ON THE INTEGRAL EQUATION GOVERNING THE DISTRIBUTION OF THE TRUE AND THE APPARENT ROTATIONAL VELOCITIES OF STARS

S. Chandrasekhar and G. Münch
Yerkes Observatory

Received October 31, 1949

ABSTRACT

In this paper the integral equation governing the distribution of the true (v) and the apparent ($v \sin i$) rotational velocities of stars is reconsidered, with the object of suggesting the most suitable methods for analyzing observed frequency functions affected by a random orientation factor, $\sin i$, in the manner of the rotational velocities of stars. First, it is shown that there is a simple relation between the moments of the true and the observed frequency functions which enables us to pass from the moments of the one to the moments of the other. The mean and the mean square of the true rotational velocities of stars can therefore be determined directly from the corresponding means of the observed distribution of $v \sin i$. When it is felt that something more should be said about the true distribution of v than its mean and mean square, it is suggested that a comparison be made between the observed distribution and those derived from certain assumed forms of the true frequency function. Reasons are given for preferring this method to an inversion of the integral equation by a numerical procedure. The form

$$f(x) = \frac{1}{\sqrt{\pi}} \{ e^{-(x-x_1)^2} + e^{-(x+x_1)^2} \}$$

suggested by Elsa van Dien has therefore been considered in some detail, and a one-parameter family of frequency functions for $x \sin i$ has been derived.

The methods of analysis suggested in this paper have been applied to a rediscussion of the rotational velocities of stars.

1. *Introduction.*—A problem of some interest which arises in discussions relating to stellar rotation concerns the manner in which allowance must be made for the random orientation of the rotational axes of the stars in deriving the true distribution of the rotational velocities.[1] The precise nature of the problem encountered here is the following:

From a study of the line profiles of a rotating star we may deduce the value of $v \sin i$, where v denotes the equatorial rotational velocity of the star and i is the inclination of its axis of rotation to the line of sight. From the distribution of $v \sin i$ determined in this fashion for a homogeneous[2] group of stars, we wish to infer the true distribution of the rotational velocities v. It is apparent that an integral equation must govern the two distributions, and the problem is essentially one of solving this integral equation.

The same integral equation as that which governs the distribution of v and $v \sin i$ occurs also in other contexts: for example, in the discussion of the mass function of binary stars.[3] Again, the integral equation which governs the distribution of star images on a photograph of a globular cluster and the true space distribution of the stars in the cluster is also the same.[4] This paper is therefore devoted to an examination of this integral equation and to outlining suitable methods of analysis of observational data affected by random orientation in the manner of the rotational velocities of stars.

[1] Cf. O. Struve, *Pop. Astr.*, **53**, 202, 259, 1945; see esp. pp. 213 and 214.

[2] "Homogeneous" in the sense that selection has not operated for certain preferred orientations of the axis of rotation.

[3] Cf. G. P. Kuiper, *Pub. A.S.P.*, **47**, 15, 1935; see esp. pp. 32–36 and Fig. 2 on p. 32.

[4] Cf. S. Chandrasekhar, *Principles of Stellar Dynamics* (Chicago: University of Chicago Press, 1942), p. 233. Eq. (5.814) given on p. 233 becomes identical with the integral eq. (9) derived in § 2 of this paper, if eq. (5.814) is expressed in terms of the number of star images $2\pi\nu(x)xdx$ in a ring between x and $x + dx$ and the number of stars $4\pi r^2 N(r)dr$ in the cluster, in a spherical shell between r and $r + dr$, instead of $\nu(x)$ and $N(r)$.

2. *The basic integral equation.*—The problem which we wish to consider may be formulated in the following manner:

A parameter x occurs with a probability distribution governed by a frequency function $f(x)$ $(0 \leqslant x < \infty)$. The quantity

$$y = x \sin i \tag{1}$$

is observed where the probability of occurrence of the inclination i between i and $i + di$ is known to be $\sin i \, di$. It is required to relate the probability distribution of y with that of x.

Let $\phi(y)$ denote the frequency function of y. Then, from the definition of probability, it follows that $\phi(y)dy$ is the surface integral of $f(x) \sin i$ over the area included between the curves

$$y = x \sin i \quad \text{and} \quad y + dy = x \sin i\,, \tag{2}$$

in the (x, i)-plane (see Fig. 1). Thus

$$\phi(y)\, dy = \iint_{\Delta S} f(x) \sin i \, dx di\,, \tag{3}$$

where ΔS denotes the area shaded in Figure 1.

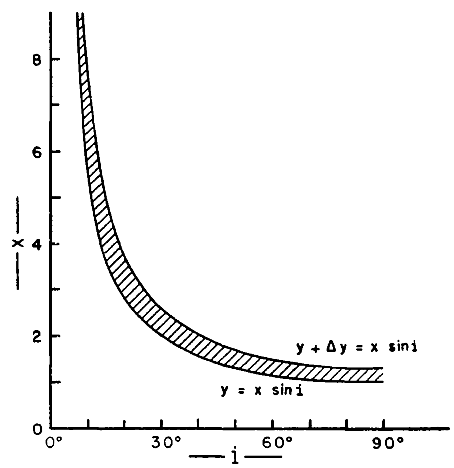

FIG. 1.—Illustrating the area in the (x, i) plane over which $f(x) \sin i$ must be integrated to give $\phi(y)dy$.

Now at a particular i, the height of the strip ΔS is given by

$$dx = \frac{dy}{\sin i}\,; \tag{4}$$

and, since y and dy are assigned, we can rewrite equation (3) in the form

$$\phi(y)\, dy = dy \int_{y = x \sin i} f(x)\, di\,, \tag{5}$$

where the integral is now extended along the curve $y = x \sin i$. Accordingly,

$$\phi(y) = \int_0^{\pi/2} f\left(\frac{y}{\sin i}\right) di\,. \tag{6}$$

Alternatively, we may also argue as follows:

At a particular x the width of the strip ΔS is

$$di = \frac{dy}{x \cos i}; \tag{7}$$

hence, we may also write

$$\phi(y)\, dy = dy \int_{y=x\sin i} f(x) \frac{\sin i}{x \cos i}\, dx, \tag{8}$$

where the integral is again extended along the curve $y = x \sin i$; using this equation of the curve to eliminate i in the integrand of equation (8), we obtain

$$\phi(y) = y \int_y^\infty \frac{f(x)}{x(x^2 - y^2)^{1/2}}\, dx. \tag{9}$$

This equation relating $\phi(y)$ and $f(x)$ is implicit in the papers of Struve on stellar rotation; and the equation occurs explicitly in Kuiper's paper to which we have already referred.[3]

It is, of course, clear that equations (6) and (9) are entirely equivalent.

