T^{1/2} n_i n_e , \tag{51}$$

[17] L. Spitzer, Jr., *Ap. J.*, **107**, 6, eq. (14), 1948.

$$\beta = \frac{h\nu_L}{kT} = \frac{158,000^\circ}{T}, \tag{52}$$

and ν_L is the frequency at the Lyman limit. For n_c we have

$$n_c = B\beta\phi_2(\beta), \tag{53}$$

where $\phi_2(\beta)$, tabulated by Spitzer,[17] is

$$\phi_2(\beta) = \sum_{m=2}^{\infty} \frac{\beta}{m^3} e^{\beta/m^2} Ei \frac{\beta}{m^2}. \tag{54}$$

The emissivity per gram per unit frequency range in the two-quantum continuum is, if we combine equations (42), (46), (49), and (53),

$$j_{2q} = \frac{Bh\beta^2}{4\pi\rho} \frac{2X\phi_2(\beta)}{3.77\beta} y\psi(y). \tag{55}$$

The recombination emission from captures on level m has emissivity

$$j_m = \frac{Bh\beta^2}{4\pi\rho} \frac{1}{m^3} 10^{-10.16\theta(y-4/3m^2)} \qquad \left(y > \frac{4}{3m^2}\right), \tag{56}$$

where

$$\lambda = \frac{1.216 \times 10^{-5}}{y} \text{ cm}. \tag{57}$$

The quantity θ is the usual reciprocal temperature, $5040^\circ/T$. The ratio j_{2q}/j_3, measuring the relative importance of two-quantum and Paschen emission, is not far from unity at λ 4800 for $X = 0.32$ and $\theta = 0.5$; j_{2q}/j_2 is about 0.03 at the Balmer limit. We define j'_H as the total intensity of the recombination emission continua up to $m = 4$ (eq. [56]). Values of j'_H are given in Table 2; series with $m > 4$ and free-free emission are neglected. The j_{2q} tabulated are obtained for the typical value $X = 0.32$ and can be scaled proportionately. The temperature dependence of the emission is small in units of $Bh\beta^2/4\pi\rho$. (The unit is proportional to $\rho T^{-3/2}$, mass-emission coefficient, and would be proportional to $\rho^2 T^{-3/2}$ per unit volume.) Figure 1 and Table 2 show $\log_{10} j$, the sum of the two sources of emission. Note how the Balmer discontinuity is reduced and how the apparent color temperature varies with frequency. The recombination emission declines quite steeply shortward of each series limit, but the decline is smoothed out by the relatively blue two-quantum continuum. If the electron temperature is derived from the slope just below a series limit, by the use of equations (43) and (44), the resultant value would be higher than the true temperature. Thus, when $\theta = 0.7$, its apparent value from the slope is 0.52 in the range λλ 3646–3040 and 0.36 in the range λλ 8200–4861; when $\theta = 0.35$ the apparent values are 0.25 and 0.14, respectively. Since the usual determinations of the electron temperature give θ apparently near 0.5, the case of $\theta = 1.00$ is particularly important. Between λ 8200 and λ 6080 the slope corresponds to $\theta = 0.72$; between λ 6080 and λ 4861 the slope corresponds to $\theta = 0.33$, and in the photographic region λλ 4861–4050 it corresponds to $\theta = 0.0$; for λλ 3646–3040, the usual limit of ultraviolet spectrophotometry, $\theta = 0.74$.

The observational material now available for study is mainly confined to the Balmer emission continuum. T. L. Page[3, 18] finds that the slope of the observed emission for $\lambda < 3646$ corresponds to $7000^\circ < T_e < 10,000^\circ$, in general, with only moderate deviations from the older theoretical prediction that j'_H varies as $\exp(-h\nu/kT)$. This is quite consistent with the results in our Table 2, which show that the two-quantum emission

[18] *M.N.*, **96**, 604, 1936.

hardly affects the Balmer emission for $\lambda > 3300$. On the basis of the slight decrease in slope that our results predict, we might suggest that the conventionally deduced electron temperatures are somewhat too high and should be reduced by about 25 per cent. One prediction based on the present analysis is that the electron temperatures deduced from spectrophotometry in the region λλ 6080–4050 should be very much higher than those deduced from either the infrared (Paschen) or ultraviolet (Balmer) emission just shortward of a series limit.

The two-quantum emission reduces the discontinuity at the Balmer limit. From Table 2 we can obtain the predicted ratio of the surface brightness j at λ 4050 to that at λ 3646. This ratio proves rather insensitive to temperature and is equal to 0.23 and 0.09 for $\theta = 0.35$ and $\theta = 1.0$, respectively. Without the two-quantum emission, the ratio would be 0.10–0.01.

