ON THE CAUSE OF CEPHEID PULSATION

Sir A. S. Eddington, O.M., F.R.S.

(Received 1941 April 2)

1. *Introduction*.—The most outstanding difficulties in the pulsation theory of Cepheid variables are—

(1) The quarter-period retardation of phase between the flow of heat in the main part of the interior and the outflow from the surface.

(2) The existence of a period-luminosity relation.

We shall find reason to believe that there is a close connection between (1) and (2), and that both depend on—

(3) The existence of a layer not far below the photosphere where hydrogen is in the mid-stage of ionisation, so that it is ionised and de-ionised in the course of the pulsation.

The critical hydrogen layer is a comparatively new factor in the problem, since it could not be deemed important until the high abundance of hydrogen in the stars was recognised in 1932. It has been investigated by Unsöld and others in connection with surface phenomena on the Sun, but I think it has not hitherto been considered in connection with Cepheid pulsation. To show the general plausibility of connecting it with (1) and (2), we remark:

(*a*) It is difficult to see any possible form of explanation of the period-luminosity relation which does not make it depend on a critical ionisation. A period-luminosity relation is effectively a density-mass relation, since the period is sensitive to changes of density and the luminosity to changes of mass. It therefore implies that, for any given mass, there is only a narrow range of density in which the star is pulsatorily unstable. The *transience* of the instability—the fact that it disappears both at slightly higher and slightly lower densities—greatly limits the field in which we can seek for an explanation. Presumably we must look for some equally transient modification of the properties of stellar material. The only kind of transient modification that is known is the sharp drop in the ratio of specific heats Γ, when the predominant element or elements are at the mid-stage of an ionisation. I considered this possibility in an early investigation *; but, following the ideas of the time, I took the predominant element to be a heavy element, such as iron, undergoing the L-ionisation. This ionisation occurs in the deep interior, and in a giant star depends mainly on the central temperature. The hypothesis that pulsatory instability occurs when the heavy elements in the interior are near the middle of the L-ionisation therefore required that the Cepheids should all have nearly the same central temperature—which is far from true. The substitution of hydrogen as the predominant element reopens the discussion, since it shifts attention from the deep interior to the layer (3) near the surface, which is now the only region in which there can be a serious drop of Γ.

(*b*) The great difficulty of accounting for the phase retardation of roughly a quarter-period can only be appreciated by a close study of the equations. In a previous investigation †, I divided the star into three regions: A, the interior region with temperature above 100,000°, where the adiabatic approximation is nearly perfect; C, the exterior region with temperature below 40,000°, with negligible capacity for storing heat; B, the intermediate region. There could be no appreciable change of phase in A and C; and a detailed study of the differential equation applicable to region B also indicated no

* *Internal Constitution of the Stars*, p. 204.

† *M.N.*, 87, 539, 1927.

change of phase.* One way—apparently the only way—of avoiding this *impasse* is by some means to add heat-storage capacity to region C. Such extra heat-storage is provided by the ionisation and de-ionisation of the critical hydrogen layer which lies in region C.

(*c*) A crucial question is whether the amount of hydrogen in the critical layer is sufficient to meet the demands of our theory. It has to store (in the form of ionisation energy) the temporary excess of radiant energy accumulating during the quarter-period delay in emission. The rough calculations are fairly encouraging, showing that the order of magnitude is not far wrong. When greater precision is sought, we encounter the difficulty that the critical layer in Cepheids corresponds to temperatures chiefly between 10,000° and 20,000°. This is a region where our knowledge is especially uncertain; it is out of reach of direct observation, and the simplifications adopted at higher temperatures are inapplicable or dubious; moreover, it is convectively unstable. The somewhat conjectural calculations already existing give too little hydrogen for our purpose. But we might argue that the estimate obtained from the Cepheid theory is the better founded, and may justifiably be used to extend our knowledge of this difficult region. The need for further light on this question has led me to a study of the effects of convection, contained in an accompanying paper.