3. *The formal solution of the integral equation.*—Equation (9) can be reduced to Abel's integral equation by the substitutions

$$y^2 = \frac{1}{\eta} \quad \text{and} \quad x^2 = \frac{1}{\xi}; \tag{10}$$

for, with these substitutions, equation (9) becomes

$$\Phi(\eta) = \int_0^\eta \frac{F(\xi)}{(\eta - \xi)^{1/2}}\, d\xi, \tag{11}$$

where

$$\Phi(\eta) \equiv \phi\left(\frac{1}{\sqrt{\eta}}\right) \quad \text{and} \quad F(\xi) \equiv \frac{1}{2\sqrt{\xi}} f\left(\frac{1}{\sqrt{\xi}}\right). \tag{12}$$

It is well known that the solution of Abel's equation (11) is given by

$$F(\xi) = \frac{1}{\pi} \frac{\partial}{\partial \xi} \int_0^\xi \frac{\Phi(\eta)}{(\xi - \eta)^{1/2}}\, d\eta. \tag{13}$$

In terms of the original variables the foregoing solution is equivalent to

$$f(x) = -\frac{2}{\pi} x^2 \frac{\partial}{\partial x} x \int_x^\infty \frac{\phi(y)}{y^2 (y^2 - x^2)^{1/2}}\, dy. \tag{14}$$

While equation (14) represents the formal solution of the problem, it is not of much practical use, since it requires differentiation and it is known that the differentiation of an observed frequency function can lead to results which are misleading unless the observations are of high precision. In practice it will therefore be advisable to use, as far as possible, only the moments of the observed frequency function, $\phi(y)$, and determine one or more parameters (such as the mean and the mean square deviation) of the true frequency function, $f(x)$. In any case, the form of the solution (14) emphasizes the fact that no artifice can really circumvent the necessity of differentiating the observed frequency function. Therefore, when the observed distribution is known in the form of a histogram (as is often the case) a numerical inversion of the integral equation (9) can hardly be expected to give trustworthy results.

4. *The relation between the moments of $\phi(y)$ and $f(x)$.*—We shall first show that there is a simple relation between the moments of $\phi(y)$ and $f(x)$.

Multiplying equation (9) by y^n and integrating over the range of y, we have

$$\int_0^\infty \phi(y)\, y^n dy = \int_0^\infty dy\, y^{n+1} \int_y^\infty dx \frac{f(x)}{x(x^2 - y^2)^{1/2}}. \tag{15}$$

Inverting the order of the integration on the right-hand side, we obtain

$$\int_0^\infty \phi(y)\, y^n dy = \int_0^\infty \frac{dx}{x} f(x) \int_0^x dy \frac{y^{n+1}}{(x^2 - y^2)^{1/2}}. \tag{16}$$

Now letting $y = tx$ in the integral over y, we have

$$\int_0^\infty \phi(y)\, y^n dy = \int_0^\infty f(x)\, x^n dx \int_0^1 \frac{t^{n+1}}{(1 - t^2)^{1/2}} dt. \tag{17}$$

Hence

$$\overline{y^n} = \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)} \overline{x^n}. \tag{18}$$

In particular,

$$\bar{x} = \frac{4}{\pi}\bar{y}; \qquad \overline{x^2} = \frac{3}{2}\overline{y^2}; \qquad \overline{x^3} = \frac{16}{3\pi}\overline{y^3}. \tag{19}$$

It has been pointed out to us by Messrs. A. Brown, Su-Shu Huang, and D. Osterbrock that a relation of the form (18) also exists, when the relation (1) is replaced by the more general $y = x\psi(i)$, where $\psi(i)$ is an arbitrary integrable function of i; for, by multiplying the equation corresponding to (3) by y^n and integrating over the whole phase space we obtain the relation

$$\overline{y^n} = \overline{x^n} \int_0^{\pi/2} \psi^n(i)\, w(i)\, di, \tag{18'}$$

where $w(i)$ is the frequency function of i.

The mean, the mean square deviation, and the skewness of the true distribution $f(x)$ can therefore be derived from the moments of the apparent distribution $\phi(y)$ according to the following formulae:

$$\bar{x} = \frac{4}{\pi}\bar{y}; \qquad \overline{(x - \bar{x})^2} = 1.5\overline{y^2} - \frac{16}{\pi^2}\bar{y}^2,$$

and

$$\overline{(x - \bar{x})^3} = \frac{16}{3\pi}\overline{y^3} - \frac{18}{\pi}\overline{y^2}\,\bar{y} + \frac{128}{\pi^3}\bar{y}^3. \tag{20}$$

As an example of the application of the foregoing formulae we shall consider the recent results of A. Slettebak[5] on the rotational velocities of the Be stars. From his table of $v \sin i$ we find that

$$\overline{v \sin i} = 2.73; \qquad \overline{v^2 \sin^2 i} = 8.242; \qquad \text{and} \qquad \overline{v^3 \sin^3 i} = 26.356, \tag{21}$$

where a unit of velocity of 100 km/sec has been adopted. From equations (20) we now deduce that

$$\bar{v} = 3.48; \qquad \overline{(v - v)^2}^{1/2} = 0.50,$$

and

$$\overline{(v - \bar{v})^3} = -0.039. \tag{22}$$

The skewness of the true distribution is therefore negligible, and the mean (~350 km/sec) and the root mean square deviation (~50 km/sec) of the true distribution of the

[5] *Ap. J.*, **110**, 498, 1949; see esp. Table 2 of this paper.

rotational velocities of the Be stars are therefore determined with the same precision as these quantities are determined for the apparent distribution.