It is tempting to interpret the so-called V_c observed by Page[3] in the region λλ 4800–

TABLE 2

THE INTENSITY* OF THE RECOMBINATION EMISSION (j'_H), THE TWO-QUANTUM EMISSION (j_{2q} FOR $X = 0.32$), AND THE TOTAL EMISSION (j)

λ (A)	y	$\theta=0.35$			$\theta=0.50$			$\theta=0.70$			$\theta=1.00$		
		$\log_{10} j'_H$	$\log_{10} j_{2q}$	$\log_{10} j$	$\log_{10} j'_H$	$\log_{10} j_{2q}$	$\log_{10} j$	$\log_{10} j'_H$	$\log_{10} j_{2q}$	$\log_{10} j$	$\log_{10} j'_H$	$\log_{10} j_{2q}$	$\log_{10} j$
8202..	0.148	−1.34	−2.04	−1.26	−1.35	−2.15	−1.29	−1.37	−2.25	−1.32	−1.39	−2.36	−1.35
6080..	.20	−1.52	−1.85	−1.35	−1.61	−1.96	−1.45	−1.74	−2.06	−1.57	−1.92	−2.17	−1.73
4861..	.25	−1.70	−1.72	−1.41	−1.87	−1.82	−1.55	−2.10	−1.92	−1.70	−2.43	−2.04	−1.89
4050..	.30	−1.88	−1.62	−1.43	−2.12	−1.72	−1.58	−2.37	−1.82	−1.71	−2.94	−1.94	−1.89
3646..	.333	−0.88	−1.56	−0.79	−0.88	−1.66	−0.82	−0.90	−1.76	−0.84	−0.90	−1.88	−0.86
3470..	.35	−0.94	−1.53	−0.84	−0.97	−1.64	−0.89	−1.02	−1.74	−0.94	−1.07	−1.85	−1.00
3040..	.40	−1.11	−1.47	−0.95	−1.22	−1.57	−1.06	−1.38	−1.67	−1.20	−1.58	−1.79	−1.37
2700..	.45	−1.31	−1.41	−1.05	−1.48	−1.51	−1.19	−1.73	−1.61	−1.37	−2.09	−1.73	−1.57
2430..	.50	−1.47	−1.36	−1.11	−1.75	−1.47	−1.28	−2.09	−1.57	−1.45	−2.59	−1.68	−1.63
2210..	.55	−1.65	−1.31	−1.15	−2.00	−1.43	−1.32	−2.44	−1.53	−1.47	−3.11	−1.64	−1.63
2020..	0.60	−1.85	−1.29	−1.18	−2.26	−1.39	−1.34	−2.80	−1.49	−1.47	−3.61	−1.61	−1.61

* Emissivities are given in units of $Bh\beta^2/4\pi\rho$.

3900 as the two-quantum emission essentially unaffected by the ordinary Paschen continuum. Unfortunately, his observations were made with a very wide slit and were uncorrected for unresolved weak emission lines. They give a V_c from one-half to one-quarter the Balmer emission. This ratio is very high compared to that predicted for ordinary recombination emission but is consistent with the present theory if $X = 1$. His observations indicate, however, that the V_c is approximately constant per wave-length interval, i.e., slightly decreasing shortward, per $d\nu$, while the Paschen continuum decreases rapidly and the two-quantum emission increases slowly shortward.

P. Swings and O. Struve[19] reproduce spectra showing a strong continuous spectrum in the Orion nebula and in IC 418. The great strength found for the continuum in IC 418, λλ 4900–3700, is quite remarkable. One peculiarity of IC 418 is the relatively low temperature and high corresponding photographic brightness of the central star. The Balmer discontinuity is very small. The Balmer discontinuity is also quite small in the Orion nebula, possibly because of the starlight scattered by interstellar dust grains.[20] The Crab nebula has a strong continuous spectrum, certainly unaffected by scattered starlight.

[19] *Ap. J.*, **96**, 310, 1942.

[20] J. L. Greenstein, *Ap. J.*, **104**, 414, 1946.