It is generally agreed that the maintaining energy of the pulsation comes from the variation with temperature and density of the rate of liberation of sub-atomic energy. Through this variation heat is added to the material when it is at a high temperature and subtracted when it is at a low temperature, so that mechanical work is produced as in a thermodynamic engine. Pulsation occurs if this maintaining energy exceeds the dissipation. We have therefore to inquire whether the transient cause of Cepheid pulsation is a sudden increase of maintaining energy (per unit amplitude) or a sudden decrease of dissipation. All suggested forms of liberation of sub-atomic energy are extremely sensitive to temperature; so that if pulsation were associated with a transient peculiarity of the maintaining energy it would almost certainly occur at an approximately fixed central temperature. For example, an increase of maintaining energy, confined to a narrow range of density, would occur if there were an especially temperature-sensitive source of sub-atomic energy which came into play when the star had contracted so as to reach an appropriate central temperature, further contraction being arrested until this particular source was exhausted.† But this would not give the existing sequence of Cepheid variables, in which the stars of short period have much higher internal temperatures than those of long period.

We conclude therefore that *Cepheid pulsation is associated with a transient decrease of dissipation.*

There is therefore a suggested triple connection between (1) the sharp decrease of dissipation which permits pulsation, (2) the low value of Γ in the critical hydrogen layer, and (3) the quarter-period retardation of phase. In §§ 2–4 two links of this connection are tightened by obtaining an expression for the dissipation which shows explicitly that it depends on the phase-retardation and on the value of Γ in the outer layers. This part of the investigation is believed to be rigorous. The third link—the dependence of the phase-retardation on the value of Γ in the critical layer—does not admit of precise treatment, and is studied in § 5 in an over-simplified model.

In earlier investigations it was expected that the 90° phase-retardation would be revealed as a *general* property of the solution of the equations of oscillation. No such

* There exist solutions of the differential equation for region A which give change of phase (discussed by M. Schwarzschild); but the difficulty is that they require boundary conditions which give a larger change of phase in region B, and this in turn requires a still larger change of phase in region C incompatible with its low capacity for heat storage. (See also § 10.)

† It seems rather likely that such sources exist; but they explain the concentration of the Cepheids near particular points on the period-luminosity curve—not the curve itself (Gamow, *Birth and Death of the Sun*, p. 150).

property was found; and the present theory takes a different form. If a star is set in oscillation by a disturbance, the phase-retardation may be anything from 0° to 180°, or more, according to the extent of the region in which the value of Γ is depressed. But there is an optimum extension of the region, with a corresponding phase-retardation, which makes the dissipation a minimum; and when, in the course of evolution of a star, the region reaches this extension, the disturbance, instead of decaying, swells into Cepheid pulsation. Thus the retardation in Cepheids is dependent on the condition for minimum dissipation. It is not likely to be exactly 90° (the mean observational value is about 70°); and it may depend to some extent on the period.

In the main the case for the present theory is that we seem able to eliminate all alternatives. There are, however, two results which give it positive support. In § 6 we find that, if the criterion for pulsatory instability is concerned with the state of ionisation of the outer layers, a very general application of the principle of homology yields a fairly correct period-luminosity relation. In § 9 we find that the theory gives an upper limit of about $0^{m}.8$ (bolometric) to the magnitude range, which is in satisfactory agreement with observation.

2. *Calculation of the Dissipation.*—Let F be the outward flow of radiation per second across a sphere of radius ξ, and let m be the mass inside the sphere. The gain of heat in an element at ξ due to the transfer of radiation is $-dF/dm$ per gm. per sec. To this must be added the heat ϵ supplied by liberation of sub-atomic energy. The gain of heat per gm. per sec. is accordingly

$$\frac{dQ}{dt}=\epsilon-\frac{dF}{dm}. \tag{2.1}$$

We fix attention on a gram of matter moving with the pulsation, so that m remains constant. We write

$$\xi=\xi_0(1+\xi_1), \qquad T=T_0(1+T_1), \qquad F=F_0(1+F_1), \qquad \epsilon=\epsilon_0(1+\epsilon_1),$$

etc., where ξ_0, T_0, . . . are the equilibrium values. Since $\epsilon_0=dF_0/dm$, (2.1) becomes