5. *Special forms of* $f(x)$.—When the frequency function $\phi(y)$ is not too well determined by the observations, then the mean and, to a less extent, the mean square are probably the only two quantities which can be determined with any degree of trustworthiness. Under these circumstances the formulae of the last section suffice to convert the mean and the mean square of observed frequency function into the corresponding means of the true frequency function, in which we are interested. However, in some cases it may be thought that $\phi(y)$ is well enough determined to say something more about $f(x)$ than its mean and mean square. In such cases attempts have been made in the literature to invert the integral equation (9) by a numerical procedure. But, as we have already indicated, such a procedure can be justified only in cases in which the observed frequency function can be differentiated with confidence. In all other cases it is preferable to assume a form for $f(x)$ involving one or more parameters and suggested by the physical nature of the quantity under discussion and to determine these parameters consist-

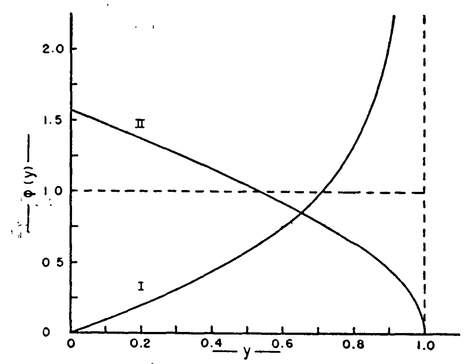

FIG. 2.—The frequency functions $\phi(y)$ derived from a δ-function centered at $x_1 = 1$ (curve *I*) and a uniform distribution of x in the interval $0 \leqslant x \leqslant 1$ (curve *II*). The same curves represent the function $\phi(y)$ for other values of x_1 if the abscissae are understood to mean y/x_1 and the ordinates $x_1\phi(x_1)$.

ently with the accuracy of the observational material. This latter procedure has the disadvantage that some form of $f(x)$ has to be assumed. Consequently, it entails a certain degree of arbitrariness; but it is inherent in the nature of the problem and cannot be avoided whenever the observed frequency function is not of a precision required for an unambiguous application of the solution given by equation (14).

In accordance with the remarks in the preceding paragraph, we shall consider certain special forms of $f(x)$ and derive the corresponding forms of $\phi(y)$.

The first form for $f(x)$ that we shall consider is when x takes only one value, say, x_1. Then

$$f(x) = \delta(x - x_1) , \tag{23}$$

where δ denotes Dirac's δ-function. From equation (9) it now follows that

$$\begin{aligned} \phi(y;\ \delta_{x_1}) &= \frac{y}{x_1(x_1^2 - y^2)^{1/2}} \qquad & y < x_1 \\ &= 0 & y > x_1 ; \end{aligned} \tag{24}$$

this distribution has therefore a singularity at $x = x_1$ (see Fig. 2).[6]

[6] The occurrence of this singularity makes the "reconstruction" of an observed $\phi(y)$ by superposing a number of distributions $\phi(y;\ \delta_{x_n})$ in the form

$$\phi(y) = \Sigma a_n \phi(y; \delta_{x_n})$$

a less valid procedure than it would be otherwise.

A second form of $f(x)$ which may be considered is the case in which all values of x less than a certain x_1 occur with equal probability. Then

$$f(x) = \frac{1}{x_1} \qquad x \leqslant x_1$$
$$= 0 \qquad x > x_1 . \tag{25}$$

For this form of $f(x)$, equation (9) leads to

$$\phi(y) = \frac{1}{x_1} \cos^{-1} \frac{y}{x_1} \qquad y \leqslant x_1$$
$$= 0 \qquad y > x_1 . \tag{26}$$

The nature of this distribution is also illustrated in Figure 2.

While cases (23) and (25) are sometimes useful, they do not provide the flexibility in the predicted forms of $\phi(y)$ which we should like to have for comparisons with observed frequency functions. A form for $f(x)$ which appears most suitable in the stellar contexts is the one suggested by Elsa van Dien,[7] namely,

$$f(x) = \frac{1}{\sqrt{\pi}} \{ e^{-(x-x_1)^2} + e^{-(x+x_1)^2} \} . \tag{27}$$

This form for $f(x)$ provides a family of frequency functions with two parameters. Also, for $x_1 = 0$, $f(x)$ represents a Gaussian distribution; and for x_1 large it is again, essentially, a Gaussian distribution; and between these two limits we have a family of "intermediate" distributions which are even.

For $f(x)$ given by equation (27) the first three moments are:

$$\overline{x} = \frac{1}{\sqrt{\pi}} e^{-x_1^2} + x_1 \Phi(x_1) ,$$
$$\overline{x^2} = x_1^2 + \tfrac{1}{2} , \tag{28}$$

and

$$\overline{x^3} = \frac{1}{\sqrt{\pi}} e^{-x_1^2} (1 + x_1^2) + (\tfrac{3}{2} + x_1^2) x_1 \Phi(x_1)$$

where

$$\Phi(x_1) = \frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-t^2} dt \tag{29}$$

is the error function. The numerical values of these moments are given in Table 1. The corresponding moments of y can be obtained in accordance with equation (19).

When $f(x)$ has the form (27), the equation for $\phi(y)$ becomes

$$\phi(y; x_1) = \frac{y}{\sqrt{\pi}} \int_y^{\infty} \frac{e^{-(x-x_1)^2} + e^{-(x+x_1)^2}}{x(x^2 - y^2)^{1/2}} dx , \tag{30}$$

or, alternatively,

$$\phi(y; x_1) = \frac{2y}{\sqrt{\pi}} e^{-x_1^2} \int_y^{\infty} \frac{e^{-x^2} \cosh 2 x x_1}{x(x^2 - y^2)^{1/2}} dx . \tag{31}$$

In equations (30) and (31) we have emphasized the dependence of $\phi(y)$ on x_1 by including it as an argument.

It does not seem that the integral defining $\phi(y; x_1)$ can be reduced to known functions. However, it will appear from the discussion of this integral in § 6 that we can get a fairly complete picture of $\phi(y; x_1)$ by considering its behavior for various ranges of its arguments.

[7] *J.R.A.S. Canada*, **42**, 249, 1948.

