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THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS*

II. THE SPEED OF FLUCTUATIONS; DYNAMICAL FRICTION; SPATIAL CORRELATIONS

S. CHANDRASEKHAR AND J. VON NEUMANN

ABSTRACT

This paper is devoted principally to a statistical analysis of the speed of fluctuations in the force per unit mass, $\boldsymbol{F}$, acting on a star which is moving with a velocity $\boldsymbol{v}$ with respect to the centroid of the near-by stars. The solution to the problem depends on the evaluation of the first and the second moments of the rate of change of $\boldsymbol{F}$ for a given value of $\boldsymbol{F}$.

The statistical problem has been solved on the assumptions of a uniform Poisson distribution of the stars and a spherical distribution of the velocities with respect to the chosen local standard of rest. No other restrictions have been introduced; in particular, proper allowance has been made for a distribution over the different masses M.

It is found that

$$\bar{\dot{\boldsymbol{F}}}_{\boldsymbol{F},\,\boldsymbol{v}} = -\tfrac{2}{3}\pi G\overline{M}nB\left(\frac{|\boldsymbol{F}|}{Q_H}\right)\left(\boldsymbol{v} - 3\,\frac{\boldsymbol{v}\cdot\boldsymbol{F}}{|\boldsymbol{F}|^2}\boldsymbol{F}\right),$$

where G denotes the constant of gravitation, $\overline{M}$ the average mass of the stars, n the number of stars per unit volume, and B a certain function of $|\boldsymbol{F}|/Q_H$ (where Q_H is a certain "normal" field strength). It is indicated how in consequence of this lack of randomness in the rate of change of $\boldsymbol{F}$ for given $\boldsymbol{F}$ and $\boldsymbol{v}$ a star may experience *dynamical friction* (i.e., a systematic tendency to be decelerated in the direction of its motion by an amount proportional to $|\boldsymbol{v}|$).

The various second moments of $\dot{\boldsymbol{F}}$ have also been calculated and lead to an estimate of the mean life of the state $|\boldsymbol{F}|$.

The closely related problem of the correlations in $\boldsymbol{F}$ acting at two very close points in a system containing a random distribution of stars is also considered.

1. *Introduction.*—In an earlier paper[1] we analyzed certain statistical features of the fluctuating gravitational force acting at some fixed point in a system containing a random distribution of stars in motion. In this paper we propose to extend this discussion to the case of the fluctuations in the force acting on a star which is moving with a definite velocity $\boldsymbol{v}$ in an appropriately chosen local standard of rest. The discussion of this case requires an essential generalization of the analysis contained in I, since for our present

* Some of the results contained in this paper were included in and formed part of an essay written by S. Chandrasekhar and entitled "New Methods in Stellar Dynamics," for which an A. Cressy Morrison Prize was awarded by the New York Academy of Sciences.

[1] *Ap. J.*, **95**, 489, 1941. This paper will be referred to as I.

problem what is relevant is the distribution of the velocities of the other stars *relative* to the one under consideration, and we cannot assume that this has a random character,[2] for the distribution of the relative velocities V will show a marked asymmetry in that there will be a preponderance of negative velocities. More exactly, if the distribution of the velocities relative to the chosen standard of rest is characterized by randomness, then

$$\overline{V} = -v \; . \tag{1}$$

This asymmetry in the distribution of the relative velocities has important physical consequences. Thus, as we shall show (§ 11), it is as a direct result of this asymmetry that a star experiences *dynamical friction*, or, expressed differently, it suffers from a systematic tendency to be decelerated in the direction of its motion by an amount proportional to $|v|$.

A second problem we shall consider in this paper concerns the correlation in the forces acting at two very close points; this problem is closely related to the one formulated in the preceding paragraph.

2. *The general formula for* W (F, f).—Consider a star moving with a velocity v. The force F acting on the star per unit mass is given by

$$F = G \sum_i M_i \frac{r_i}{|r_i|^3} , \tag{2}$$

where M_i denotes the mass of a typical field star and r_i its instantaneous position vector relative to the star under consideration. Accordingly, the rate of change of F is given by

$$f = \frac{dF}{dt} = G \sum_i M_i \left(\frac{V_i}{|r_i|^3} - 3 \frac{r_i [r_i \cdot V_i]}{|r_i|^5} \right) , \tag{3}$$

where V_i denotes the velocity of a field star relative to the one under consideration.

It is clear that the speed of the fluctuations can be specified in terms of the distribution function

$$W(F, f) , \tag{4}$$

which gives the simultaneous probability of a given force F acting and an associated rate of change of F of amount f. The general expression for this probability can be readily written down following Markoff's method outlined in I, § 2. We have (cf. I, eqs. [18] and [19])

$$W(F, f) = \frac{1}{64\pi^6} \int_{|\rho|=0}^{\infty} \int_{|\sigma|=0}^{\infty} e^{-i(\rho \cdot F + \sigma \cdot f)} A(\rho, \sigma) \, d\rho \, d\sigma , \tag{5}$$

where

$$A(\rho, \sigma) = \lim_{R \to \infty} \left[\frac{3}{4\pi R^3} \int_{0<M<\infty} \int_{|r|<R} \int_{|V|<\infty} e^{i(\rho \cdot \phi + \sigma \cdot \psi)} \tau \, dr \, dV \, dM \right]^{4\pi R^3 n/3} . \tag{6}$$

In equations (5) and (6) ρ and σ are two auxiliary vectors, n denotes the average number of stars per unit volume,

$$\phi = GM \frac{r}{|r|^3} ; \qquad \psi = GM \left(\frac{V}{|r|^3} - 3 \frac{r [r \cdot V]}{|r|^5} \right) , \tag{7}$$

and

$$\tau \, dV \, dM \equiv \tau (V; M) \, dV \, dM \tag{8}$$

[2] We use the word "random" in the sense defined in S. Chandrasekhar, *Principles of Stellar Dynamics*, p. 8, University of Chicago Press, 1942.

gives the probability that a star with a relative velocity in the range $(V, V + dV)$ and with a mass between M and $M + dM$ will be found. It should be further noted that in writing down equations (5) and (6) we have supposed that the fluctuations in the stellar distribution which occur are subject only to the restriction of a constant average density.

Since

$$\frac{3}{4\pi R^3}\int_{0<M<\infty}\int_{|r|<R}\int_{|V|<\infty} \tau\, dr dV dM = 1\,, \tag{9}$$

we can re-write equation (6) as

$$A(\rho, \sigma) = \lim_{R\to\infty}\left[1 - \frac{3}{4\pi R^3}\int_{0<M<\infty}\int_{|r|<R}\int_{|V|<\infty} \times \{1 - e^{i(\rho\cdot\phi + \sigma\cdot\psi)}\}\, \tau\, dr dV dM\right]^{4\pi R^3 n/3}. \tag{10}$$

We replace formula (10) by

$$A(\rho, \sigma) = \lim_{R\to\infty}\left[1 - \frac{3}{4\pi R^3}\int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{1 - e^{i(\rho\cdot\phi + \sigma\cdot\psi)}\}\, \tau\, dr dV dM\right]^{4\pi R^3 n/3}. \tag{11}$$

The integral which occurs in the foregoing equation is conditionally convergent. It should be noted that (among others) the integral $\int_{-\infty}^{+\infty} \ldots dr$ is a triple integral, since r is a vector. The origin of the expression should make it plausible that one must integrate over the two polar angles ϑ and ω of r first and over $|r|$ last. (Indeed, this $\int_{-\infty}^{+\infty} \ldots dr$ or, rather, $\int_0^\infty\int_0^\pi\int_0^{2\pi} \ldots |r|^2 \sin\vartheta d\omega d\vartheta d|r|$ originates from the $\int_{|r|<R} \ldots dr$ or, rather, $\int_0^R\int_0^\pi\int_0^{2\pi} \ldots |r|^2 \sin\vartheta d\omega d\vartheta d|r|$, of equation (10) with $R\to\infty$.) A justification of this plausible procedure will be given elsewhere, by means of complex integration. This point is of importance, because an improper order of integrations may lead to incorrect results.

Now equation (11) can also be written as

$$A(\rho, \sigma) = e^{-nC(\rho, \sigma)}\,, \tag{12}$$

where

$$C(\rho, \sigma) = \int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{1 - e^{i(\rho\cdot\phi + \sigma\cdot\psi)}\}\, \tau\, dr dV dM\,. \tag{13}$$

In the foregoing expression for $C(\rho, \sigma)$ we shall introduce ϕ as the variable of integration instead of r. Since ϕ and r have the same polar co-ordinates and since $|\phi|$ and $|r|$ determine each other, our earlier remarks concerning the r-integration apply equally to the ϕ-integration. We have (cf. I, eqs. [22]–[24])

$$dr = -\tfrac{1}{2}(GM)^{3/2}|\phi|^{-9/2} d\phi\,. \tag{14}$$

We can re-write equation (13) as

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{1}{2}G^{3/2}\int_0^{\infty}\int_{-\infty}^{+\infty} dM\, dV\, \tau M^{3/2}\left[\int_{-\infty}^{+\infty}\{1 - e^{i(\boldsymbol{\rho}\cdot\boldsymbol{\phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}\}\,|\boldsymbol{\phi}|^{-9/2}d\boldsymbol{\phi}\right]. \tag{15}$$

In equation (15) $\boldsymbol{\sigma}\cdot\boldsymbol{\psi}$ will have to be expressed in terms of $\boldsymbol{\phi}$. Thus,

$$\boldsymbol{\sigma}\cdot\boldsymbol{\psi} = (GM)^{-1/2}\{\,|\boldsymbol{\phi}|^{3/2}(\boldsymbol{\sigma}\cdot\boldsymbol{V}) - 3\,|\boldsymbol{\phi}|^{-1/2}(\boldsymbol{\phi}\cdot\boldsymbol{V})\,(\boldsymbol{\phi}\cdot\boldsymbol{\sigma})\,\}\,. \tag{16}$$

If we now put

$$\boldsymbol{\sigma} = (GM)^{1/2}\boldsymbol{\sigma}_1\,, \tag{17}$$

$\boldsymbol{\sigma}\cdot\boldsymbol{\psi}$ can be expressed more conveniently as

$$\boldsymbol{\sigma}\cdot\boldsymbol{\psi} = |\boldsymbol{\phi}|^{3/2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{V}) - 3\,|\boldsymbol{\phi}|^{-1/2}(\boldsymbol{\phi}\cdot\boldsymbol{V})\,(\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_1)\,. \tag{18}$$

Returning to equation (15), we write it in the form

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{1}{2}G^{3/2}\int_0^{\infty}\int_{-\infty}^{+\infty} dM\, dV\, \tau M^{3/2} D(\boldsymbol{\rho}, \boldsymbol{\sigma})\,, \tag{19}$$

where

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \int_{-\infty}^{+\infty}\{1 - e^{i(\boldsymbol{\rho}\cdot\boldsymbol{\phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}\}\,|\boldsymbol{\phi}|^{-9/2}d\boldsymbol{\phi}\,. \tag{20}$$

An alternative form for $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ is

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \int_{-\infty}^{+\infty}(1 - e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}})\,|\boldsymbol{\phi}|^{-9/2}d\boldsymbol{\phi} + \int_{-\infty}^{+\infty} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}}(1 - e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\psi}})\,|\boldsymbol{\phi}|^{-9/2}d\boldsymbol{\phi}\,. \tag{21}$$

The first of the two integrals which occur in the foregoing equation is equivalent to one we have already evaluated in I (eqs. [55]–[58]). We thus have

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{8}{15}(2\pi)^{3/2}|\boldsymbol{\rho}|^{3/2} + \int_{-\infty}^{+\infty} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}}(1 - e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\psi}})\,|\boldsymbol{\phi}|^{-9/2}d\boldsymbol{\phi}\,. \tag{22}$$

Equations (5), (12), (18), (19), and (22) formally solve the problem of the determination of $W(\boldsymbol{F}, \boldsymbol{f})$. However, an explicit evaluation of $W(\boldsymbol{F}, \boldsymbol{f})$ would require a complete knowledge of the function $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$. But if we are interested only in the moments of $\boldsymbol{f}$ for a given $\boldsymbol{F}$, we need only the behavior of $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}| \to 0$ or, according to equations (12) and (19), in the behavior of $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}| \to 0$. We can therefore expand

$$1 - e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\psi}}\,, \tag{23}$$

which occurs under the integral sign in equation (22) in a power series in $\boldsymbol{\sigma}$. Retaining only the first two terms in this expansion, we obtain

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{8}{15}(2\pi)^{3/2}|\boldsymbol{\rho}|^{3/2} - D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) + D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) + O(|\boldsymbol{\sigma}|^3)\,, \tag{24}$$

where

$$D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = i\int_{-\infty}^{+\infty} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}}(\boldsymbol{\sigma}\cdot\boldsymbol{\psi})\,|\boldsymbol{\phi}|^{-9/2}d\boldsymbol{\phi}\,, \tag{25}$$

and

$$D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}} (\boldsymbol{\sigma}\cdot\boldsymbol{\psi})^2 |\boldsymbol{\phi}|^{-9/2} d\boldsymbol{\phi} . \tag{26}$$

3. *The evaluation of* $D_1(\boldsymbol{\rho}, \boldsymbol{\sigma})$.—Substituting for $\boldsymbol{\sigma}\cdot\boldsymbol{\psi}$ from equation (18) in equation (25), we have

$$D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = i\int_{-\infty}^{+\infty} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}} \left\{ (\boldsymbol{\sigma}_1\cdot V) - 3\,\frac{(\boldsymbol{\phi}\cdot V)\,(\boldsymbol{\phi}\cdot\boldsymbol{\sigma}_1)}{|\boldsymbol{\phi}|^2} \right\} |\boldsymbol{\phi}|^{-3} d\boldsymbol{\phi} . \tag{27}$$