$$\frac{dQ}{dt}=\epsilon_0\epsilon_1-\frac{d(F_0F_1)}{dm}. \tag{2.2}$$

If a steady pulsation is maintained, the energy in the element is the same at the beginning and end of a cycle. Hence, during a complete cycle, the work done on the element by surrounding pressures is

$$W=-\int_C \frac{dQ}{dt}dt. \tag{2.31}$$

The entropy of the element is also the same after a complete cycle; hence by the second law of thermodynamics

$$0=\int_C \frac{1}{T}\frac{dQ}{dt}dt. \tag{2.32}$$

Since $1/T=1/T_0(1+T_1)=1/T_0-T_1/T_0$ to the first order in the amplitude, (2.32) gives

$$0=\frac{1}{T_0}\int_C \frac{dQ}{dt}dt-\frac{1}{T_0}\int_C T_1\frac{dQ}{dt}dt$$

correct to the second order in the amplitudes. Hence by (2.31)

$$W=-\int_C T_1\frac{dQ}{dt}dt \tag{2.33}$$

correct to the second order.

The advantage of using (2.33) rather than (2.31) is that in order to obtain W (which is a second order quantity) we need only calculate dQ/dt to the first order, whereas (2.31) requires a long investigation of the second order terms in dQ/dt.*

The quantities T_1, F_1, ϵ_1, etc., contain time factors $\cos(nt+\alpha)$ with various phases α. We shall denote by $(X)_Y$ the component of a periodic quantity X in the phase of the periodic quantity Y, and by $(X)_{iY}$ the component in the phase 90° greater. With this notation the time-factors are dropped, so that T_1, F_1, etc., are the semi-amplitudes. Then (2.33) gives for the average rate of doing work on a gram of matter

$$\begin{aligned}\frac{dW}{dt} &= -\tfrac{1}{2}T_1\left(\frac{dQ}{dt}\right)_T \\ &= -\tfrac{1}{2}T_1\left(\epsilon_0\epsilon_1-\frac{d(F_0F_1)}{dm}\right)_T. \end{aligned} \tag{2.4}$$

The two terms on the right represent respectively the maintaining energy (with reversed sign) and the dissipation. Hence for the whole star the maintaining energy E and the dissipation D are (per second)

$$E=\tfrac{1}{2}\int T_1\epsilon_0(\epsilon_1)_T dm, \tag{2.51}$$

$$D=\tfrac{1}{2}\int T_1\left(\frac{d(F_0F_1)}{dm}\right)_T dm. \tag{2.52}$$

Another form of (2.52) is useful. Let the *adiabatic* variation of temperature with pressure be given by

$$T \propto P^\theta \tag{2.6}$$

so that in adiabatic conditions $T_1=\theta P_1$. In general conditions, we separate T_1 into two parts, namely $T_1'=\theta P_1$ resulting directly from the change of pressure, and $T_1''=T_1-\theta P_1$ resulting from the accession of heat Q. Since T_1'' has the same phase as Q, it differs 90° in phase from dQ/dt, and therefore contributes nothing to (2.33). We may therefore replace T_1 by T_1' or θP_1 in the subsequent steps, and (2.52) then becomes

$$D=\tfrac{1}{2}\int \theta P_1\left(\frac{d(F_0F_1)}{dm}\right)_P dm. \tag{2.7}$$

3. *Approximation in the Non-adiabatic Region.*—Throughout most of the interior the adiabatic approximation is nearly perfect, and we may take $\frac{1}{20}$ of the star's radius as a generous estimate of the thickness of the region in which it is inadequate. In parts of this region T_1, F_1, ρ_1 may be altogether different from the adiabatically computed values; but we shall show that the deviations of ξ_1, P_1 are on a much smaller scale.†

Let $\Delta\rho_1$ be the deviation of ρ_1 from the adiabatically computed value. If $\Delta\rho_1$ were constant throughout the non-adiabatic region, its volume would be changed in the ratio $(1-\Delta\rho_1)$ and its thickness in nearly the same ratio. This would give a modification $\Delta\xi_1$ of ξ_1 ranging from 0 at the inner boundary of the non-adiabatic region to $-\frac{1}{20}\Delta\rho_1$ at the surface. Allowing for the actual variation of $\Delta\rho_1$ in the region, the *maximum* value of $\Delta\xi_1$ is not greater than $\frac{1}{20}$ of the *mean* value of $\Delta\rho_1$; and a homogeneous comparison of mean with mean, or maximum with maximum, would give a ratio considerably less than $\frac{1}{20}$.