6. *The behavior of $\phi(y; x_1)$ for various ranges of its arguments.* (i) *The expansion of $\phi(y; x_1)$ in a power series in x_1.*—When $x_1 \leqslant 1$, we can obtain a rapidly convergent expansion for $\phi(y; x_1)$ by replacing cosh $2xx_1$ in equation (31) by its series and integrating term by term; thus,

$$\phi(y; x_1) = \frac{2}{\sqrt{\pi}} e^{-x_1^2} \sum_{n=0}^{\infty} \frac{(2x_1)^{2n}}{2n!} y \int_y^{\infty} \frac{e^{-x^2} x^{2n-1}}{(x^2 - y^2)^{1/2}} dx . \tag{32}$$

Letting

$$I_n(y) = y \int_y^{\infty} \frac{e^{-x^2} x^{2n-1}}{(x^2 - y^2)^{1/2}} dx , \tag{33}$$

we can rewrite equation (32) in the form

$$\phi(y; x_1) = \frac{2}{\sqrt{\pi}} e^{-x_1^2} \sum_{n=0}^{\infty} \frac{2^{2n}}{2n!} x_1^{2n} I_n(y) . \tag{34}$$

TABLE 1

CONSTANTS RELATING TO THE FREQUENCY FUNCTION

$$f(x) = \frac{1}{\sqrt{\pi}} [e^{-(x-x_1)^2} + e^{-(x+x_1)^2}]$$

x_1	$\bar{x}$	$\overline{x^2}$	$\overline{x^3}$	$\bar{x}/[2(\overline{x^2}-\bar{x}^2)]^{1/2}$
0.0	0.5642	0.50	0.564	0.93593
0.1	0.5698	0.51	0.581	0.93598
0.2	0.5866	0.54	0.632	0.93719
0.3	0.6142	0.59	0.719	0.9418
0.4	0.6521	0.66	0.842	0.9518
0.5	0.6996	0.75	1.00	0.9693
0.6	0.7559	0.86	1.21	0.9951
0.7	0.8201	0.99	1.46	1.0292
0.8	0.8912	1.14	1.76	1.072
0.9	0.9682	1.31	2.11	1.122
1.0	1.0502	1.50	2.52	1.179
1.2	1.226	1.94	3.54	1.314
1.4	1.413	2.46	4.85	1.466
1.6	1.606	3.06	6.45	1.636
1.8	1.802	3.74	8.53	1.819
2.0	2.001	4.50	11.0	2.009
2.2	2.200	5.34	13.9	2.204
2.4	2.400	6.26	17.4	2.402
2.6	2.600	7.26	21.5	2.601
2.8	2.800	8.34	26.2	2.800
3.0	3.000	9.50	31.5	3.000

The integrals $I_n(y)$ ($n = 0, 1, \ldots$) which occur in this expansion can all be expressed in terms of known functions in the following manner:

First, we may note that, with the substitution

$$x^2 = y^2 + u^2 , \tag{35}$$

I_n becomes

$$I_n = y\, e^{-y^2} \int_0^{\infty} e^{-u^2} (y^2 + u^2)^{n-1} du . \tag{36}$$

Next, we observe that the I_n's satisfy the recursion formula,

$$I_{n+1} = (n - \tfrac{1}{2} + y^2) I_n - (n-1) y^2 I_{n-1} \qquad (n = 1, \ldots) . \tag{37}$$

This formula can be derived by writing I_n in the form

$$I_n = y \int_y^\infty e^{-x^2} x^{2n-2} \frac{d}{dx} (x^2 - y^2)^{1/2} dx \tag{38}$$

and integrating by parts; thus

$$\begin{aligned} I_n &= 2y \int_y^\infty e^{-x^2} x^{2n-3} (x^2 - n + 1)(x^2 - y^2)^{1/2} dx \\ &= 2y \int_y^\infty \frac{e^{-x^2}}{(x^2 - y^2)^{1/2}} x^{2n-3} [x^4 - x^2 (n - 1 + y^2) + (n-1) y^2] \, dx \\ &= 2 [I_{n+1} - (n - 1 + y^2) I_n + (n-1) y^2 I_{n-1}] ; \end{aligned} \tag{39}$$

and this is equivalent to formula (37).

Using the recursion formula (37), we can reduce the evaluation of I_n to I_1 and I_0; and both these can be reduced to known functions. Thus (cf. eq. [36])

$$I_1 = y e^{-y^2} \int_0^\infty e^{-u^2} du = \tfrac{1}{2} \sqrt{\pi} \, e^{-y^2} y . \tag{40}$$

On the other hand,

$$I_0(y) = y e^{-y^2} \int_0^\infty \frac{e^{-u^2}}{y^2 + u^2} du ; \tag{41}$$

and we recognize I_0 as a standard integral which occurs in the theory of the line-absorption coefficient in stellar atmospheres;[8] and using known results we have

$$I_0(y) = \tfrac{1}{2}\pi [1 - \Phi(y)] , \tag{42}$$

where $\Phi(y)$ denotes the error function (cf. eq. 29]).

All the integrals I_n can therefore be evaluated successively in terms of I_0 and I_1.

For purposes of practical calculation it is convenient to rewrite the expansion (34) in the form

$$\phi(y; x_1) = e^{-x_1^2} \sum_{n=0}^{\infty} F_n(y) \, x_1^{2n} , \tag{43}$$

where

$$F_n(y) = \frac{2^{2n+1}}{2n! \sqrt{\pi}} I_n(y) . \tag{44}$$

The functions $F_n(y)$ for $n = 0, \ldots, 6$, are tabulated (see Table 2). Using this table, we can evaluate $\phi(y; x_1)$ for all $x_1 \leqslant 1$ with entirely sufficient accuracy.

ii) *The expansion of $\phi(y; x_1)$ in a power series in y.*—An expansion of $\phi(y; x_1)$ in y analogous to the one in x_1 found in the preceding subsection can be obtained in the following manner:

First, using the relation

$$e^{-p^2} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \cos 2pu \, du , \tag{45}$$

[8] Cf. D. L. Harris, *Ap. J.*, **108**, 112, 1948. In Harris' notation

$$I_0(y) = \tfrac{1}{2}\pi \, e^{-y^2} H(y, 0) ;$$

and, according to equation (2) of his paper,

$$H(y, 0) = \frac{2}{\sqrt{\pi}} e^{y^2} \int_y^\infty e^{-t^2} dt ;$$

the result (42) quoted in the text follows from this.