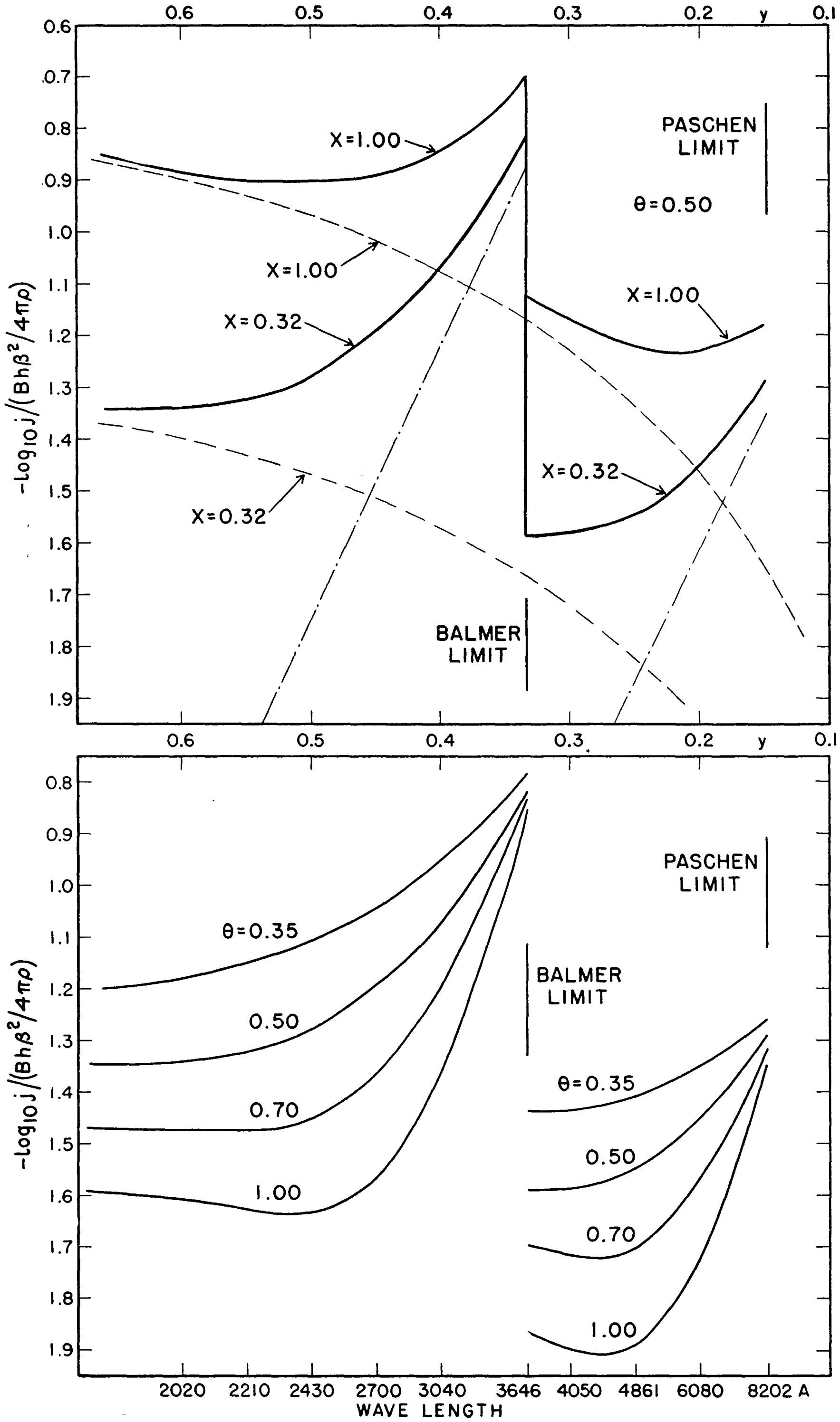

FIG. 1.—*a*, Upper section. Plotted are: the straight lines (*dots and dashes*), representing the hydrogen recombination emission, j_H'; dashed-line curves for the two-quantum emission, j_{2q} when $X = 0.32$ and 1.00; solid-line curves, showing their sum, j, the total hydrogen emission. All are given for electron temperature 10,000°, $\theta = 0.5$. *b*, Lower section. The total emission, j, including two-quantum radiation for $X = 0.32$ at various electron temperatures.

R. Minkowski[21] finds that in the Crab nebula the Balmer jump is very small, perhaps only 15 per cent, and the electron temperature deduced from the energy distribution in the visual region is about $T_e = 36{,}000°$. The apparent high T_e reminds one of the results predicted for j_{2q} in our Figure 1. But at $T_e = 36{,}000°$ the Balmer jump is still quite appreciable. Thus Minkowski's hypothesis of free-free emission in the Crab nebula seems preferable. It is apparent that further accurate spectrophotometry of continua in planetary nebula is greatly to be desired.

We are grateful to Drs. O. C. Wilson and R. Minkowski for a number of illuminating discussions on this problem.

[21] *Mt. W. Contr.*, No. 666; *Ap. J.*, **96**, 199, 1942.