To evaluate this integral we shall first choose a Cartesian system of co-ordinates with the z-axis in the direction of $\boldsymbol{\rho}$. Let the vectors $\boldsymbol{\sigma}_1$ and V in this system of co-ordinates be

$$\boldsymbol{\sigma}_1 = (s_1, s_2, s_3) ; \qquad V = (V_1, V_2, V_3) . \tag{28}$$

Further, let

$$\mathbf{1}_{\boldsymbol{\phi}} = (l, m, n) = (\sin\vartheta \cos\omega, \sin\vartheta \sin\omega, \cos\vartheta) \tag{29}$$

be a unit vector in the direction of $\boldsymbol{\phi}$. Equation (27) now becomes

$$\left.\begin{aligned} D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= i\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} d|\boldsymbol{\phi}|\, d\vartheta d\omega\, |\boldsymbol{\phi}|^{-1} \sin\vartheta\, e^{i|\boldsymbol{\rho}||\boldsymbol{\phi}|\cos\vartheta} \\ &\times [(s_1V_1 + s_2V_2 + s_3V_3) - 3\,(l^2 s_1 V_1 + m^2 s_2 V_2 + n^2 s_3 V_3 + lm[s_2 V_1 + s_1 V_2] \\ &\qquad + mn[s_3 V_2 + s_2 V_3] + nl[s_1 V_3 + s_3 V_1])] . \end{aligned}\right\} \tag{30}$$

The integral over ω is readily performed, and we obtain

$$\left.\begin{aligned} D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = 2\pi i\int_0^{\infty}\int_{-1}^{+1} e^{i|\boldsymbol{\rho}||\boldsymbol{\phi}|t} \{ [s_3 V_3 - \tfrac{1}{2}(s_1 V_1 + s_2 V_2)] \\ + [\tfrac{3}{2}(s_1 V_1 + s_2 V_2) - 3 s_3 V_3] t^2 \} |\boldsymbol{\phi}|^{-1} dt d|\boldsymbol{\phi}| , \end{aligned}\right\} \tag{31}$$

where we have written $t = \cos\vartheta$. Without altering its value we can clearly replace t by $-t$ in the foregoing expression. But this replacement changes

$$e^{i|\boldsymbol{\rho}||\boldsymbol{\phi}|t} \quad \text{into} \quad e^{-i|\boldsymbol{\rho}||\boldsymbol{\phi}|t} \tag{32}$$

under the integral sign; taking the arithmetic mean of the two resulting integrands, we obtain

$$\left.\begin{aligned} D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = 4\pi i\int_0^{\infty}\int_0^{1} \cos(|\boldsymbol{\rho}|\,|\boldsymbol{\phi}|t) \{ [s_3 V_3 - \tfrac{1}{2}(s_1 V_1 + s_2 V_2)] \\ + [\tfrac{3}{2}(s_1 V_1 + s_2 V_2) - 3 s_3 V_3] t^2 \} |\boldsymbol{\phi}|^{-1} dt d|\boldsymbol{\phi}| , \end{aligned}\right\} \tag{33}$$

or, writing $x = |\boldsymbol{\rho}|\,|\boldsymbol{\phi}|$, we have

$$\left.\begin{aligned} D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = 4\pi i\int_0^{\infty}\int_0^{1} \cos(xt) \{ [s_3 V_3 - \tfrac{1}{2}(s_1 V_1 + s_2 V_2)] \\ + [\tfrac{3}{2}(s_1 V_1 + s_2 V_2) - 3 s_3 V_3] t^2 \} x^{-1} dt dx . \end{aligned}\right\} \tag{34}$$

Both the x- and the t-integrations can now be carried out. According to the remarks made after equations (11) and (13), we must carry out the integration over t first and x

later. (We may note here that actually the reverse order of carrying out the integration over x first and t later leads to the value 0; this paradoxical result is, of course, caused by the conditional convergence of the multiple integral.) The t-integration gives (cf. I, eqs. [137])

$$\left.\begin{aligned} D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= 4\pi i \int_0^\infty \Big\{ (s_3 V_3 - \tfrac{1}{2}[s_1 V_1 + s_2 V_2]) \frac{\sin x}{x} \\ &\quad + (\tfrac{3}{2}[s_1 V_1 + s_2 V_2] - 3 s_3 V_3)\left(\frac{\sin x}{x} - \frac{2}{x^3}[\sin x - x \cos x]\right)\Big\} \frac{dx}{x} \\ &= 4\pi i (s_1 V_1 + s_2 V_2 - 2 s_3 V_3) \int_0^\infty (x^2 \sin x - 3 \sin x + 3x \cos x) \frac{dx}{x^4} . \end{aligned}\right\} \quad (35)$$

Now

$$\left.\begin{aligned} &\int_0^\infty (x^2 \sin x - 3 \sin x + 3x \cos x) \frac{dx}{x^4} \\ &\qquad = -\frac{1}{3} \int_0^\infty (x^2 \sin x - 3 \sin x + 3x \cos x) \frac{d}{dx}\left(\frac{1}{x^3}\right) dx \\ &\qquad = \frac{1}{3} \int_0^\infty (x \cos x - \sin x) \frac{dx}{x^2} \\ &\qquad = \frac{1}{3} \int_0^\infty \frac{d}{dx}\left(\frac{\sin x}{x}\right) dx = -\frac{1}{3} . \end{aligned}\right\} \quad (36)$$

Hence, combining equations (35) and (36), we have

$$D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -\tfrac{4}{3}\pi i (s_1 V_1 + s_2 V_2 - 2 s_3 V_3) , \quad (37)$$

where, it will be recalled, s_1, s_2, s_3 and V_1, V_2, V_3 are the components of $\boldsymbol{\sigma}_1$ and V in a system of co-ordinates in which the z-axis is in the direction of $\boldsymbol{\rho}$. If $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ in the same system, then, according to equation (17), we can re-write equation (37) as

$$D_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -\tfrac{4}{3}\pi i (GM)^{-1/2} (\sigma_1 V_1 + \sigma_2 V_2 - 2 \sigma_3 V_3) . \quad (38)$$

4. *The evaluation of* $D_2(\boldsymbol{\rho}, \boldsymbol{\sigma})$.—According to equations (18) and (26),

$$D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}} [(\boldsymbol{\sigma}_1 \cdot V) - 3(\mathbf{1}_{\boldsymbol{\phi}} \cdot V)(\mathbf{1}_{\boldsymbol{\phi}} \cdot \boldsymbol{\sigma}_1)]^2 |\boldsymbol{\phi}|^{-3/2} d\boldsymbol{\phi} , \quad (39)$$

where, as in equation (29), $\mathbf{1}_{\boldsymbol{\phi}}$ is a unit vector in the direction of $\boldsymbol{\phi}$. Using polar co-ordinates, we can express $D_2(\boldsymbol{\rho}, \boldsymbol{\sigma})$ as

$$\left.\begin{aligned} D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{1}{2} \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} e^{i|\boldsymbol{\rho}||\boldsymbol{\phi}|t} [(\boldsymbol{\sigma}_1 \cdot V) - 3(\mathbf{1}_{\boldsymbol{\phi}} \cdot V)(\mathbf{1}_{\boldsymbol{\phi}} \cdot \boldsymbol{\sigma}_1)]^2 \\ \times |\boldsymbol{\phi}|^{1/2} d\omega dt d|\boldsymbol{\phi}| , \end{aligned}\right\} \quad (40)$$

or, introducing a new variable z, defined as

$$z = |\boldsymbol{\rho}| |\boldsymbol{\phi}| , \quad (41)$$

we have

$$D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{1}{2}\,|\boldsymbol{\rho}|^{-3/2}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} e^{izt}[\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{V}) - 3\,(\mathbf{1}_\Phi\cdot\boldsymbol{V})\,(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)\,]^2 z^{1/2} d\omega\, dt\, dz\,. \qquad (42)$$

After performing the integration over ω, we obtain

$$D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \pi\,|\boldsymbol{\rho}|^{-3/2}\int_0^\infty\int_{-1}^{+1} e^{izt}\ \overline{[\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{V}) - 3\,(\mathbf{1}_\Phi\cdot\boldsymbol{V})\,(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)\,]^2}\ z^{1/2} dt\, dz\,, \qquad (43)$$

where we have used the bar over $[(\boldsymbol{\sigma}_1\cdot \boldsymbol{V}) - 3(\mathbf{1}_\Phi\cdot \boldsymbol{V})(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)]^2$ to indicate that the averaging over ω has been carried out.

In order now to evaluate the integral in equation (43) we first note that, since $\overline{[\boldsymbol{\sigma}_1\cdot \boldsymbol{V} - 3(\mathbf{1}_\Phi\cdot \boldsymbol{V})(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)]^2}$ is an even function in t (cf. eqs. [47] and [49] below), $D_2(\boldsymbol{\rho},\boldsymbol{\sigma})$ has the alternative form

$$D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \pi\,|\boldsymbol{\rho}|^{-3/2}\int_0^\infty\int_{-1}^{+1} \cos\,(zt)\ \overline{[\boldsymbol{\sigma}_1\cdot\boldsymbol{V} - 3\,(\mathbf{1}_\Phi\cdot\boldsymbol{V})\,(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)\,]^2}\ z^{1/2} dt\, dz\,. \qquad (44)$$

The integration over z and t are now best performed by regarding them as complex variables and integrating along appropriately chosen contours. Thus, writing equation (44) as

$$D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \pi\,|\boldsymbol{\rho}|^{-3/2}\mathcal{R}\int_{-1}^{+1}\int_0^\infty e^{izt}\ \overline{[\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{V}) - 3\,(\mathbf{1}_\Phi\cdot\boldsymbol{V})\,(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)\,]^2}\ z^{1/2} dz\, dt \qquad (45)$$

and choosing for z and t paths of integrations as in the appendix to our first paper, we obtain

$$\left.\begin{aligned} D_2(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -\pi\Gamma\left(\tfrac{3}{2}\right)\,|\boldsymbol{\rho}|^{-3/2}\mathcal{R}\,e^{-i\pi/4}\int_{-1}^{+1} & \overline{[\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{V}) - 3\,(\mathbf{1}_\Phi\cdot\boldsymbol{V})\,(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)\,]^2} \\ & \times t^{-3/2} dt\,. \end{aligned}\right\} \qquad (46)$$

In equation (46) the integration over t has to be carried out along a curve from -1 to $+1$ in the complex t-plane, which lies entirely in the domain $\mathcal{I}t \geqslant 0$ and $|t| \leqslant 1$. In order to carry out this integration we must first expand $[(\boldsymbol{\sigma}_1\cdot \boldsymbol{V}) - 3(\mathbf{1}_\Phi\cdot \boldsymbol{V})(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)]^2$ and average over ω. We find that

$$\left.\begin{aligned} & \overline{[\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{V}) - 3\,(\mathbf{1}_\Phi\cdot\boldsymbol{V})\,(\mathbf{1}_\Phi\cdot\boldsymbol{\sigma}_1)\,]^2} \\ & = \overline{[\,(s_1V_1 + s_2V_2 + s_3V_3) - 3[\![\, l^2 s_1V_1 + m^2 s_2V_2 + n^2 s_3V_3 + lm\,(s_1V_2} \\ & \qquad \overline{+ s_2V_1) + mn\,(s_2V_3 + s_3V_2) + nl\,(s_3V_1 + s_1V_3)\,]\!]\,]^2} \\ & = (s_1V_1 + s_2V_2 + s_3V_3)^2 - 6\,(s_1V_1 + s_2V_2 + s_3V_3)\,(\overline{l^2}\, s_1V_1 + \overline{m^2}\, s_2V_2 \\ & \qquad + \overline{n^2}\, s_3V_3) \\ & + 9\,(\overline{l^4}\, s_1^2V_1^2 + \overline{m^4}\, s_2^2V_2^2 + \overline{n^4}\, s_3^2V_3^2) + 9\overline{l^2m^2}\,(s_2^2V_1^2 + s_1^2V_2^2 + 4\,s_1s_2V_1V_2) \\ & + 9\overline{m^2n^2}\,(s_2^2V_3^2 + s_3^2V_2^2 + 4\,s_2s_3V_2V_3) + 9\overline{n^2l^2}\,(s_3^2V_1^2 + s_1^2V_3^2 + 4\,s_3s_1V_3V_1)\,, \end{aligned}\right\} \qquad (47)$$

where

$$l=\sin\vartheta\cos\omega\ ;\qquad m=\sin\vartheta\sin\omega\ ;\qquad n=\cos\vartheta\,(=t)\,. \tag{48}$$

Since

$$\left.\begin{aligned}\overline{l^2}=\overline{m^2}&=\tfrac{1}{2}(1-t^2)\ ; & \overline{n^2}&=t^2\ ,\\ \overline{n^2l^2}=\overline{n^2m^2}&=\tfrac{1}{2}t^2(1-t^2)\ ; & \overline{l^2m^2}&=\tfrac{1}{8}(1-t^2)^2\ ,\\ \overline{l^4}=\overline{m^4}&=\tfrac{3}{8}(1-t^2)^2\ ; & \overline{n^4}&=t^4\ ,\end{aligned}\right\} \tag{49}$$

the integral we have to evaluate is

$$\left.\begin{aligned}I=\int_{-1}^{+1}\{&(s_1V_1+s_2V_2+s_3V_3)^2t^{-3/2}-6(s_1V_1+s_2V_2+s_3V_3)\\ &\times[\![\tfrac{1}{2}(s_1V_1+s_2V_2)(1-t^2)t^{-3/2}+s_3V_3t^{1/2}]\!]+\tfrac{27}{8}(s_1^2V_1^2+s_2^2V_2^2)\\ &\times(1-t^2)^2t^{-3/2}+9s_3^2V_3^2t^{5/2}+\tfrac{9}{8}(s_2^2V_1^2+s_1^2V_2^2+4s_2s_1V_1V_2)\\ &\times(1-t^2)^2t^{-3/2}+\tfrac{9}{2}(s_2^2V_3^2+s_3^2V_2^2+s_3^2V_1^2+s_1^2V_3^2+4s_2s_3V_2V_3\\ &\qquad+4s_3s_1V_3V_1)(1-t^2)t^{1/2}\}\,dt\,.\end{aligned}\right\} \tag{50}$$