* The short method was used in the author's investigations (*M.N.*, 79, 177, 1919; *I.C.S.*, § 134). Later writers seem to have overlooked this simplification, and have unnecessarily made the direct calculation of the second order terms in dQ/dt.

† The approximation which consists in neglecting $\Delta\xi_1$, ΔP_1 in comparison with $\Delta\rho_1$, ΔT_1, ΔF_1 was first used in treating the outer layers of a Cepheid in *M.N.*, 87, 542, 1927.

By the rigorous formula (*I.C.S.*, equation (131.1))

$$\frac{d(P_0P_1)}{dP_0} = -(4+n^2\xi_0/g_0)\xi_1 \tag{3.1}$$

we have

$$P_1 = -\frac{1}{P_0}\int_0^{P_0}(4+n^2\xi_0/g_0)\xi_1 dP_0. \tag{3.2}$$

The factor $(4+n^2\xi_0/g_0)$ is about 5·5. It is nearly the same in all Cepheids, and changes very little in the non-adiabatic region. Thus P_1 is approximately $-5·5\bar{\xi}_1$, where $\bar{\xi}_1$ is the mean value (averaged for equal steps of P_0) outside the point considered; and for the deviation ΔP_1 from the adiabatic value P_1

$$\Delta P_1/P_1 = \overline{\Delta\xi_1}/\bar{\xi}_1. \tag{3.3}$$

The proportionate deviation of P_1 will therefore not exceed that of ξ_1, and (owing to the averaging) it will generally be considerably less.

It is therefore a consistent approximation in the non-adiabatic region to neglect ΔP_1 and $\Delta\xi_1$ whilst retaining $\Delta\rho_1$, ΔT_1 and ΔF_1. The order of magnitude of ΔT_1 and $\Delta\rho_1$ is the same; in fact, ΔP_1 being negligible, the perfect-gas law gives $\Delta T_1 = -\Delta\rho_1$.*

We may therefore accept the values of ξ_1 and P_1 given by the usual equations of adiabatic pulsation as approximately valid up to the surface, the deviation in the non-adiabatic region being confined to density, temperature, flux of heat, and quantities derived from them. The error of P_1 can scarcely exceed 2 or 3 per cent. Since the adiabatic solution gives a constant phase for P_1 and $-\xi_1$ in all parts of the star, we shall distinguish this constant phase by the suffix $_C$. The dissipation formula (2.7) accordingly becomes

$$D = \tfrac{1}{2}\int \theta P_1\left(\frac{d(F_0F_1)}{dm}\right)_C dm \tag{3.4}$$

$$= -\tfrac{1}{2}\int \frac{d(\theta P_1)}{dm}F_0(F_1)_C dm + \tfrac{1}{2}[\theta P_1 F_0 (F_1)_C] \tag{3.5}$$

by integration by parts. Note that the integration by parts is only possible with a phase-suffix $_C$ which represents *constant* phase; for example, it cannot be applied to (2.52).

4. *The Retardation of Phase.*—Consider a pulsating star in which—

(*a*) The index θ is constant throughout.

(*b*) Maximum radiation occurs a quarter period after minimum radius.

We shall show that D is negative.

Condition (*b*) asserts that at the surface F_1 differs 90° in phase from $-\xi_1$, so that $(F_1)_C = 0$. Hence the second term in (3.5) vanishes, and

$$-D = \tfrac{1}{2}\int \frac{d(\theta P_1)}{dm}F_0(F_1)_C dm. \tag{4.1}$$

The solutions of the equation of adiabatic pulsation show that P_1 increases continually from the centre outwards. According to our approximation (§ 3) this continues to apply in the non-adiabatic region. Hence, θ being constant, $d(\theta P_1)/dm$ is positive throughout.