we write

$$\frac{1}{\sqrt{\pi}}\left[e^{-(x-x_1)^2}+e^{-(x+x_1)^2}\right]=\frac{4}{\pi}\int_0^\infty e^{-u^2}\cos 2x_1u\cos 2xu\,du\,. \tag{46}$$

Then, inserting this expression in equation (30) and inverting the order of the integrations, we obtain

$$\phi(y;\,x_1)=\frac{4}{\pi}\int_0^\infty du\,e^{-u^2}\cos 2x_1u\int_y^\infty dx\,\frac{y\cos 2xu}{x(x^2-y^2)^{1/2}} \tag{47}$$

or, letting $x = yt$ in the integral over x, we have

$$\phi(y;\,x_1)=\frac{4}{\pi}\int_0^\infty du\,e^{-u^2}\cos 2x_1u\int_1^\infty dt\,\frac{\cos 2uyt}{t(t^2-1)^{1/2}}. \tag{48}$$

TABLE 2

THE FUNCTIONS $F_n(y)$ FOR $n=0, 1, \ldots, 6$

y	F_0	F_1	F_2	F_3	F_4	F_5	F_6
0.0	1.7725	0	0	0	0	0	0
0.2	1.3777	0.3843	0.0692	0.0135	0.0024	0.0004	0.000005
0.4	1.0131	.6817	.1500	.0283	.0049	.0008	.000010
0.6	0.7021	.8372	.2400	.0461	.0078	.0012	.000016
0.8	0.4571	.8437	.3206	.0675	.0112	.0017	.000022
1.0	0.2788	.7358	.3679	.0899	.0155	.0022	.000029
1.2	0.1590	.5686	.3677	.1078	.0202	.0030	.000037
1.4	0.08457	.39440	.32341	.11484	.02451	.00380	.000048
1.6	0.04192	.24738	.25232	.10844	.02689	.00460	.000061
1.8	0.01933	.14099	.17577	.09078	.02637	.00510	.000073
2.0	0.00829	.07326	.10989	.06756	.02300	.00509	.000081
2.2	0.00330	.03479	.06193	.04487	.01781	.00453	.000081
2.4	0.00122	.01513	.03156	.02668	.01228	.00359	.000073
2.6	0.000418	0.006028	0.014588	0.014255	0.007550	0.002525	0.000058

Now from the identity

$$J_0(z)=\frac{2}{\pi}\int_1^\infty\frac{\sin zt}{(t^2-1)^{1/2}}\,dt, \tag{49}$$

where $J_0(z)$ denotes the Bessel function of order zero, it follows that

$$\int_0^z J_0(z)\,dz=1-\frac{2}{\pi}\int_1^\infty\frac{\cos zt}{t(t^2-1)^{1/2}}\,dt. \tag{50}$$

Using this result in equation (48), we obtain (cf. eq. [45])

$$\phi(y;\,x_1)=\sqrt{\pi}\,e^{-x_1^2}-2\int_0^\infty e^{-u^2}\left\{\int_0^{2uy}J_0(z)\,dz\right\}\cos 2x_1u\,du\,. \tag{51}$$

We can now obtain an expansion of $\phi(y; x_1)$ as a power series in y by expanding the integral over the Bessel function in equation (51) as a power series in $2uy$. Thus, from the known series for $J_0(z)$ it follows that

$$\int_0^{2uy}J_0(z)\,dz=2\sum_{n=0}^{\infty}(-1)^n\frac{(uy)^{2n+1}}{(2n+1)(n!)^2}. \tag{52}$$

Therefore, defining

$$D_n(x_1)=\frac{2}{n!}\,e^{x_1^2}\int_0^\infty e^{-u^2}u^{2n+1}\cos 2x_1u\,du\,, \tag{53}$$

we can rewrite equation (51) in the form

$$\phi(y;\, x_1) = \sqrt{\pi}\, e^{-x_1^2}\left[1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{D_n(x_1)}{(2n+1)\, n!}\, y^{2n+1}\right]. \tag{54}$$

The integrals $D_n(x_1)$ ($n = 0, 1, \ldots$) which occur in this expansion can be reduced to the evaluation of D_0 by the use of the following recursion formulae, which are readily established by successive integration by parts:

$$(n+1)\, D_{n+1}(x_1) = (2n + \tfrac{3}{2} - x_1^2)\, D_n(x_1) - (n + \tfrac{1}{2})\, D_{n-1}(x_1)$$

and

$$D_n(x_1) = \left(1 - \frac{1}{n}\, x_1^2\right) D_{n-1}(x_1) + \frac{x_1}{2n} \frac{d D_{n-1}(x_1)}{d x_1} \qquad (n = 1, 2, \ldots). \tag{55}$$

By transformations similar to those used in the reduction of the integral which occurs in the theory of the line-absorption coefficient which combines Doppler effect and damping, we find that

$$D_0(x_1) = e^{x_1^2}\left\{1 - 2 x_1 e^{-x_1^2} \int_0^{x_1} e^{t^2} dt\right\}. \tag{56}$$

The expression on the right-hand side is simply related to a quantity which has been tabulated by Harris;[8] thus, in Harris' notation,

$$D_0(x_1) = -\tfrac{1}{2} \sqrt{\pi}\, e^{x_1^2} H_1(x_1). \tag{57}$$

Thus the functions D_n can all be evaluated. However, the series (54) does not extend the range of the expansion (43) by a large amount for $x_1 > 1$. Thus, for $x_1 = 1.5$, it is found that seven terms in the series (54) are required to evaluate ϕ at $y = 1.3$; for larger values of x_1 the range of y in which the series can be profitably used becomes less. Nevertheless, the series (54) does enable us to bridge the gap between the series (43) good for $x_1 \leqslant 1$ (and all y) and the asymptotic expansions found below for $x_1 > 2$.

iii) *An asymptotic expansion for $\phi(y; x_1)$ for x_1 large and $y < x_1$.*—When $x_1 > 2$, we can neglect the term in $e^{-(x+x_1)^2}$ in equation (30); the maximum error introduced by this is less than 2 per cent for $x_1 = 2$; and for larger values of x_1 the error introduced by this simplification decreases very rapidly—in fact, like $e^{-x_1^2}$! For $x_1 > 2$, we may therefore write

$$\phi(y;\, x_1) = \frac{y}{\sqrt{\pi}} \int_y^{\infty} \frac{e^{-(x-x_1)^2}}{x\,(x^2 - y^2)^{1/2}}\, dx \qquad (x_1 > 2). \tag{58}$$