We readily verify that the various complex integrals which occur in the foregoing expression have the following values:

$$\left.\begin{aligned}\int_{-1}^{+1}t^{-3/2}dt&=-2(1+i)\ ; & \int_{-1}^{+1}(1-t^2)t^{-3/2}dt&=-\tfrac{8}{3}(1+i)\,,\\ \int_{-1}^{+1}t^{1/2}dt&=+\tfrac{2}{3}(1+i)\ ; & \int_{-1}^{+1}(1-t^2)^2t^{-3/2}dt&=-\tfrac{64}{21}(1+i)\,,\\ \int_{-1}^{+1}t^{5/2}dt&=+\tfrac{2}{7}(1+i)\ ; & \int_{-1}^{+1}(1-t^2)t^{1/2}dt&=+\tfrac{8}{21}(1+i)\,.\end{aligned}\right\} \tag{51}$$

Substituting these values in equation (50) and after some minor rearranging of the terms, we obtain

$$\left.\begin{aligned}I=-\tfrac{6}{7}(1+i)\,[\,&s_1^2(5V_1^2+4V_2^2-2V_3^2)+s_2^2(5V_2^2+4V_1^2-2V_3^2)\\ &+s_3^2(4V_3^2-2V_1^2-2V_2^2)-8s_2s_3V_2V_3-8s_3s_1V_3V_1+2s_1s_2V_1V_2]\,.\end{aligned}\right\} \tag{52}$$

Finally, combining equations (46) and (52) and remembering that

$$\mathcal{R}e^{-i\pi/4}(1+i)=\sqrt{2} \tag{53}$$

and returning to our original variable $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ according to equation (17), we find that

$$\left.\begin{aligned}D_2(\boldsymbol{\rho},\boldsymbol{\sigma})=\tfrac{3}{14}(2\pi)^{3/2}(GM)^{-1}|\boldsymbol{\rho}|^{-3/2}[\,&\sigma_1^2(5V_1^2+4V_2^2-2V_3^2)+\sigma_2^2(5V_2^2\\ &+4V_1^2-2V_3^2)+\sigma_3^2(4V_3^2-2V_1^2-2V_2^2)-8\sigma_2\sigma_3V_2V_3-8\sigma_3\sigma_1V_3V_1\\ &+2\sigma_1\sigma_2V_1V_2]\,.\end{aligned}\right\} \tag{54}$$

5. *The expression for* $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$ *for* $|\boldsymbol{\sigma}| \to 0$.—Combining equations (24), (38), and (54), we have

$$\left.\begin{aligned} D(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= \tfrac{8}{15}(2\pi)^{3/2}|\boldsymbol{\rho}|^{3/2} + \tfrac{4}{3}\pi i(GM)^{-1/2}(\sigma_1 V_1 + \sigma_2 V_2 - 2\sigma_3 V_3) \\ &+ \tfrac{3}{14}(2\pi)^{3/2}(GM)^{-1}|\boldsymbol{\rho}|^{-3/2}[(5\sigma_1^2 + 4\sigma_2^2 - 2\sigma_3^2)V_1^2 + (4\sigma_1^2 + 5\sigma_2^2 - 2\sigma_3^2)V_2^2 \\ &+ (4\sigma_3^2 - 2\sigma_1^2 - 2\sigma_2^2)V_3^2 - 8\sigma_2\sigma_3 V_2 V_3 - 8\sigma_3\sigma_1 V_3 V_1 + 2\sigma_1\sigma_2 V_1 V_2] \\ &+ O(|\boldsymbol{\sigma}|^3). \end{aligned}\right\} \tag{55}$$

Substituting this expression for $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in equation (19), we obtain

$$\left.\begin{aligned} C(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= \tfrac{4}{15}(2\pi)^{3/2}G^{3/2}\overline{M^{3/2}}|\boldsymbol{\rho}|^{3/2} + \tfrac{2}{3}\pi i G(\sigma_1\overline{MV_1} + \sigma_2\overline{MV_2} - 2\sigma_3\overline{MV_3}) \\ &+ \tfrac{3}{28}(2\pi)^{3/2}G^{1/2}|\boldsymbol{\rho}|^{-3/2}[(5\sigma_1^2 + 4\sigma_2^2 - 2\sigma_3^2)\overline{M^{1/2}V_1^2} + (4\sigma_1^2 + 5\sigma_2^2 - 2\sigma_3^2) \\ &\times \overline{M^{1/2}V_2^2} + (4\sigma_3^2 - 2\sigma_1^2 - 2\sigma_2^2)\overline{M^{1/2}V_3^2} - 8\sigma_2\sigma_3\overline{M^{1/2}V_2V_3} - 8\sigma_3\sigma_1\overline{M^{1/2}V_3V_1} \\ &+ 2\sigma_1\sigma_2\overline{M^{1/2}V_1V_2}] + O(|\boldsymbol{\sigma}|^3), \end{aligned}\right\} \tag{56}$$

where we have used bars to indicate that the corresponding quantities have been averaged with the weight function $\tau(\boldsymbol{V}; M)$.

In all the foregoing equations $\boldsymbol{V} = (V_1, V_2, V_3)$ represents, of course, the velocity of a typical field star relative to the one under consideration. If, now, $\boldsymbol{u}$ and $\boldsymbol{v}$ denote, respectively, the velocities of the field star and the star under consideration in the chosen standard of rest, then

$$\boldsymbol{V} = \boldsymbol{u} - \boldsymbol{v}. \tag{57}$$

We shall now introduce the assumption that the distribution of the velocities $\boldsymbol{u}$ among the stars is spherical,[3] i.e., the distribution function $\Psi(\boldsymbol{u})$ has the form

$$\Psi(\boldsymbol{u}) \equiv \Psi[j^2(M)|\boldsymbol{u}|^2], \tag{58}$$

where Ψ is an arbitrary function of the argument specified and the parameter j (of the dimensions of [velocity]$^{-1}$) is allowed to be a function of the mass of the star. This assumption concerning the distribution of the "peculiar" velocities $\boldsymbol{u}$ implies that our probability function $\tau(V; M)$ must be expressible as

$$\tau(V; M) \equiv \Psi[j^2(M)|\boldsymbol{V} + \boldsymbol{v}|^2]\chi(M), \tag{59}$$

where $\chi(M)$ governs the distribution over the different masses. For a function τ of this form we clearly have

$$\left.\begin{aligned} \overline{MV_\mu} = -\overline{M}v_\mu; \qquad \overline{M^{1/2}V_\mu^2} &= \tfrac{1}{3}\overline{M^{1/2}|\boldsymbol{u}|^2} + \overline{M^{1/2}}v_\mu^2, \qquad (\mu = 1, 2, 3) \\ \overline{M^{1/2}V_\mu V_\nu} &= \overline{M^{1/2}}v_\mu v_\nu \qquad (\mu, \nu = 1, 2, 3, \mu \neq \nu). \end{aligned}\right\} \tag{60}$$

Substituting these values in equation (56) and after some minor reductions, we find that

$$\left.\begin{aligned} C(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= \tfrac{4}{15}(2\pi)^{3/2}G^{3/2}\overline{M^{3/2}}|\boldsymbol{\rho}|^{3/2} - \tfrac{2}{3}\pi i G\overline{M}(\sigma_1 v_1 + \sigma_2 v_2 - 2\sigma_3 v_3) \\ &+ \tfrac{1}{4}(2\pi)^{3/2}G^{1/2}\overline{M^{1/2}|\boldsymbol{u}|^2}|\boldsymbol{\rho}|^{-3/2}(\sigma_1^2 + \sigma_2^2) + \tfrac{3}{28}(2\pi)^{3/2}G^{1/2}\overline{M^{1/2}}|\boldsymbol{\rho}|^{-3/2} \\ &\times [\sigma_1^2(5v_1^2 + 4v_2^2 - 2v_3^2) + \sigma_2^2(4v_1^2 + 5v_2^2 - 2v_3^2) + \sigma_3^2(4v_3^2 - 2v_1^2 - 2v_2^2) \\ &- 8\sigma_2\sigma_3 v_2 v_3 - 8\sigma_3\sigma_1 v_3 v_1 + 2\sigma_1\sigma_2 v_1 v_2] + O(|\boldsymbol{\sigma}|^3), \end{aligned}\right\} \tag{61}$$

[3] It would be entirely feasible at this stage of our work to introduce a more general distribution of the velocities (e.g., an ellipsoidal distribution); but we shall not consider these refinements in this paper.

where we may recall that $(\sigma_1, \sigma_2, \sigma_3)$ and (v_1, v_2, v_3) are the components of $\boldsymbol{\sigma}$ and $\boldsymbol{v}$ in a system of co-ordinates in which the z-axis is in the direction of $\boldsymbol{\rho}$. We shall now specialize the co-ordinate system still further by arranging that the vector $\boldsymbol{v}$ lies in the xz-plane (see Fig. 1). With this choice of the co-ordinate system

$$v_1 = |\boldsymbol{v}| \sin\gamma\,; \qquad v_2 = 0\,; \qquad v_3 = |\boldsymbol{v}|\cos\gamma\,, \tag{62}$$

where

$$\gamma = \measuredangle(\boldsymbol{\rho}, \boldsymbol{v})\,. \tag{63}$$

The expression for $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ now simplifies to

$$\left.\begin{aligned} C(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= \tfrac{4}{15}(2\pi)^{3/2}G^{3/2}\overline{M^{3/2}}\,|\boldsymbol{\rho}|^{3/2} - \tfrac{2}{3}\pi i G\overline{M}\,|\boldsymbol{v}|\,(\sigma_1\sin\gamma - 2\sigma_3\cos\gamma) \\ &+ \tfrac{1}{4}(2\pi)^{3/2}G^{1/2}\overline{M^{1/2}|\boldsymbol{u}|^2}\,|\boldsymbol{\rho}|^{-3/2}(\sigma_1^2+\sigma_2^2) + \tfrac{3}{28}(2\pi)^{3/2}G^{1/2}\overline{M^{1/2}}\,|\boldsymbol{v}|^2|\boldsymbol{\rho}|^{-3/2} \\ &\times [\sigma_1^2(5\sin^2\gamma - 2\cos^2\gamma) + \sigma_2^2(4\sin^2\gamma - 2\cos^2\gamma) + \sigma_3^2(4\cos^2\gamma - 2\sin^2\gamma) \\ &\qquad - 8\sigma_1\sigma_3\sin\gamma\cos\gamma] + O(|\boldsymbol{\sigma}|^3)\,. \end{aligned}\right\} \tag{64}$$

Substituting this expression for $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in equation (12), which defines $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$, we obtain

$$A(\boldsymbol{\rho}, \boldsymbol{\sigma}) = e^{-a|\boldsymbol{\rho}|^{3/2} + igP(\boldsymbol{\sigma}) - b|\boldsymbol{\rho}|^{-3/2}[Q(\boldsymbol{\sigma}) + kR(\boldsymbol{\sigma})] + O(|\boldsymbol{\sigma}|^3)}\,, \tag{65}$$

where we have written

$$\left.\begin{aligned} a &= \tfrac{4}{15}(2\pi)^{3/2}G^{3/2}\overline{M^{3/2}}\,n\,; \qquad & b &= \tfrac{1}{4}(2\pi)^{3/2}G^{1/2}\overline{M^{1/2}|\boldsymbol{u}|^2}\,n\,, \\ g &= \tfrac{2}{3}\pi G\overline{M}\,|\boldsymbol{v}|\,n\,; & k &= \tfrac{3}{7}\,\frac{\overline{M^{1/2}}\,|\boldsymbol{v}|^2}{\overline{M^{1/2}|\boldsymbol{u}|^2}}\,, \end{aligned}\right\} \tag{66}$$

[4]

and

$$\left.\begin{aligned} P(\boldsymbol{\sigma}) &= \sigma_1\sin\gamma - 2\sigma_3\cos\gamma\,; \qquad Q(\boldsymbol{\sigma}) = \sigma_1^2 + \sigma_2^2\,, \\ R(\boldsymbol{\sigma}) &= \sigma_1^2(5\sin^2\gamma - 2\cos^2\gamma) + \sigma_2^2(4\sin^2\gamma - 2\cos^2\gamma) \\ &\qquad + \sigma_3^2(4\cos^2\gamma - 2\sin^2\gamma) - 8\sigma_3\sigma_1\sin\gamma\cos\gamma\,. \end{aligned}\right\} \tag{67}$$

An alternative form for $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$, which we shall find useful, may be noted here:

$$\left.\begin{aligned} A(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= e^{-a|\boldsymbol{\rho}|^{3/2}}[1 + igP(\boldsymbol{\sigma}) - \tfrac{1}{2}g^2[\![P(\boldsymbol{\sigma})]\!]^2 \\ &\qquad - b|\boldsymbol{\rho}|^{-3/2}[\![Q(\boldsymbol{\sigma}) + kR(\boldsymbol{\sigma})]\!] + O(|\boldsymbol{\sigma}|^3)]\,. \end{aligned}\right\} \tag{68}$$