It is also found that $(F_1)_C$ is positive in the adiabatic region; that is to say, the flow of heat is greatest at the time of greatest contraction.† Although this result is not estab-

* In the critical hydrogen layer the relation $\Delta T_1 = -\Delta\rho_1$ is inaccurate since the change of molecular weight through ionisation is appreciable.

† See, for example, the table in *I.C.S.*, p. 197; the quantity tabulated as F_1 is $-(F_1)_C$. The table on p. 202 illustrates a reversal of sign if there is a deviation from the usual law of opacity; but the value $\gamma' = 1·355$ is now known to be much too small, and the region where the sign is reversed could not be so extensive as in the table.

lished with entire generality, being to some extent dependent on the adopted stellar model and the approximate law of opacity, the possible loopholes are, I think, not such as we should be likely to accept. In any case the region where $(F_1)_C$ might in certain circumstances have reversed sign is limited to the central part of the star, and the negative values are outweighed in the integral in (4.1) by large positive values farther out. We shall therefore disregard this contingency. In the non-adiabatic region $(F_1)_C$ falls from its positive value at the junction to its surface value zero.

Thus the elements of the integral in (4.1) are positive, and D is then negative.

The negative D means that pulsation will occur spontaneously even if no maintaining energy is provided by sub-atomic sources. If the conditions (*a*) and (*b*) hold in actual Cepheids, it follows that these stars maintain their pulsation by negative dissipation and not by sub-atomic stimulation—contrary to the general belief. Conversely, if negative dissipation is inadmissible, it follows that the conditions (*a*) and (*b*) are incompatible; in other words, the observed phase retardation is only possible because θ is not constant throughout the star.

The hypothesis of negative dissipation is not likely to be advocated except as a last resource. It is not necessarily contrary to thermodynamic principles; because the *steady* supply of energy ϵ_0 at high temperature in the interior could be converted into mechanical energy of pulsation if the constitution of the star provided a suitable "valve mechanism." * But, unless the conditions are widely different from what we suppose, there appears to be no mechanism which would regulate the flow of heat in the way required. Our result gives no new support to the hypothesis; it only shows that, in arbitrarily prescribing conditions (*a*) and (*b*), we have implicitly introduced it.

We shall assume that the hypothesis of negative dissipation can be ruled out. Our theorem then becomes a proof, by *reductio ad absurdum*, that the conditions (*a*) and (*b*) are incompatible. By (2.6), θ is connected with the adiabatic constant Γ. (If radiation pressure and variation of molecular weight are neglected, $\theta=(\Gamma-1)/\Gamma$.) Our conclusion is therefore that the quarter-period retardation of phase is impossible without a change of Γ in some part of the star. As explained in § 1, the great abundance of hydrogen makes the critical hydrogen layer the only region in which Γ can deviate seriously from its normal value $\frac{5}{3}$.

Arguing directly from (2.6), a low value of θ means that an adiabatic compression produces a relatively small increase of temperature; and therefore a given increase of temperature requires more than the normal amount of energy. In Cepheid conditions the only employment for the extra energy is in producing ionisation. The amount that can be so employed is small compared with the heat energy except in a region where the hydrogen is being ionised.

The present conclusion that phase-retardation cannot occur unless there is a region in which θ is abnormally small agrees with the investigation referred to in § 1 (*b*). The differential equations were solved for the case in which θ is constant (or a slowly varying parameter), and the phase-retardation was found to be insignificant. This negative result is now seen to be due to the fact that the condition essential for phase-retardation was left out, namely an abnormal drop of θ, or equivalently a region with abnormal capacity for storing heat. The study of the differential equation brings out the further point that, in order to be effective, the drop of θ must occur in the non-adiabatic region. It is therefore due to an ionisation which takes place at a temperature below 100,000°. This is a further proof, if any were needed, that the ionisation to be considered is that of hydrogen.