With the substitution

$$x - x_1 = t\,, \tag{59}$$

equation (58) becomes

$$\phi(y;\, x_1) = \frac{y}{\sqrt{\pi}} \int_{-(x_1-y)}^{\infty} \frac{e^{-t^2} dt}{(t + x_1)\,(t + x_1 + y)^{1/2}\,(t + x_1 - y)^{1/2}}. \tag{60}$$

Now let

$$y = a x_1 \qquad (a < 1). \tag{61}$$

Then

$$\phi(y;\, x_1) = \frac{y}{x_1 (x_1^2 - y^2)^{1/2}} \frac{1}{\sqrt{\pi}} \times \int_{-(1-a)x_1}^{\infty} \frac{e^{-t^2} dt}{(1 + t/x_1)\,[1 + t/x_1(1+a)]^{1/2}[1 + t/x_1(1-a)]^{1/2}}. \tag{62}$$

Now, if

$$(1 - a)\, x_1 > 2\,, \tag{63}$$

we may expand the denominator in formula (62) in a power series in $1/x_1$ and also extend the range of integration over t from $+\infty$ to $-\infty$. In this manner we shall obtain an asymptotic expansion valid for $x_1(1-a)\to\infty$. By the procedure indicated we find:

$$\phi(y;\,x_1)\simeq\frac{y}{x_1(x_1^2-y^2)^{1/2}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}\left\{1-\frac{t}{x_1}\frac{2-a^2}{1-a^2}+\frac{t^2}{x_1^2}\frac{6-5a^2+2a^4}{2(1-a^2)^2}-\dots\right\}dt\,, \tag{64}$$

or

$$\phi(y;\,x_1)\simeq\frac{y}{x_1(x_1^2-y^2)^{1/2}}\left\{1+\frac{6y^4-5y^2x_1^2+2x_1^4}{4x_1^2(x_1^2-y^2)^2}+\dots\right\}. \tag{65}$$

It will be noticed that the dominant term in the asymptotic expansion (65) is the contour for a δ-function in x centered at x_1; this is, of course, what we should have expected.

The expansion (65) is satisfactory for determining $\phi(y;\,x_1)$ for x_1 large and $y<x_1$; thus for $x_1=10$ it gives sufficient accuracy for $y<8$.

iv) *An asymptotic expansion for $\phi(x_1;\,x_1)$ for x_1 large.*—The expansion found in the preceding section ceases to be valid when $y\to x_1$. However, for $y=x_1$ we can find a satisfactory expansion in the following manner:

For $y=x_1$ equation (58) is

$$\phi(x_1;\,x_1)=\frac{x_1}{\sqrt{\pi}}\int_{x_1}^{\infty}\frac{e^{-(x-x_1)^2}}{x(x^2-x_1^2)^{1/2}}\,dx\,. \tag{66}$$

Now let

$$(x-x_1)^2=t\,. \tag{67}$$

Equation (66) becomes

$$\phi(x_1;\,x_1)=\frac{1}{(8\pi x_1)^{1/2}}\int_0^{\infty}\frac{e^{-t}dt}{t^{3/4}(1+\sqrt{t}/x_1)(1+\sqrt{t}/2x_1)^{1/2}}. \tag{68}$$

We can now obtain an asymptotic expansion for the integral on the right-hand side by expanding the denominator in a power series in $1/x_1$ and integrating term by term; thus

$$\phi(x_1;\,x_1)\simeq\frac{1}{(8\pi x_1)^{1/2}}\left[\Gamma(0.25)-\frac{5}{4x_1}\Gamma(0.75)+\frac{43}{32x_1^2}\Gamma(1.25)-\frac{177}{128x_1^3}\Gamma(1.75)+\frac{2867}{2048x_1^4}\Gamma(2.25)-\dots\right], \tag{69}$$

or, inserting the values of the Γ-functions, we find

$$\phi(x_1;\,x_1)\simeq\frac{0.72320}{\sqrt{x_1}}\left\{1-\frac{0.42249}{x_1}+\frac{0.33594}{x_1^2}-\frac{0.35053}{x_1^3}+\frac{0.43747}{x_1^4}-\dots\right\}. \tag{70}$$

This series can be trusted to give the value of $\phi(x_1;\,x_1)$ with sufficient accuracy for $x_1>2$.

The values of $\phi(x_1;\,x_1)$ for $x_1<2$, determined in accordance with equation (70), are listed in Table 3.

v) *An asymptotic formula for $\phi(y;\,x_1)$ for x_1 large and $y>x_1$.*—When x_1 is large and $y>x_1$, we write

$$y=\beta x_1\quad(\beta\gg 1)\qquad\text{and}\qquad t=x-y \tag{71}$$

in equation (58). We then find

$$\phi(y;\, x_1) = \frac{1}{(2\pi y)^{1/2}} \int_0^\infty \frac{\exp\{-[t^2 + 2(\beta-1)x_1 t + x_1^2(\beta-1)^2]\}}{(1+t/\beta x_1)(1+t/2\beta x_1)^{1/2} t^{1/2}}\, dt\,. \tag{72}$$

When $\beta \gg 1$, the main contribution to the integral on the right-hand side comes from the neighborhood of $t = 0$. Accordingly, we may write

$$\phi(y;\, x_1) \simeq \frac{e^{-x_1^2(\beta-1)^2}}{(2\pi y)^{1/2}} \int_0^\infty e^{-2(\beta-1)x_1 t} \frac{dt}{\sqrt{t}} \qquad (x_1 \text{ large and } \beta \gg 1)\,. \tag{73}$$

TABLE 3

$\phi(x_1;\, x_1)$

x_1	$\phi(x_1;\, x_1)$	x_1	$\phi(x_1;\, x_1)$	x_1	$\phi(x_1;\, x_1)$	x_1	$\phi(x_1;\, x_1)$
2.0	0.4379	3.5	0.3485	5	0.2998	8	0.2434
2.5	.3995	4.0	.3296	6	.2768	9	.2306
3.0	0.3712	4.5	0.3136	7	0.2585	10	0.2197