Now, according to equation (5), $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$ is the six-dimensional Fourier transform of the distribution function $W(\boldsymbol{F}, \boldsymbol{f})$. Consequently, for the purposes of this equation the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ should be referred to a fixed system of co-ordinates. But the expressions for $P(\boldsymbol{\sigma})$, $Q(\boldsymbol{\sigma})$, and $R(\boldsymbol{\sigma})$ (eq. [67]) involve the components of $\boldsymbol{\sigma}$ referred to a variable

[4] It will be noted that our present definitions of a and b agree with those of our earlier paper (I, eq. [28]).

system of co-ordinates, depending on the direction of $\boldsymbol{\rho}$. We shall now give the linear transformation required to pass from this variable xyz-system to a fixed $\xi\eta\zeta$-system (see Fig. 1). This fixed $\xi\eta\zeta$-system is so chosen that the ζ-axis is in the direction of $\boldsymbol{F}$ and

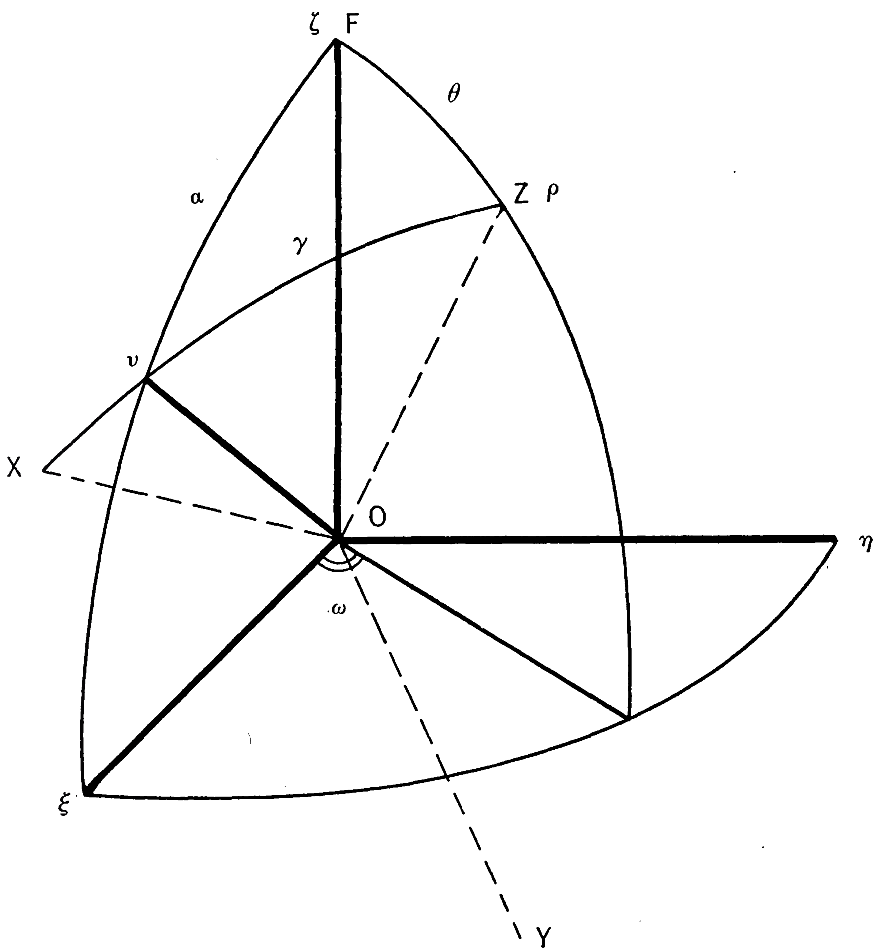

Fig. 1

the $\xi\zeta$-plane contains the vector $\boldsymbol{v}$. The accompanying table gives the direction cosines of the axes belonging to one system with respect to the axes belonging to the other.

	$O\xi$	$O\eta$	$O\zeta$
Ox..........	$\lambda_1 = \frac{\sin\alpha - l\cos\gamma}{\sin\gamma}$	$\mu_1 = -\frac{m\cos\gamma}{\sin\gamma}$	$\nu_1 = \frac{\cos\alpha - n\cos\gamma}{\sin\gamma}$
Oy..........	$\lambda_2 = \frac{m\cos\alpha}{\sin\gamma}$	$\mu_2 = \frac{n\sin\alpha - l\cos\alpha}{\sin\gamma}$	$\nu_2 = -\frac{m\sin\alpha}{\sin\gamma}$
Oz..........	$\lambda_3 = \sin\vartheta\cos\omega = l$	$\mu_3 = \sin\vartheta\sin\omega = m$	$\nu_3 = \cos\vartheta = n$

(69)

Here α is the angle between $\boldsymbol{F}$ and $\boldsymbol{v}$:

$$\alpha = \sphericalangle(\boldsymbol{F}, \boldsymbol{v}), \tag{70}$$

and

$$\cos\gamma = \cos \measuredangle (\boldsymbol{\rho}, \boldsymbol{v}) = n\cos\alpha + l\sin\alpha\,. \tag{71}$$

Thus the required linear transformation is

$$\sigma_i = \lambda_i\sigma_\xi + \mu_i\sigma_\eta + \nu_i\sigma_\zeta\,, \qquad (i = 1, 2, 3) \tag{72}$$

where σ_ξ, σ_η, and σ_ζ are the components of $\boldsymbol{\sigma}$ referred to the fixed system of co-ordinates.

6. *The evaluation of the first moment of* $\boldsymbol{f}$.—To determine the average value of $\boldsymbol{f}$ (for a given $\boldsymbol{F}$) in any specified direction, it would clearly be sufficient to evaluate the first moments of the components of $\boldsymbol{f}$ (namely, f_ξ, f_η, and f_ζ) along the three principal directions of the $\xi\eta\zeta$-system defined in § 5. We shall consider first the moment of f_ξ.

According to equation (5),

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F}, \boldsymbol{f})\, f_\xi d\boldsymbol{f} = \frac{1}{64\pi^6}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(\boldsymbol{\rho}\cdot\boldsymbol{F}+\boldsymbol{\sigma}\cdot\boldsymbol{f})} A(\boldsymbol{\rho}, \boldsymbol{\sigma})\, f_\xi d\boldsymbol{\rho} d\boldsymbol{\sigma} d\boldsymbol{f}\,. \tag{73}$$

But

$$\frac{1}{8\pi^3}\int_{-\infty}^{+\infty} e^{-i\boldsymbol{\sigma}\cdot\boldsymbol{f}} f_\xi d\boldsymbol{f} = i\,\delta'(\sigma_\xi)\,\delta(\sigma_\eta)\,\delta(\sigma_\zeta)\,, \tag{74}$$

where the δ's denote Dirac's δ-functions and δ' the first derivative of the δ-function. Remembering that

$$\int_{-\infty}^{+\infty} f(x)\,\delta'(x)\,dx = -f'(0)\,, \tag{75}$$

we can reduce equation (73) to

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F}, \boldsymbol{f})\, f_\xi d\boldsymbol{f} = -\frac{i}{8\pi^3}\int_{-\infty}^{+\infty} e^{-i\boldsymbol{\rho}\cdot\boldsymbol{F}}\left[\frac{\partial}{\partial\sigma_\xi} A(\boldsymbol{\rho}, \boldsymbol{\sigma})\right]_{|\boldsymbol{\sigma}|=0} d\boldsymbol{\rho}\,. \tag{76}$$

But, according to equation (68),

$$\left[\frac{\partial}{\partial\sigma_\xi} A(\boldsymbol{\rho}, \boldsymbol{\sigma})\right]_{|\boldsymbol{\sigma}|=0} = i g\, e^{-a|\boldsymbol{\rho}|^{3/2}}\frac{\partial P(\boldsymbol{\sigma})}{\partial\sigma_\xi}\,, \tag{77}$$

since P is linear in $\boldsymbol{\sigma}$. Thus,

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F}, \boldsymbol{f})\, f_\xi d\boldsymbol{f} = \frac{g}{8\pi^3}\int_{-\infty}^{+\infty} e^{-a|\boldsymbol{\rho}|^{3/2}-i\boldsymbol{\rho}\cdot\boldsymbol{F}}\left(\frac{\partial P}{\partial\sigma_\xi}\right) d\boldsymbol{\rho}\,, \tag{78}$$

or, choosing polar co-ordinates (see Fig. 1),

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F}, \boldsymbol{f})\, f_\xi d\boldsymbol{f} = \frac{g}{8\pi^3}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{-a|\boldsymbol{\rho}|^{3/2}-i|\boldsymbol{\rho}||\boldsymbol{F}|\cos\vartheta}\left(\frac{\partial P}{\partial\sigma_\xi}\right)|\boldsymbol{\rho}|^2\sin\vartheta\, d|\boldsymbol{\rho}|\, d\vartheta\, d\omega\,. \tag{79}$$

Performing the integration over ω, we obtain

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F}, \boldsymbol{f})\, f_\xi d\boldsymbol{f} = \frac{g}{4\pi^2}\int_0^\infty\int_{-1}^{+1} e^{-a|\boldsymbol{\rho}|^{3/2}-i|\boldsymbol{\rho}||\boldsymbol{F}|t}\left(\overline{\frac{\partial P}{\partial\sigma_\xi}}\right)|\boldsymbol{\rho}|^2 d|\boldsymbol{\rho}|\, dt\,, \tag{80}$$

where we have used a bar over $\partial P/\partial\sigma_\xi$ to denote that the averaging over ω has been carried out; further, in equation (80) we have changed from the variable ϑ to $t = \cos\vartheta$. Putting (cf. I, eq. [134])

$$|\boldsymbol{\rho}|\,|\boldsymbol{F}| = x\,; \qquad |\boldsymbol{F}| = a^{2/3}\beta \tag{81}$$

in equation (80) and remembering that $\overline{\partial P/\partial\sigma_\xi}$ is even in t (cf. eq. [87] below), we obtain

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F}, \boldsymbol{f})\, f_\xi df = \frac{g}{2\pi^2 a^2\beta^3}\int_0^\infty\int_0^1 e^{-(x/\beta)^{3/2}}\overline{\left(\frac{\partial P}{\partial\sigma_\xi}\right)}\, x^2\cos xt\,dx\,dt\,. \tag{82}$$

The first moment of f_ξ is now obtained by dividing the foregoing equation by $W(\boldsymbol{F})$ (I, eq. [117]). We thus have

$$\overline{f_\xi} = \frac{2g}{\pi\beta H(\beta)}\int_0^\infty\int_0^1 e^{-(x/\beta)^{3/2}}\overline{\left(\frac{\partial P}{\partial\sigma_\xi}\right)}\, x^2\cos xt\,dx\,dt\,. \tag{83}$$

We have similar expressions for $\overline{f_\eta}$ and $\overline{f_\zeta}$.

We shall now evaluate $\overline{\partial P/\partial\sigma_\xi}$, etc., According to equation (67),

$$P = \sigma_1\sin\gamma - 2\sigma_3\cos\gamma\,, \tag{84}$$

or, transforming to the $\xi\eta\zeta$-system according to equation (72), we have

$$P = (\lambda_1\sigma_\xi + \mu_1\sigma_\eta + \nu_1\sigma_\zeta)\sin\gamma - 2(\lambda_3\sigma_\xi + \mu_3\sigma_\eta + \nu_3\sigma_\zeta)\cos\gamma\,. \tag{85}$$

Using the table of direction cosines (69) and after some rearranging of the terms, we obtain

$$\left.\begin{aligned} P = \sigma_\xi(1-3l^2)\sin\alpha + \sigma_\zeta(1-3n^2)\cos\alpha - 3ln(\sigma_\xi\cos\alpha + \sigma_\zeta\sin\alpha)\\ - 3\sigma_\eta m(l\sin\alpha + n\cos\alpha)\,.\end{aligned}\right\} \tag{86}$$

From this equation we readily find that

$$\left.\begin{aligned} \overline{\left(\frac{\partial P}{\partial\sigma_\xi}\right)} &= (1-3\overline{l^2})\sin\alpha - 3\overline{ln}\cos\alpha = -\tfrac{1}{2}(1-3t^2)\sin\alpha\,,\\ \overline{\left(\frac{\partial P}{\partial\sigma_\eta}\right)} &= -3(\overline{ml}\sin\alpha + \overline{mn}\cos\alpha) = 0\,,\\ \overline{\left(\frac{\partial P}{\partial\sigma_\zeta}\right)} &= (1-3\overline{n^2})\cos\alpha - 3\overline{ln}\sin\alpha = +(1-3t^2)\cos\alpha\,.\end{aligned}\right\} \tag{87}$$

Thus, according to equations (83) and (87), we have

$$\overline{f_\xi} = -\frac{g}{\pi\beta H(\beta)}\mathcal{J}(\beta)\sin\alpha\,; \qquad \overline{f_\eta} = 0\,; \qquad \overline{f_\zeta} = \frac{2g}{\pi\beta H(\beta)}\mathcal{J}(\beta)\cos\alpha\,, \tag{88}$$

where we have written

$$\mathcal{J}(\beta) = \int_0^\infty\int_0^1 e^{-(x/\beta)^{3/2}}x^2(1-3t^2)\cos xt\,dx\,dt\,. \tag{89}$$

We shall now evaluate $\mathcal{J}(\beta)$. Performing the integration over t, we obtain

$$\mathcal{J}(\beta)=\int_0^\infty e^{-(x/\beta)^{3/2}}x^2\left\{\frac{\sin x}{x}-\frac{3}{x^3}(2x\cos x+x^2\sin x-2\sin x)\right\}dx \quad (90)$$

or, after some rearranging of the terms,

$$\mathcal{J}(\beta)=6\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x-x\cos x)\frac{dx}{x}-2\int_0^\infty e^{-(x/\beta)^{3/2}}x\sin x\,dx\,. \quad (91)$$