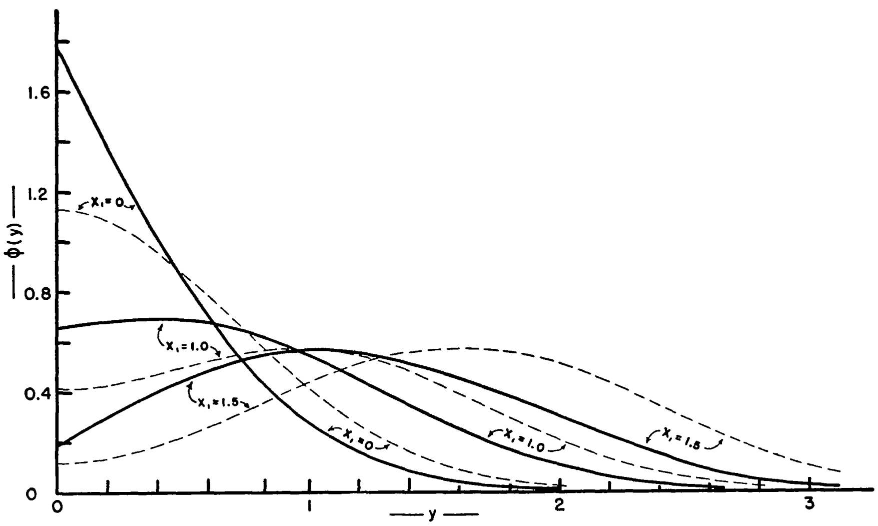

FIG. 3.—The frequency functions $\phi(y;\, x_1)$ (*full-line curves*) derived from assumed distributions of x of the form given by equation (27) (*dashed curves*) for values of $x_1 \leqslant 1.5$. For $x_1 = 0$, $f(x)$ is Gaussian, and $\phi(y)$ is explicitly known; for $x_1 = 1.0$, $\phi(y)$ was derived from the expansion (43); for $x_1 = 1.5$ the the expansion (54) was used for $y \leqslant 1.3$; and, as the asymptotic form for large y (eq. [74]) is also known, the curve for $y > 1.3$ can be drawn without much ambiguity, since the area under the curve should be unity.

Hence

$$\phi(y;\, x_1) \simeq \frac{1}{2} \frac{e^{-(y-x_1)^2}}{\sqrt{y(y-x_1)}} \qquad (y \gg x_1)\,. \tag{74}$$

7. *The one-parameter family of frequency functions* $\phi(y;\, x_1)$.—Using the various expansions for $\phi(y;\, x_1)$ obtained in §6, we can readily sketch the form of the frequency functions for various values of x_1. This is done in Figures 3 and 4.

8. *The true distribution of the rotational velocities of stars of different spectral types.*—We have already explained why, when we wish to say something more about the distribution of the rotational velocities of stars than its mean and mean square, it is preferable to compare the observed distribution of $v \sin i$ with that predicted on some assumed form of $f(v)$, instead of inverting the integral equation (9) by a numerical procedure. We shall now show that the particular form of $f(x)$ examined in the preceding sections allows us to represent the known observations on the rotational velocities of stars very satisfactorily.

Now, in order to make a comparison between an observed distribution of $v \sin i$ and the one-parameter family of frequency functions $\phi(y; x_1)$ derived in §§6 and 7, it is necessary that we determine *two* quantities from the observations, namely, (1) the *unit* in

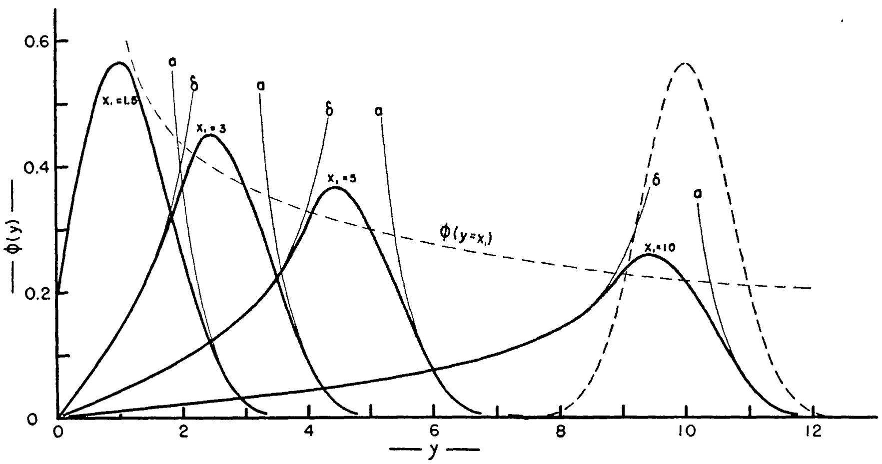

FIG. 4.—The frequency functions $\phi(y; x_1)$ (*heavy full-line curves*) derived from assumed distributions of x of the form given by equation (27) for $x_1 \geqslant 1.5$. In sketching the curves, use was made of the asymptotic expansion for $y < x_1$ (the curves marked δ), the expansion for $y \gg x_1$ (the curves marked a), and the value of $\phi(y; x_1)$ at $y = x_1$ given by equation (70) (*dashed curve*); these expansions, together with the condition that the area under the curve in each case should be unity, leave little ambiguity for drawing in the complete curves. For $x_1 \geqslant 2$ the frequency function $f(x)$ from which $\phi(y)$ is derived is, for all practical purposes, a Gaussian distribution centered at $x = x_1$ (the curve for $x_1 = 10$ is illustrated).

which x measures the velocity and (2) the parameter x_1 in the frequency function (30): for our assumption concerning $f(v)$ is, strictly,

$$f(v) = \frac{j}{\sqrt{\pi}} \left[e^{-j^2(v-v_1)^2} + e^{-j^2(v+v_1)^2} \right] , \tag{75}$$

where $1/j$ is of the dimensions of a velocity. However, with the substitutions

$$j\, v = x \qquad \text{and} \qquad j\, v_1 = x_1 , \tag{76}$$

the frequency function governing x reduces to $f(x)$ as we have defined it in equation (30).