The second of the two integrals which occur in equation (91) is seen to be related very simply to the Holtsmark function $H(\beta)$, defined in I, equation (116). Further, denoting by $K(\beta)$ the integral

$$K(\beta)=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x-x\cos x)\frac{dx}{x}, \quad (92)$$

we have

$$\mathcal{J}(\beta)=3\pi K(\beta)-\pi\beta H(\beta)\,. \quad (93)$$

Now

$$\left.\begin{aligned}\frac{dK}{d\beta}&=\frac{3}{\pi\beta^{5/2}}\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x-x\cos x)\,x^{1/2}dx\\&=-\frac{2}{\pi\beta}\int_0^\infty\frac{d}{dx}\left(e^{-(x/\beta)^{3/2}}\right)(\sin x-x\cos x)\,dx\,,\end{aligned}\right\} \quad (94)$$

or, after integrating by parts, we have

$$\frac{dK}{d\beta}=\frac{2}{\pi\beta}\int_0^\infty e^{-(x/\beta)^{3/2}}x\sin x\,dx=H(\beta)\,. \quad (95)$$

Hence

$$K(\beta)=\int_0^\beta H(\beta)\,d\beta\,. \quad (96)$$

Now, combining equations (88), (93), and (96), we have

$$\overline{f_\xi}=-gB(\beta)\sin\alpha\,;\qquad \overline{f_\eta}=0\,;\qquad \overline{f_\zeta}=2gB(\beta)\cos\alpha\,, \quad (97)$$

where

$$B(\beta)=3\,\frac{\int_0^\beta H(\beta)\,d\beta}{\beta H(\beta)}-1\,. \quad (98)$$

The asymptotic properties of $B(\beta)$ are easily derived from those of $H(\beta)$ (I, eqs. [118] and [119]). We find that

$$B(\beta)=\tfrac{1}{15}\Gamma\left(\tfrac{10}{3}\right)\beta^2+O(\beta^4)\,,\qquad(\beta\to 0) \quad (99)$$

and

$$B(\beta)\backsim\frac{8}{5}\sqrt{\frac{\pi}{2}}\,\beta^{3/2}\,.\qquad(\beta\to\infty) \quad (100)$$

Consider now an arbitrary direction (l, m, n). Then

$$\overline{f_{l,m,n}} = l\overline{f_\xi} + n\overline{f_\zeta}\,, \tag{101}$$

or, according to equation (97),

$$\overline{f_{l,m,n}} = -\tfrac{2}{3}\pi G\overline{M}nB(\beta)\,|\boldsymbol{v}|\,(l\sin\alpha - 2n\cos\alpha)\,, \tag{102}$$

where we have substituted for g from equation (66). Remembering that the direction cosines of $\boldsymbol{v}$ are $(\sin\alpha, 0, \cos\alpha)$, we have

$$\left.\begin{aligned} |\boldsymbol{v}|\,(l\sin\alpha - 2n\cos\alpha) &= |\boldsymbol{v}|\,(l\sin\alpha + n\cos\alpha) - 3n|\boldsymbol{v}|\cos\alpha \\ &= \boldsymbol{v}\cdot\mathbf{1}_{l,m,n} - 3\,\frac{(\boldsymbol{v}\cdot\boldsymbol{F})}{|\boldsymbol{F}|^2}\,\boldsymbol{F}\cdot\mathbf{1}_{l,m,n}\,, \end{aligned}\right\} \tag{103}$$

where $\mathbf{1}_{l,m,n}$ stands for a unit vector in the direction (l, m, n). Hence we can re-write equation (102) as

$$\bar{\boldsymbol{f}}\cdot\mathbf{1}_{l,m,n} = -\tfrac{2}{3}\pi G\overline{M}nB(\beta)\left[\boldsymbol{v} - 3\,\frac{(\boldsymbol{v}\cdot\boldsymbol{F})}{|\boldsymbol{F}|^2}\,\boldsymbol{F}\right]\cdot\mathbf{1}_{l,m,n} \tag{104}$$

or, more simply, as

$$\bar{\boldsymbol{f}} = -\tfrac{2}{3}\pi G\overline{M}nB\left(\frac{|\boldsymbol{F}|}{Q_H}\right)\left(\boldsymbol{v} - 3\,\frac{(\boldsymbol{v}\cdot\boldsymbol{F})}{|\boldsymbol{F}|^2}\,\boldsymbol{F}\right). \tag{105}$$

We shall return to this equation in § 11.

7. *The evaluation of the second moment of* ***f***.—There are in general six independent moments of the second order that we have to consider, namely,

$$\overline{f_\xi^2}\,,\quad \overline{f_\eta^2}\,,\quad \overline{f_\zeta^2}\,,\quad \overline{f_\xi f_\eta}\,,\quad \overline{f_\eta f_\zeta}\,,\quad \text{and}\quad \overline{f_\zeta f_\xi}\,. \tag{106}$$

In terms of these six moments, the second moment of the resolved component of $\boldsymbol{f}$ along any arbitrary direction can be specified. For, if $f_{l,m,n}$ denotes the component of $\boldsymbol{f}$ in the direction (l, m, n), then

$$\left.\begin{aligned} \overline{f_{l,m,n}^2} &= \overline{(lf_\xi + mf_\eta + nf_\zeta)^2} \\ &= l^2\overline{f_\xi^2} + m^2\overline{f_\eta^2} + n^2\overline{f_\zeta^2} + 2lm\overline{f_\xi f_\eta} + 2mn\overline{f_\eta f_\zeta} + 2nl\overline{f_\zeta f_\xi}\,. \end{aligned}\right\} \tag{107}$$

In our particular problem we should expect on symmetry grounds (and this is verified to to be the case) that two of the six moments (eq. [106]), namely, $\overline{f_\xi f_\eta}$ and $\overline{f_\eta f_\zeta}$, vanish identically (cf. eq. [97], according to which $\overline{f_\eta} = 0$). Accordingly, equation (107) reduces in our case to

$$\overline{f_{l,m,n}^2} = l^2\overline{f_\xi^2} + m^2\overline{f_\eta^2} + n^2\overline{f_\zeta^2} + 2nl\overline{f_\zeta f_\xi}\,. \tag{107'}$$

We shall first consider the moments $\overline{f_\xi^2}$, $\overline{f_\eta^2}$, and $\overline{f_\zeta^2}$. We have (cf. I, eq. [123])

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F},\boldsymbol{f})\,f_\tau^2 d\boldsymbol{f} = -\frac{1}{8\pi^3}\int_{-\infty}^{+\infty} e^{-i\boldsymbol{\rho}\cdot\boldsymbol{F}}\left[\frac{\partial^2}{\partial\sigma_\tau^2}\,A(\boldsymbol{\rho},\boldsymbol{\sigma})\right]_{|\boldsymbol{\sigma}|=0} d\boldsymbol{\rho}\,, \tag{108}$$

where we have used τ to denote either ξ, η, or ζ. Now, according to equation (68),

$$\left[\frac{\partial^2}{\partial\sigma_\tau^2}A(\boldsymbol{\rho},\boldsymbol{\sigma})\right]_{|\boldsymbol{\sigma}|=0} = -e^{-a|\boldsymbol{\rho}|^{3/2}}\left\{\tfrac{1}{2}g^2\frac{\partial^2}{\partial\sigma_\tau^2}P^2(\boldsymbol{\sigma}) + b|\boldsymbol{\rho}|^{-3/2}\left[\frac{\partial^2}{\partial\sigma_\tau^2}Q(\boldsymbol{\sigma}) + k\frac{\partial^2}{\partial\sigma_\tau^2}R(\boldsymbol{\sigma})\right]\right\}. \tag{109}$$

Hence,

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F},\boldsymbol{f})\,f_\tau^2 d\boldsymbol{f} = \frac{1}{8\pi^3}\int_{-\infty}^{+\infty} e^{-a|\boldsymbol{\rho}|^{3/2}-i\boldsymbol{\rho}\cdot\boldsymbol{F}} \times\left\{\tfrac{1}{2}g^2\frac{\partial^2 P^2}{\partial\sigma_\tau^2} + b|\boldsymbol{\rho}|^{-3/2}\left[\frac{\partial^2 Q}{\partial\sigma_\tau^2} + k\frac{\partial^2 R}{\partial\sigma_\tau^2}\right]\right\}d\boldsymbol{\rho}, \tag{110}$$

or, choosing polar co-ordinates in the (ξ, η, ζ)-system,

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F},\boldsymbol{f})\,f_\tau^2 d\boldsymbol{f} = \frac{1}{8\pi^3}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} e^{-a|\boldsymbol{\rho}|^{3/2}-i|\boldsymbol{\rho}||\boldsymbol{F}|t} \times\left\{\tfrac{1}{2}g^2\frac{\partial^2 P}{\partial\sigma_\tau^2} + b|\boldsymbol{\rho}|^{-3/2}\left[\frac{\partial^2 Q}{\partial\sigma_\tau^2} + k\frac{\partial^2 R}{\partial\sigma_\tau^2}\right]\right\}|\boldsymbol{\rho}|^2 d\omega\,dt\,d|\boldsymbol{\rho}|. \tag{111}$$

Performing the integration over the azimuthal angle ω and introducing the variable

$$x = |\boldsymbol{\rho}|\,|\boldsymbol{F}|; \qquad |\boldsymbol{F}| = a^{2/3}\beta, \tag{112}$$

equation (111) becomes

$$\int_{-\infty}^{+\infty} W(\boldsymbol{F},\boldsymbol{f})\,f_\tau^2 d\boldsymbol{f} = \frac{1}{2\pi^2 a^2\beta^3}\int_0^\infty\int_0^1 e^{-(x/\beta)^{3/2}}\left\{\tfrac{1}{2}g^2\left(\overline{\frac{\partial^2 P^2}{\partial\sigma_\tau^2}}\right) + ab\beta^{3/2}\left[\left(\overline{\frac{\partial^2 Q}{\partial\sigma_\tau^2}}\right) + k\left(\overline{\frac{\partial^2 R}{\partial\sigma_\tau^2}}\right)\right]x^{-3/2}\right\}x^2\cos xt\,dt\,dx, \tag{113}$$

where the bars over $\partial^2P^2/\partial\sigma_\tau^2$, etc., indicate that the averaging over ω has been carried out. We shall presently show that $\overline{\partial^2P^2/\partial\sigma_\tau^2}$, etc., have the forms

$$\left.\begin{aligned}\left(\overline{\frac{\partial^2 P^2}{\partial\sigma_\tau^2}}\right) &= \mathcal{P}_{0,\tau} + \mathcal{P}_{2,\tau}t^2 + \mathcal{P}_{4,\tau}t^4,\\ \left(\overline{\frac{\partial^2 Q}{\partial\sigma_\tau^2}}\right) &= \mathcal{Q}_{0,\tau} + \mathcal{Q}_{2,\tau}t^2,\\ \left(\overline{\frac{\partial^2 R}{\partial\sigma_\tau^2}}\right) &= \mathcal{R}_{0,\tau} + \mathcal{R}_{2,\tau}t^2 + \mathcal{R}_{4,\tau}t^4,\end{aligned}\right\}\quad(\tau = \xi, \eta, \zeta) \tag{114}$$

where $\mathcal{P}_{0,\tau}, \ldots, \mathcal{R}_{4,\tau}$ are constants with respect to the variables of integration in equation (113). Substituting these equivalents for $\overline{\partial^2P^2/\partial\sigma_\tau^2}$, etc., in equation (113) and dividing throughout by $W(\boldsymbol{F})$ according to I, equation (117), we obtain for the moment of f_τ^2 the expression

$$\left.\begin{aligned}\overline{f_\tau^2}=&\frac{2ab\beta^{1/2}}{\pi H(\beta)}\int_0^\infty\int_0^1 e^{-(x/\beta)^{3/2}}(\mathcal{L}_{0,\tau}+\mathcal{L}_{2,\tau}t^2+\mathcal{L}_{4,\tau}t^4)\,x^{1/2}\cos(xt)\,dx\,dt\\&+\frac{g^2}{\pi\beta H(\beta)}\int_0^\infty\int_0^1 e^{-(x/\beta)^{3/2}}(\mathcal{P}_{0,\tau}+\mathcal{P}_{2,\tau}t^2+\mathcal{P}_{4,\tau}t^4)\,x^2\cos(xt)\,dx\,dt\,,\end{aligned}\right\}\quad(115)$$

where, for the sake of brevity, we have written

$$\mathcal{L}_{0,\tau}=\mathcal{Q}_{0,\tau}+k\mathcal{R}_{0,\tau}\,;\quad \mathcal{L}_{2,\tau}=\mathcal{Q}_{2,\tau}+k\mathcal{R}_{2,\tau}\,;\quad \mathcal{L}_{4,\tau}=k\mathcal{R}_{4,\tau}\quad(\tau=\xi,\eta,\zeta)\,.\quad(116)$$

In equation (115) the integration over t can now be effected. Using the elementary formulae

$$\left.\begin{aligned}&\int_0^1\cos xt\,dt=\frac{\sin x}{x}\,,\\&\int_0^1 t^2\cos xt\,dt=\frac{\sin x}{x}-\frac{2}{x^3}(\sin x-x\cos x)\,,\\&\int_0^1 t^4\cos xt\,dt=\frac{\sin x}{x}-\frac{4}{x^5}(-x^3\cos x+6x\cos x+3x^2\sin x-6\sin x)\,,\end{aligned}\right\}\quad(117)$$

we readily obtain from equation (115)

$$\left.\begin{aligned}\overline{f_\tau^2}=&\,ab\frac{\beta^{1/2}}{H(\beta)}\left\{(\mathcal{L}_{0,\tau}+\mathcal{L}_{2,\tau}+\mathcal{L}_{4,\tau})G(\beta)-2\mathcal{L}_{2,\tau}I(\beta)-4\mathcal{L}_{4,\tau}J(\beta)\right\}\\&+\frac{g^2}{2\beta H(\beta)}\left\{(\mathcal{P}_{0,\tau}+\mathcal{P}_{2,\tau}+\mathcal{P}_{4,\tau})\beta H(\beta)-2\mathcal{P}_{2,\tau}K(\beta)-4\mathcal{P}_{4,\tau}L(\beta)\right\},\end{aligned}\right\}\quad(118)$$

where G, H, I, J, K, and L are functions of β defined by

$$\left.\begin{aligned}G(\beta)&=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-1/2}\sin x\,dx\,,\\H(\beta)&=\frac{2}{\pi\beta}\int_0^\infty e^{-(x/\beta)^{3/2}}x\sin x\,dx\,,\\I(\beta)&=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-5/2}(\sin x-x\cos x)\,dx\,,\\J(\beta)&=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-9/2}(-x^3\cos x+6x\cos x+3x^2\sin x-6\sin x)\,dx\,,\\K(\beta)&=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-1}(\sin x-x\cos x)\,dx\,,\\L(\beta)&=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-3}(-x^3\cos x+6x\cos x+3x^2\sin x-6\sin x)\,dx\,.\end{aligned}\right\}\quad(119)^5$$

[5] We may draw attention to the fact that we have not changed here our definition of $H(\beta)$; it represents the Holtsmark function as hitherto; similarly, our present definitions of $G(\beta)$ and $K(\beta)$ agree with those we have given earlier (I, eq. [159] and eq. [92] of this paper). However, the functions $I(\beta)$, $J(\beta)$, and $L(\beta)$ are introduced here for the first time.

It remains to obtain the expressions for $\mathcal{L}_{0,\tau}, \ldots, \mathcal{P}_{4,\tau}$ ($\tau = \xi, \eta, \zeta$). The calculations are straightforward but somewhat tedious. But we shall illustrate the method by outlining the derivation of $\mathcal{R}_{0,\xi}$, $\mathcal{R}_{2,\xi}$, and $\mathcal{R}_{4,\xi}$:

According to equation (67)

$$\left.\begin{aligned}\frac{\partial^2 R}{\partial \sigma_\xi^2} = \frac{\partial^2}{\partial \sigma_\xi^2}(\sigma_1^2[5 \sin^2\gamma - 2\cos^2\gamma] + \sigma_2^2[4\sin^2\gamma - 2\cos^2\gamma] \\ + \sigma_3^2[4\cos^2\gamma - 2\sin^2\gamma] - 8\sigma_3\sigma_1 \sin\gamma\cos\gamma)\,.\end{aligned}\right\} \quad (120)$$

To carry out the differentiation we must first apply to ($\sigma_1, \sigma_2, \sigma_3$) the linear transformation (72). It is, however, not necessary to carry out this transformation explicitly. For, since clearly

$$\frac{\partial^2 \sigma_i^2}{\partial \sigma_\xi^2} = 2\lambda_i^2 \qquad (i = 1, 2, 3)\,; \qquad \frac{\partial^2(\sigma_3\sigma_1)}{\partial \sigma_\xi^2} = 2\lambda_1\lambda_3\,, \quad (121)$$

we have

$$\left.\begin{aligned}\frac{\partial^2 R}{\partial \sigma_\xi^2} = 2[\lambda_1^2(5\sin^2\gamma - 2\cos^2\gamma) + \lambda_2^2(4\sin^2\gamma - 2\cos^2\gamma) \\ + \lambda_3^2(4\cos^2\gamma - 2\sin^2\gamma) - 8\lambda_1\lambda_3 \sin\gamma\cos\gamma]\end{aligned}\right\} \quad (122)$$

or, after some minor reductions,

$$\frac{\partial^2 R}{\partial \sigma_\xi^2} = 2[-2 + 6\lambda_3^2\cos^2\gamma + (7\lambda_1^2 + 6\lambda_2^2)\sin^2\gamma - 8\lambda_1\lambda_3\sin\gamma\cos\gamma]\,. \quad (123)$$

Substituting for λ_1, λ_2, and λ_3 from the table of direction cosines (69) and using for $\cos\gamma$ its equivalent (71), we obtain, after some rearranging of the terms,

$$\left.\begin{aligned}\frac{\partial^2 R}{\partial \sigma_\xi^2} = 2[-2 + 21l^4\sin^2\alpha + 21l^2n^2\cos^2\alpha + 42l^3n\sin\alpha\cos\alpha + 7\sin^2\alpha \\ + 6m^2\cos^2\alpha - 22l^2\sin^2\alpha - 22ln\sin\alpha\cos\alpha]\,,\end{aligned}\right\} \quad (124)$$

where it will be recalled that α is the angle between $\boldsymbol{F}$ and $\boldsymbol{v}$. Averaging the quantity on the right-hand side of equation (142), we finally obtain

$$\overline{\left(\frac{\partial^2 R}{\partial \sigma_\xi^2}\right)} = (2 + \tfrac{7}{4}\sin^2\alpha) + (15 - \tfrac{49}{2}\sin^2\alpha)\,t^2 + (-21 + \tfrac{147}{4}\sin^2\alpha)\,t^4\,, \quad (125)$$

from which the values of $\mathcal{R}_{0,\xi}$, $\mathcal{R}_{2,\xi}$, and $\mathcal{R}_{4,\xi}$ can at once be read. The evaluation of the other quantities proceeds similarly. Collecting all the results, we obtain the following table of values:

$$\left.\begin{aligned}
&\mathcal{P}_{0,\xi} = \tfrac{11}{4}\sin^2\alpha\,; && \mathcal{P}_{2,\xi} = +9 - \tfrac{33}{2}\sin^2\alpha\,; && \mathcal{P}_{4,\xi} = -9 + \tfrac{63}{4}\sin^2\alpha\,,\\
&\mathcal{P}_{0,\eta} = \tfrac{9}{4}\sin^2\alpha\,; && \mathcal{P}_{2,\eta} = +9 - \tfrac{27}{2}\sin^2\alpha\,; && \mathcal{P}_{4,\eta} = -9 + \tfrac{45}{4}\sin^2\alpha\,,\\
&\mathcal{P}_{0,\zeta} = 2\cos^2\alpha\,; && \mathcal{P}_{2,\zeta} = -12 + 21\sin^2\alpha\,; && \mathcal{P}_{4,\zeta} = +18 - 27\sin^2\alpha\,;
\end{aligned}\right\} \quad (126)$$

$$\left.\begin{aligned}
&\mathcal{Q}_{0,\xi} = 1\,; && \mathcal{Q}_{2,\xi} = +1\,; && \mathcal{Q}_{4,\xi} = 0\,,\\
&\mathcal{Q}_{0,\eta} = 1\,; && \mathcal{Q}_{2,\eta} = +1\,; && \mathcal{Q}_{4,\eta} = 0\,,\\
&\mathcal{Q}_{0,\zeta} = 2\,; && \mathcal{Q}_{2,\zeta} = -2\,; && \mathcal{Q}_{4,\zeta} = 0\,;
\end{aligned}\right\} \quad (127)$$

and

$$\left.\begin{aligned}
&\mathcal{R}_{0,\,\xi}=2\;\;+\tfrac{7}{4}\sin^2\alpha\,; &&\mathcal{R}_{2,\,\xi}=+15-\tfrac{49}{2}\sin^2\alpha\,; &&\mathcal{R}_{4,\,\xi}=-21+\tfrac{147}{4}\sin^2\alpha\,,\\
&\mathcal{R}_{0,\,\eta}=2\;\;-\tfrac{3}{4}\sin^2\alpha\,; &&\mathcal{R}_{2,\,\eta}=+15-\tfrac{27}{2}\sin^2\alpha\,; &&\mathcal{R}_{4,\,\eta}=-21+\tfrac{105}{4}\sin^2\alpha\,,\\
&\mathcal{R}_{0,\,\zeta}=10-8\sin^2\alpha\,; &&\mathcal{R}_{2,\,\zeta}=-44+59\sin^2\alpha\,; &&\mathcal{R}_{4,\,\zeta}=+42-\;63\sin^2\alpha\,.
\end{aligned}\right\}\tag{128}$$

We have still to evaluate the cross-moments $\overline{f_\xi f_\eta}$, $\overline{f_\eta f_\zeta}$, and $\overline{f_\zeta f_\xi}$. As we have already stated, the first two vanish identically, and we have only to consider $\overline{f_\zeta f_\xi}$. Analogous to equation (110) we now have

$$\left.\begin{aligned}
\int_{-\infty}^{+\infty} W(\boldsymbol{F},\boldsymbol{f})\, f_\xi f_\zeta df &= \frac{1}{8\pi^3}\int_{-\infty}^{+\infty} e^{-a|\boldsymbol{\rho}|^{3/2}-i\boldsymbol{\rho}\cdot\boldsymbol{F}}\\
&\times\left\{\tfrac{1}{2}g^2\frac{\partial^2 P^2}{\partial\sigma_\xi\partial\sigma_\zeta}+b\,|\boldsymbol{\rho}|^{-3/2}\left[\frac{\partial^2 Q}{\partial\sigma_\xi\partial\sigma_\zeta}+k\,\frac{\partial^2 R}{\partial\sigma_\xi\partial\sigma_\zeta}\right]\right\}d\boldsymbol{\rho}\,,
\end{aligned}\right\}\tag{129}$$

or, changing to polar co-ordinates and integrating over ω, we find (cf. eq. [113])

$$\left.\begin{aligned}
\int_{-\infty}^{+\infty} W(\boldsymbol{F},\boldsymbol{f})\, f_\xi f_\zeta df &= \frac{1}{2\pi^2 a^2\beta^3}\int_0^\infty\!\!\int_0^1 e^{-(x/\beta)^{3/2}}\left\{\tfrac{1}{2}g^2\left(\overline{\frac{\partial^2 P}{\partial\sigma_\xi\partial\sigma_\zeta}}\right)\right.\\
&\left.+ab\beta^{3/2}\left[\left(\overline{\frac{\partial^2 Q}{\partial\sigma_\xi\partial\sigma_\zeta}}\right)+k\left(\overline{\frac{\partial^2 R}{\partial\sigma_\zeta\partial\sigma_\zeta}}\right)\right]x^{-3/2}\right\}x^2\cos xt\,dt\,dx\,,
\end{aligned}\right\}\tag{130}$$

where we have made a further change of variables according to equation (112). In equation (130) the bars are again to be understood as indicating that the corresponding quantities have been averaged over ω. An elementary, but somewhat lengthy, calculation yields

$$\left.\begin{aligned}
\left(\overline{\frac{\partial^2 P^2}{\partial\sigma_\xi\partial\sigma_\zeta}}\right) &= (-1+15t^2-18t^4)\sin\alpha\cos\alpha\,,\\
\left(\overline{\frac{\partial^2 R}{\partial\sigma_\xi\partial\sigma_\zeta}}\right) &= (-3+37t^2-42t^4)\sin\alpha\cos\alpha\,.
\end{aligned}\right\}\tag{131}$$

Further,

$$\left(\overline{\frac{\partial^2 Q}{\partial\sigma_\xi\partial\sigma_\zeta}}\right)=0\,.\tag{132}$$

Substituting these formulae in equations (130) and dividing throughout by $W(\boldsymbol{F})$ according to I, equation (117), we obtain

$$\left.\begin{aligned}
\overline{f_\xi f_\zeta} = \Bigg\{&\frac{2abk\beta^{1/2}}{\pi H(\beta)}\int_0^\infty\!\!\int_0^1 e^{-(x/\beta)^{3/2}}(-3+37t^2-42t^4)\,x^{1/2}\cos xt\,dt\,dx\\
&+\frac{g^2}{\pi\beta H(\beta)}\int_0^\infty\!\!\int_0^1 e^{-(x/\beta)^{3/2}}(-1+15t^2-18t^4)\,x^2\cos xt\,dt\,dx\Bigg\}\sin\alpha\cos\alpha\,,
\end{aligned}\right\}\tag{133}$$

or, integrating over t and introducing the auxiliary functions $G, \ldots, L$, according to equation (119), we have

$$\overline{f_\xi f_\zeta} = \left\{ abk \frac{\beta^{1/2}}{H(\beta)} [-8G(\beta) - 74I(\beta) + 168J(\beta)] + \frac{g^2}{2\beta H(\beta)} [-4\beta H(\beta) - 30K(\beta) + 72L(\beta)] \right\} \sin\alpha \cos\alpha . \quad (134)$$

Equations (118) and (134), together with

$$\overline{f_\xi f_\eta} \equiv 0 ; \qquad \overline{f_\eta f_\zeta} \equiv 0 , \quad (135)$$

represent, then, our expressions for the six independent second-order moments.