Since the mean and (to a less extent) the mean square are probably the only two quantities which are determined with any precision, it appears that, in making comparisons between the observed distributions of $v \sin i$ and those derived from the function (75), we determine the parameters j and v_1 of the distribution in the following manner:

From the observed values of the mean and the mean square of $v \sin i$ we first determine the values of $\bar{v}$ and $\overline{v^2}$ according to the moment relations of §4 (eqs. [19] and [20]). From these mean values we next evaluate the quantity

$$\frac{\bar{v}}{[2(\overline{v^2} - \bar{v}^2)]^{1/2}} . \tag{77}$$

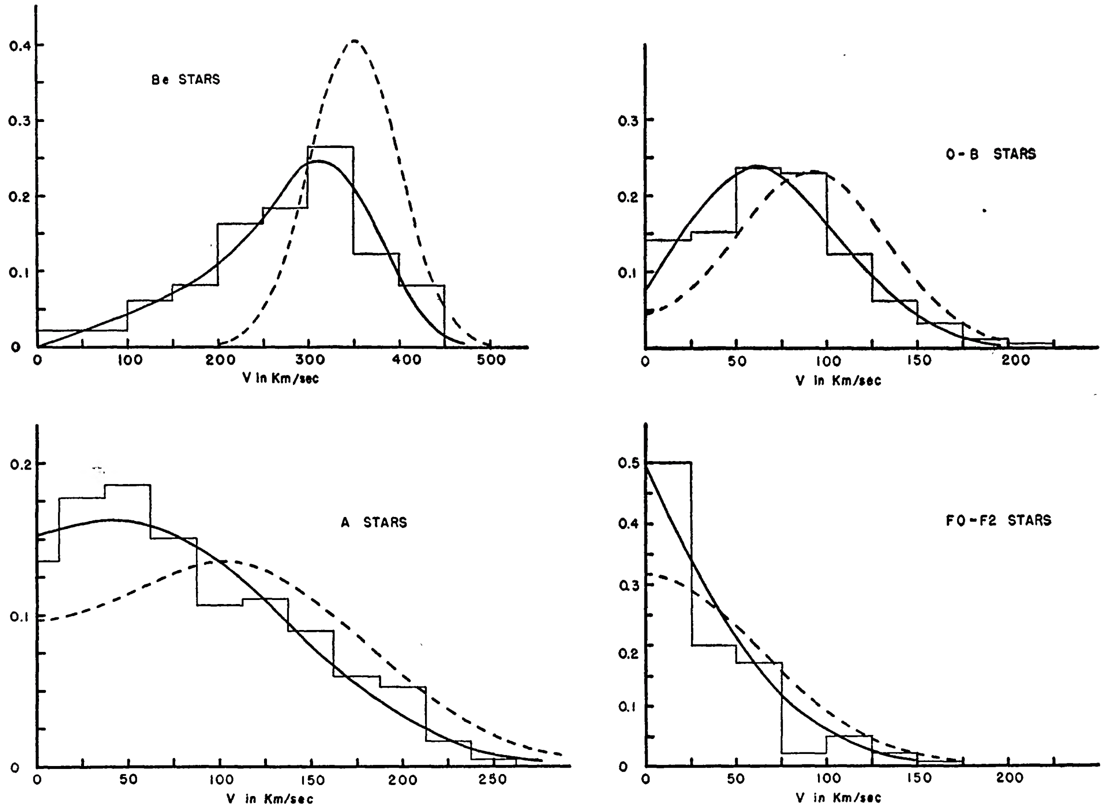

FIG. 5.—The comparison of the observed distributions of the rotational velocities of stars of different spectral types with those derived on the assumption of a true distribution of velocities of the form given by equation (75) and for values of the parameters listed in Table 4. In each case the observed distribution of $v \sin i$ is represented by the histogram, while the full-line curve represents the predicted distribution of $v \sin i$ for a true distribution of v given by the dashed curve.

TABLE 4

REDUCTION OF DATA ON ROTATIONAL VELOCITIES OF STARS OF DIFFERENT SPECTRAL TYPES

	Spectral Type			
	Be	O–B	A	F0–F2
$\overline{v \sin i}$ (km/sec)	273	74	88	40
$\overline{v^2 \sin^2 i}$ (km/sec)^2	82,420	7238	11,254	2978
$\bar{v}$ (km/sec)	348	94	112	51
$\overline{v^2}$ (km/sec)^2	123,600	10,860	16,880	4470
$\bar{v}/(2[\overline{v^2}-\bar{v}^2])^{1/2}$	4.90	1.47	1.20	0.82
x_1 computed	4.9	1.41	1.03	....
x_1 adopted	5.0	1.5	1.0	0.0
j^{-1} (km/sec)	70	63	107	90

This quantity is clearly independent of the unit in which the velocity is measured; accordingly, for the distribution (75), it depends only on x_1 and assumes the values listed in the last column of Table 1 for the various values of x_1. Interpolating in this column of Table 1, we can therefore determine x_1 when the value of the quantity (77) has been deduced from observations. Once x_1 has been determined, the unit, $1/j$, in which x measures the velocity, follows from the relation

$$\bar{x} = j\bar{v} , \tag{78}$$

since $\bar{x}$ as a function of x_1 is known (eq. [28] and Table 1). With $1/j$ and x_1 determined in this fashion, the comparison between the observed distribution of $v \sin i$ and the member of the family $\phi'y; x_1)$ belonging to the derived value of x_1 is a straightforward matter.

The known data on the rotational velocities of the stars[9] have been analyzed in the manner described in the preceding paragraphs. The results are summarized in Table 4 and Figure 5. It is seen that the observations are represented entirely satisfactorily by the assumption (75) regarding the distribution of the rotational velocities and with the parameters listed in Table 4.

[9] For the Be stars we have used the results of Slettebak (*op. cit.*), while for the stars of the other spectral types we have used the data compiled by Struve (*op. cit.*, p. 259) and C. Westgate (*Ap. J.*, **77**, 141, 1933; **78**, 46, 1933; **79**, 357, 1934). Dr. Struve has pointed out to us that for the O and the B stars Miss Westgate's values for the rotational velocities may be too low; however, he believes that the relative distribution may be reliable. That Miss Westgate's rotational velocities for the O and the B stars are systematically too low would seem to be confirmed by Slettebak's work.