8. *The moment* $|f|^2$ *and its average* $\overline{|f|^2}$. *The mean life of the state* **F**.—In § 7 we evaluated all the independent second-order moments that exist and, as we have already stated, the second moment of the resolved component of f along any direction can be specified in terms of them. However, the most important quantity is the average value of $|f|^2$. This is clearly given by

$$\overline{|f|^2} = \sum_{\tau=\xi,\eta,\zeta} \overline{f_\tau^2} \quad (136)$$

or, according to equation (118), by

$$\overline{|f|^2} = ab \frac{\beta^{1/2}}{H(\beta)} \left\{ \left(\sum_{i=0,2,4} \sum_{\tau=\xi,\eta,\zeta} \mathcal{L}_{i,\tau} \right) G(\beta) - 2 \left(\sum_{\tau=\xi,\eta,\zeta} \mathcal{L}_{2,\tau} \right) I(\beta) - 4 \left(\sum_{\tau=\xi,\eta,\zeta} \mathcal{L}_{4,\tau} \right) J(\beta) \right\} + \frac{g^2}{2\beta H(\beta)} \left\{ \left(\sum_{i=0,2,4} \sum_{\tau=\xi,\eta,\zeta} \mathcal{P}_{i,\tau} \right) \beta H(\beta) - 2 \left(\sum_{\tau=\xi,\eta,\zeta} \mathcal{P}_{2,\tau} \right) K(\beta) - 4 \left(\sum_{\tau=\xi,\eta,\zeta} \mathcal{P}_{4,\tau} \right) L(\beta) \right\} . \quad (137)$$

On the other hand, from equations (126)–(128) we find that

$$\begin{aligned} &\sum_\tau \mathcal{P}_{0,\tau} = 5\sin^2\alpha + 2\cos^2\alpha ; && \sum_\tau \mathcal{P}_{2,\tau} = 6 - 9\sin^2\alpha ; && \sum_\tau \mathcal{P}_{4,\tau} = 0 , \\ &\sum_\tau \mathcal{Q}_{0,\tau} = 4 ; && \sum_\tau \mathcal{Q}_{2,\tau} = 0 ; && \sum_\tau \mathcal{Q}_{4,\tau} = 0 , \\ &\sum_\tau \mathcal{R}_{0,\tau} = 14 - 7\sin^2\alpha ; && \sum_\tau \mathcal{R}_{2,\tau} = -14 + 21\sin^2\alpha ; && \sum_\tau \mathcal{R}_{4,\tau} = 0 . \end{aligned} \quad (138)$$

Again, according to equations (138),

$$\sum_i \sum_\tau \mathcal{P}_{i,\tau} = 8 - 6\sin^2\alpha ; \quad \sum_i \sum_\tau \mathcal{Q}_{i,\tau} = 4 ; \quad \sum_i \sum_\tau \mathcal{R}_{i,\tau} = 14\sin^2\alpha . \quad (139)$$

We thus have

$$\overline{|f|^2}_{F,v} = 2ab \frac{\beta^{1/2}}{H(\beta)} \left\{ 2G(\beta) + 7k [\sin^2\alpha\, G(\beta) - (3\sin^2\alpha - 2) I(\beta)] \right\} + \frac{g^2}{\beta H(\beta)} \left\{ (4 - 3\sin^2\alpha)\beta H(\beta) + 3(3\sin^2\alpha - 2) K(\beta) \right\} . \quad (140)$$

The foregoing expression gives the mean square value of the rate of change, f, to be expected in the intensity of the force, F, acting on a star when it is further known that the direction of F makes an angle α with the direction of motion. But the possible directions of F, for a given direction of motion, are distributed uniformly over the unit sphere. Hence, the mean square value of f for a given value of F and for all relative orientations of the two vectors F and v is obtained by simply averaging equation (140) over α. We thus obtain

$$\overline{\overline{|f|^2}}_{|F|,\,|v|} = 4ab\left\{\frac{\beta^{1/2}G(\beta)}{H(\beta)}\left(1+\tfrac{7}{3}k\right)+\frac{g^2}{2ab}\right\}; \tag{141}$$

or, substituting for k and $g^2/2ab$ from equation (66), we find that

$$\overline{\overline{|f|^2}}_{|F|,\,|v|} = 4ab\left\{\frac{\beta^{1/2}G(\beta)}{H(\beta)}\left(1+\frac{\overline{M^{1/2}}\,|v|^2}{\overline{M^{1/2}}\,\overline{|u|^2}}\right)+\frac{5}{12\pi}\frac{\overline{M^2}\,|v|^2}{\overline{M^{3/2}}\,\overline{M^{1/2}\,|u|^2}}\right\}. \tag{142}$$

If $|v| \rightarrow 0$, we recover the formula of our earlier paper (cf. I, eq. [158]).

The formula (142) has an immediate application for estimating the mean life of the state of fluctuation in which a force F per unit mass acts on a star moving with a speed $|v|$. For, arguing as in I, § 9, we may define the mean life by the equation

$$T_{|F|,\,|v|} = \frac{|F|}{\sqrt{\overline{\overline{|f|^2}}_{|F|,\,|v|}}}. \tag{143}$$

According to equations (112) and (142), we therefore have

$$\left.\begin{aligned} T_{|F|,\,|v|} = \left[\frac{a^{1/3}}{4b}\frac{\beta^{3/2}H(\beta)}{G(\beta)}\right]^{1/2}\Bigg[1+\frac{\overline{M^{1/2}}\,|v|^2}{\overline{M^{1/2}}\,\overline{|u|^2}}+\frac{5}{12\pi}\frac{\overline{M^2}\,|v|^2}{\overline{M^{3/2}}\,\overline{M^{1/2}\,|u|^2}}&\\ \times\frac{H(\beta)}{\beta^{1/2}G(\beta)}\Bigg]^{-1/2}&. \end{aligned}\right\} \tag{144}$$

From this equation we derive the relation

$$T_{|F|,\,|v|} = T_{|F|,\,0}\left[1+\frac{\overline{M^{1/2}}\,|v|^2}{\overline{M^{1/2}}\,\overline{|u|^2}}+\frac{5}{12\pi}\frac{\overline{M^2}\,|v|^2}{\overline{M^{3/2}}\,\overline{M^{1/2}\,|u|^2}}\frac{H(\beta)}{\beta^{1/2}G(\beta)}\right]^{-1/2}. \tag{145}$$

9. *The auxiliary functions* G, H, I, J, K, *and* L.—In §§ 6, 7, and 8 we have seen that our expressions for the various moments involve some or all of the functions $G(\beta)$, $H(\beta)$, $I(\beta)$, $J(\beta)$, $K(\beta)$, and $L(\beta)$. These functions are all defined by certain definite integrals (eq. [119]), the integrands of which contain β as a parameter. We shall now show how the functions $G(\beta)$, $I(\beta)$, $J(\beta)$, $K(\beta)$, and $L(\beta)$ can all be expressed in terms of the Holtsmark function, $H(\beta)$. We shall also establish certain relations which exist among them.

We have already seen that the functions $G(\beta)$ and $K(\beta)$ are related to $H(\beta)$, according to the formulae (eq. [96] and I, eq. [162]),

$$G(\beta) = \frac{3}{2}\int_0^\beta \beta^{-3/2}H(\beta)\,d\beta \tag{146}$$

and

$$K(\beta) = \int_0^\beta H(\beta)\,d\beta\,. \tag{147}$$

It remains to establish similar relations for I, J, and L.

Consider, first, $I(\beta)$. Writing the integral, defining it in the form

$$I(\beta) = -\frac{4}{3\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x - x\cos x)\frac{d}{dx}(x^{-3/2})\,dx\,, \tag{148}$$

and integrating by parts, we find

$$I(\beta) = \frac{4}{3\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-1/2}\sin x\,dx - \frac{2}{\pi\beta^{3/2}}\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x - x\cos x)\frac{dx}{x}\,, \tag{149}$$

or, remembering our definition of $G(\beta)$ (eq. [119]), we have

$$I(\beta) = \tfrac{2}{3}G(\beta) - \frac{2}{\pi\beta^{3/2}}\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x - x\cos x)\,x^{-1}dx\,. \tag{150}$$

But

$$\frac{dI}{d\beta} = \frac{3}{\pi\beta^{5/2}}\int_0^\infty e^{-(x/\beta)^{3/2}}(\sin x - x\cos x)\,x^{-1}dx\,. \tag{151}$$

Hence,

$$I(\beta) = \tfrac{2}{3}G(\beta) - \tfrac{2}{3}\beta\frac{dI}{d\beta}\,; \tag{152}$$

in other words, $I(\beta)$ satisfies the differential equation

$$\beta\frac{dI}{d\beta} + \tfrac{3}{2}I = G(\beta)\,. \tag{153}$$

The solution of equation (153) appropriate for us is

$$I(\beta) = \beta^{-3/2}\int_0^\beta \beta^{1/2}G(\beta)\,d\beta\,. \tag{154}$$

This formula is useful for the purposes of numerically evaluating the function $I(\beta)$.

We shall now establish for $J(\beta)$ a relation similar to equation (154). We have

$$\left.\begin{aligned} J(\beta) &= -\frac{4}{7\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}(-x^3\cos x + 6x\cos x + 3x^2\sin x - 6\sin x) \\ &\qquad\qquad\times\frac{d}{dx}(x^{-7/2})\,dx \\ &= \frac{4}{7\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{-1/2}\sin x\,dx \\ &\quad - \frac{6}{7\pi\beta^{3/2}}\int_0^\infty e^{-(x/\beta)^{3/2}}(-x^3\cos x + 6x\cos x + 3x^2\sin x - 6\sin x)\frac{dx}{x^3} \\ &= \tfrac{2}{7}G(\beta) - \tfrac{2}{7}\beta\frac{dJ}{d\beta}\,. \end{aligned}\right\} \tag{155}$$

Hence $J(\beta)$ satisfies the differential equation

$$\beta\frac{dJ}{d\beta} + \tfrac{7}{2}J = G(\beta)\,. \tag{156}$$

Consequently,

$$J(\beta) = \beta^{-7/2} \int_0^\beta \beta^{5/2} G(\beta)\, d\beta\,. \tag{157}$$

This is the required formula.

By following a procedure exactly similar to that adopted for treating $I(\beta)$ and $J(\beta)$ we can show that $L(\beta)$ satisfies the differential equation

$$\beta \frac{dL}{d\beta} + 2L = \beta H(\beta) \tag{158}$$

and that therefore

$$L(\beta) = \beta^{-2} \int_0^\beta \beta^2 H(\beta)\, d\beta\,. \tag{159}$$

Further, according to equations (150) and (155), we have the relations

$$I(\beta) = \tfrac{2}{3} G(\beta) - \beta^{-3/2} K(\beta) \tag{160}$$

and

$$J(\beta) = \tfrac{2}{7} G(\beta) - \tfrac{3}{7} \beta^{-3/2} L(\beta)\,. \tag{161}$$

Finally, we may note the following asymptotic forms for the various functions:

$$\left.\begin{array}{ll}
\beta \to 0 & \beta \to \infty \\
G(\beta) \to \dfrac{4}{3\pi} \beta^{3/2} & G(\beta) \to \sqrt{\dfrac{2}{\pi}} \\
H(\beta) \to \dfrac{4}{3\pi} \beta^{2} & H(\beta) \to \dfrac{15}{8}\sqrt{\dfrac{2}{\pi}}\, \beta^{-5/2} \\
I(\beta) \to \dfrac{4}{9\pi} \beta^{3/2} & I(\beta) \to \dfrac{2}{3}\sqrt{\dfrac{2}{\pi}} \\
J(\beta) \to \dfrac{4}{15\pi} \beta^{3/2} & J(\beta) \to \dfrac{2}{7}\sqrt{\dfrac{2}{\pi}} \\
K(\beta) \to \dfrac{4}{9\pi} \beta^{3} & K(\beta) \to 1 \\
L(\beta) \to \dfrac{4}{15\pi} \beta^{3} & L(\beta) \to \dfrac{15}{4}\sqrt{\dfrac{2}{\pi}}\, \beta^{-3/2} \\
B(\beta) \to \tfrac{1}{15}\Gamma\left(\tfrac{10}{3}\right) \beta^{2} & B(\beta) \to \dfrac{8}{5}\sqrt{\dfrac{\pi}{2}}\, \beta^{3/2}
\end{array}\right\} \tag{162}$$

10. *The correlations in* $\boldsymbol{F}$ *acting at two very close points*.—The formal theory developed in the preceding sections has direct applications to a different problem, namely, that of the correlations in the force acting at two very close points, for the difference between the values of $\boldsymbol{F}$ acting at two points distant $\delta \boldsymbol{r}$ from each other is given by

$$\Delta \boldsymbol{F} = G \sum_i M_i \left\{ \frac{\delta \boldsymbol{r}}{|\boldsymbol{r}_i|^3} - 3 \frac{\boldsymbol{r}_i (\boldsymbol{r}_i \cdot \delta \boldsymbol{r}_i)}{|\boldsymbol{r}_i|^5} \right\}, \tag{163}$$ [6]

where we have assumed that one of the points is at the origin of our system of coordinates. Comparing equations (3) and (163), we see that formally the problems of specifying the distributions $W(\boldsymbol{F}, \boldsymbol{f})$ and $W(\boldsymbol{F}, \Delta \boldsymbol{F})$ differ only to the extent that, while in the first case we have to allow for a distribution over the relative velocities $\boldsymbol{V}$, in our present problem $\delta \boldsymbol{r}$ is a fixed constant vector.

[6] We discuss later in this section the validity of this formula.

Thus, expressing $W(F, \Delta F)$ in the form

$$W(F, \Delta F) = \frac{1}{64\pi^6} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(\rho\cdot F + \sigma\cdot \Delta F)} A(\rho, \sigma)\, d\rho d\sigma\,, \tag{164}$$

we have (cf. eq. [12])

$$A(\rho, \sigma) = e^{-nC(\rho, \sigma)}\,, \tag{165}$$

where (cf. eq. [15])

$$C(\rho, \sigma) = \tfrac{1}{2} G^{3/2} \int_0^\infty dM \chi(M) M^{3/2} \int_{-\infty}^{+\infty} \{1 - e^{i(\rho\cdot\phi + \sigma\cdot\psi)}\} |\phi|^{-9/2} d\phi\,. \tag{166}$$

In equation (166) ψ now stands for (cf. eq. [16])

$$\psi = (GM)^{-1/2} \{ |\phi|^{3/2} \delta r - 3 |\phi|^{-1/2} (\phi\cdot \delta r) \phi \}\,. \tag{167}$$

From this point on, the analysis proceds exactly as in §§ 3, 4, and 5. It is thus seen that equation (64) is now replaced by (cf. eq. [56])

$$\left.\begin{aligned} C(\rho, \sigma) &= \tfrac{4}{15}(2\pi)^{3/2} G^{3/2} \overline{M^{3/2}} |\rho|^{3/2} + \tfrac{2}{3}\pi i G\overline{M} |\delta r| (\sigma_1 \sin\gamma - 2\sigma_3 \cos\gamma) \\ &+ \tfrac{3}{28}(2\pi)^{3/2} G^{1/2} \overline{M^{1/2}} |\delta r|^2 |\rho|^{-3/2} [\sigma_1^2 (5 \sin^2\gamma - 2\cos^2\gamma) + \sigma_2^2 (4\sin^2\gamma \\ &- 2\cos^2\gamma) + \sigma_3^2 (4\cos^2\gamma - 2\sin^2\gamma) - 8\sigma_1\sigma_3 \sin\gamma\cos\gamma] + O(|\sigma|^3)\,, \end{aligned}\right\} \tag{168}$$

where

$$\gamma = \measuredangle(\rho, \delta r) \tag{169}$$

and the co-ordinate system has been so chosen that the z-axis is in the direction of ρ and δr lies in the xz-plane (see Fig. 1). Hence,

$$A(\rho, \sigma) = e^{-a|\rho|^{3/2} - ig' P(\sigma) - b'|\rho|^{-3/2} R(\sigma) + O(|\sigma|^3)}\,, \tag{170}$$

where a, $P(\sigma)$, and $R(\sigma)$ have the same meanings as in equations (66) and (67), while b' and g' now stand for

$$b' = \tfrac{3}{28}(2\pi)^{3/2} G^{1/2} \overline{M^{1/2}} |\delta r|^2 n\,; \qquad g' = \tfrac{2}{3}\pi G\overline{M} |\delta r| n\,. \tag{171}$$

The evaluation of the first and the second moments of ΔF proceeds as in §§ 6, 7, and 8. Thus, analogous to equation (105), we now have

$$\overline{\Delta F}_{F,\,\delta r} = \tfrac{2}{3}\pi G\overline{M} n B\left(\frac{|F|}{Q_H}\right)\left(\delta r - 3\frac{(F\cdot\delta r)}{|F|^2} F\right). \tag{172}$$

Similarly the moment of $|\Delta F|^2$ is given by (cf. eq. [140])

$$\left.\begin{aligned} \overline{|\Delta F|^2}_{F,\,\delta r} &= 14 a b' \frac{\beta^{1/2}}{H(\beta)} [\sin^2 \alpha G(\beta) - (3\sin^2\alpha - 2) I(\beta)] \\ &+ \frac{g'^2}{\beta H(\beta)} [(4 - 3\sin^2\alpha)\beta H(\beta) + 3(3\sin^2\alpha - 2) K(\beta)]\,, \end{aligned}\right\} \tag{173}$$

or, substituting for a, b', and g' from equations (66) and (171), we have

$$\left.\begin{aligned}\overline{|\Delta F|^2_{F,\,\delta r}} = \frac{16\pi^3}{5} G^2\overline{M^{3/2}}\,\overline{M^{1/2}}n^2|\delta r|^2\frac{\beta^{1/2}}{H(\beta)}[\sin^2\alpha G(\beta) \\ -(3\sin^2\alpha-2)I(\beta)] + \frac{4\pi^2}{9}G^2\overline{M}^2 n^2|\delta r|^2\frac{1}{\beta H(\beta)}[(4-3\sin^2\alpha) \\ \times\beta H(\beta)+3(3\sin^2\alpha-2)K(\beta)]\,.\end{aligned}\right\} \quad (174)$$

Averaging the foregoing expression over all α, we find

$$\overline{\overline{|\Delta F|^2_{|F|,\,|\delta r|}}} = \frac{32\pi^3}{15}G^2\overline{M^{3/2}}\,\overline{M^{1/2}}n^2|\delta r|^2\frac{\beta^{1/2}G(\beta)}{H(\beta)} + \frac{8\pi^2}{9}G^2\overline{M}^2n^2|\delta r|^2\,. \quad (175)$$

Now, according to equation (172) for a fixed $|\delta r|$, $\overline{\Delta F}$ tends to infinity as $|F| \to \infty$. This is contrary to what we should expect on physical grounds, namely, that $\overline{\Delta F}$ should tend to zero as $|F| \to \infty$, for, since the highest fields are approximately produced by the nearest neighbor,[7] it follows that, as $|F| \to \infty$, the particular star which effectively produces the field must be so close to one of the two points considered that no correlations in the *directions* of F acting at the two points can be expected. In other words, $\overline{\Delta F}$ should tend to vanish as $|F| \to \infty$. But this same argument shows why our present theory of spatial correlations fails as $|F| \to \infty$, for, given a $|\delta r|$, however small, we can always choose a $|F|$ so large that on the first-neighbor approximation, the "nearest neighbor" will be closer to one of the points than $|\delta r|$. Under these circumstances the contribution to ΔF arising from this nearest neighbor will no longer be represented to any degree of accuracy by a term of the series (163)— the Taylor expansion of $r/|r|^3$ on which this series is based will cease to be valid for at least that particular term corresponding to the nearest neighbor producing a $|F| \to \infty$. We therefore conclude that our present method gives only the asymptotic behavior of the true spatial correlations in the sense that, given a $|F|$, however large, we can choose a $|\delta r|$ sufficiently small for our formulae to be valid for $|F|$ less than the specified limit.

11. *Dynamical friction.*—We shall now return to equation (105) for a discussion of its implications for general dynamical theory. According to this equation,

$$\bar{\dot{F}} = \overline{\left(\frac{dF}{dt}\right)}_{F,\,v} = -\tfrac{2}{3}\pi G\overline{M}nB\left(\frac{|F|}{Q_H}\right)\left(v - 3\frac{v\cdot F}{|F|^2}F\right), \quad (176)$$

where $B(\beta)$ is defined in equation (98). We shall first derive certain formal consequences of this equation.

Multiplying equation (176) scalarly with F, we obtain

$$F\cdot\overline{\left(\frac{dF}{dt}\right)}_{F,\,v} = \tfrac{4}{3}\pi G\overline{M}nB\left(\frac{|F|}{Q_H}\right)(v\cdot F)\,. \quad (177)$$

But

$$F\cdot\overline{\left(\frac{dF}{dt}\right)}_{F,\,v} = |F|\overline{\left(\frac{d|F|}{dt}\right)}_{F,\,v}\,. \quad (178)$$

Hence

$$\overline{\left(\frac{d|F|}{dt}\right)}_{F,\,v} = \tfrac{4}{3}\pi G\overline{M}nB\left(\frac{|F|}{Q_H}\right)\frac{v\cdot F}{|F|}\,. \quad (179)$$

[7] S. Chandrasekhar, *Ap. J.*, **94**, 511, 1941 (§§ 3 and 4).

On the other hand, if F_j denotes the component of $\boldsymbol{F}$ in an arbitrary direction at right angles to the direction of $\boldsymbol{v}$, then, according to equation (176),

$$\overline{\left(\frac{dF_j}{dt}\right)}_{F,\,v}=2\pi G\overline{M}nB\left(\frac{|F|}{Q_H}\right)\frac{v\cdot F}{|F|^2}F_j\,. \tag{180}$$

Combining equations (179) and (180), we have

$$\frac{1}{F_j}\overline{\left(\frac{dF_j}{dt}\right)}_{F,\,v}=\frac{3}{2}\frac{1}{|F|}\overline{\left(\frac{d|F|}{dt}\right)}_{F,\,v}\,. \tag{181}$$

Equation (181) is clearly equivalent to

$$\overline{\left[\frac{d}{dt}(\log|F_j|-\tfrac{3}{2}\log|F|)\right]}_{F,\,v}=0\,. \tag{182}$$

In other words, we have proved that

$$\overline{\left[\frac{d}{dt}\left(\frac{F_j}{|F|^{3/2}}\right)\right]}_{F,\,v}=0\,. \tag{183}$$

We shall now examine the physical consequences of equation (176) more closely. In words, the meaning of this equation is that the component of

$$-\tfrac{2}{3}\pi G\overline{M}nB\left(\frac{|F|}{Q_H}\right)\left(v-3\frac{v\cdot F}{|F|^2}F\right) \tag{184}$$

along any particular direction gives the average value of the rate of change of the force per unit mass acting on a star that is to be expected in the specified direction when the star is moving with a velocity $\boldsymbol{v}$ in an appropriately chosen local standard of rest. Stated in this manner, the essential difference is at once seen in the stochastic variations of $\boldsymbol{F}$ with time in the two cases $|v|=0$ and $|v|\neq 0$. In the former case $\overline{\dot{F}}\equiv 0$; but this is not generally true when $|v|\neq 0$. Or, expressed somewhat differently, when $|v|=0$ the changes in $\boldsymbol{F}$ occur with equal probability in all directions, while this is not the case when $|v|\neq 0$. The exact nature of this difference is brought out quite clearly when we consider

$$\overline{\left(\frac{d|F|}{dt}\right)}_{F,\,v} \tag{185}$$

according to equation (179). Remembering that $B(\beta)\geqslant 0$ for $\beta\geqslant 0$, we conclude from equation (179) that

$$\overline{\left(\frac{d|F|}{dt}\right)}_{F,\,v}>0 \quad \text{if} \quad v\cdot F>0 \tag{186}$$

and

$$\overline{\left(\frac{d|F|}{dt}\right)}_{F,\,v}<0 \quad \text{if} \quad v\cdot F<0\,. \tag{187}$$

In other words, if $\boldsymbol{F}$ has a positive component in the direction of $\boldsymbol{v}$, $|F|$ increases on the average; while, if $\boldsymbol{F}$ has a negative component in the direction of $\boldsymbol{v}$, $|F|$ decreases on the average. We shall now show that it is this essential asymmetry introduced by the direction of $\boldsymbol{v}$ that gives rise to the phenomenon of dynamical friction.

The characteristic aspects of the situation governed by equation (179) are best understood when we contrast it with the case $\bar{F} \equiv 0$. Under these circumstances we can visualize the motion of the representative point in the velocity space somewhat as follows:[8] The representative point suffers small random displacements in a manner that can be adequately described by the theory of random flights.[9] More specifically, the star may be assumed to suffer a large number of discrete increases in velocity of amounts $\boldsymbol{F}T(\boldsymbol{F})$, where T is the "mean life" of the state $\boldsymbol{F}$; these increases are further assumed to take place in random directions. On these assumptions it readily follows that the mean square increase in the velocity, which we may expect the star to suffer in time t, is given by

$$\overline{|\Delta \boldsymbol{v}|^2} = \overline{|\boldsymbol{F}|^2 T} t . \tag{188}$$

An alternative way of describing the same situation is to assert that the function $P(\boldsymbol{v}, t; \boldsymbol{v}_0)$, which gives the probability that the star has a velocity $\boldsymbol{v}$ at time t, given that $\boldsymbol{v} = \boldsymbol{v}_0$ at $t = 0$, satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = D \left(\frac{\partial^2 P}{\partial v_1^2} + \frac{\partial^2 P}{\partial v_2^2} + \frac{\partial^2 P}{\partial v_3^2} \right), \tag{189}$$

with the "coefficient of diffusion," D, having the value

$$D = \tfrac{1}{6} \overline{|\boldsymbol{F}|^2 T} . \tag{190}$$

The solution of equation (189) appropriate for our purposes is

$$P(\boldsymbol{v}, t; \boldsymbol{v}_0) = \frac{1}{(4\pi D t)^{3/2}} e^{-|\boldsymbol{v}-\boldsymbol{v}_0|^2/4Dt} . \tag{191}$$

The formula (188) is seen to follow immediately from the foregoing solution.

Returning to the discussion of the case governed by equations (176) and (179), we see at once that the idealization of the motion of the representative point in the velocity space as a true problem in random flights can no longer be valid. Moreover, according to equations (183), (186), and (187), during a given state of fluctuation the star is likely to suffer a greater amount of acceleration in the direction of $-\boldsymbol{v}$ when $(\boldsymbol{v} \cdot \boldsymbol{F})$ is negative than in the direction of $+\boldsymbol{v}$ when $(\boldsymbol{v} \cdot \boldsymbol{F})$ is positive. But the a priori probabilities for $(\boldsymbol{v} \cdot \boldsymbol{F})$ to be positive or negative are equal. Hence, when integrated over a large number of fluctuations, the star must suffer cumulatively a larger absolute amount of acceleration in the direction opposite to its direction of motion than in the direction of motion. In other words, there is a tendency for the star to be relatively decelerated in the direction of its motion; further, this tendency is proportional to $|\boldsymbol{v}|$. But these are exactly what are implied by the existence of dynamical friction.[10]

YERKES OBSERVATORY
WILLIAMS BAY, WISCONSIN
AND
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY

[8] Cf. *ibid.*, §§ 2 and 7.

[9] See, e.g., Lord Rayleigh, *Phil. Mag.*, 6th ser., **37**, 321, 1919.

[10] See two forthcoming papers by one of us (S.C.) on "Dynamical friction" in an early issue of the *Ap. J.*