To show how negative dissipation can be avoided by admitting a low value of θ in the non-adiabatic region, we return to (3.4) and calculate separately the dissipations D_a, D_n in the adiabatic and non-adiabatic regions. Values at the junction of the two

* The "maintaining energy" of the pulsation, which depends on the *temperature-sensitiveness* of ϵ, must not be confused with the energy which maintains the star's heat, *i.e.* ϵ itself.

regions will be indicated by an accent, and surface values by an asterisk. We consider a star with quarter-period phase-retardation (so that $(F_1^*)_C = 0$), and with different constant values θ_a, θ_n of θ in the two regions.

In the adiabatic region $P_1 < P_1'$, and $d(F_0F_1)/dm$ is positive. Hence by (3.4)

$$0 < D_a < \tfrac{1}{2}\theta_a P_1' F_0' (F_1')_C. \tag{4.2}$$

It is easily seen that D_a is *substantially* less than the upper limit given.

Since there is little change of P_1 in the shallow non-adiabatic layer, we have approximately

$$D_n = \tfrac{1}{2}\theta_n P_1' F_0' \{(F_1^*)_C - (F_1')_C\} \tag{4.3}$$

$$= -\tfrac{1}{2}\theta_n P_1' F_0' (F_1')_C \tag{4.4}$$

since $(F_1^*)_C = 0$. By (4.2) and (4.4), $D_a + D_n$ will be negative unless θ_n is substantially less than θ_a.

This concludes the first stage of our investigation. It seems an almost inescapable conclusion that the critical hydrogen layer is the key factor on which the phase-retardation, the pulsatory instability and the period-luminosity relation depend. I therefore approach the remaining problems with a rather strong conviction that no other solution is possible, and that there must be a way out of the various difficulties which may appear when the details of the solution are considered.

5. *Phase-Retardation in the Critical Layer.*—If F_1 travels through the non-adiabatic region without loss of amplitude, we have

$$(F_1^*)_C = F_1' \cos\phi,$$

where ϕ is the phase-retardation. Hence by (4.3)

$$-D_n = \tfrac{1}{2} P_1' F_0' F_1' \theta_n (1 - \cos\phi). \tag{5.1}$$

Thus the condition for minimum dissipation is

$$\theta_n(1 - \cos\phi) \text{ is a maximum,} \tag{5.2}$$

and according to the foregoing theory pulsation will occur only in the neighbourhood of this maximum. We could therefore determine the phase-retardation ϕ in Cepheids from (5.2) if we knew the law of dependence of ϕ on θ_n. This dependence is the third leg of our three-cornered relation. We can scarcely hope to obtain a definite formula, but we shall discuss the main features of the problem in a simplified model.

We suppose the star in adiabatic pulsation to be surrounded by a non-adiabatic "blanket" in which θ is very small. By § 3, P_1 in the blanket is not abnormal, and therefore the temperature oscillation θP_1, produced directly by the expansion and contraction, is very small. The main temperature oscillation in the blanket is produced by the difference between the influx F_0F_1' and efflux $F_0F_1^*$ of radiation. Let s be the specific heat, and set

$$z = -\int s T_0 \, dm. \tag{5.31}$$

Then in the blanket

$$-d(F_0F_1) = s\,dm \, . \frac{d}{dt}(T_0T_1),$$

so that, if T_1 varies as e^{int},

$$\frac{dF_1}{dz} = \frac{1}{F_0}\frac{dT_1}{dt} = \frac{in}{F_0}T_1. \tag{5.32}$$

Since F depends on the gradient of T, we might as a rough approximation set F_1 proportional to dT_1/dz. This would give an equation of the form $d^2T_1/dz^2 = i\alpha T_1$, familiar in the theory of conduction of heat—as, for example, in the propagation downwards into the Earth of the annual temperature variation at the Earth's surface. The solution gives a temperature wave propagated with change of phase but with a very rapid decay of amplitude, the decrement being $e^{-\frac{1}{2}\pi}(=1/4{\cdot}8)$ for a phase-retardation of 90°. It is therefore important—

(1) To show definitely, by comparison of theory and observation, that there is no great difference of amplitude of F_1' and F_1^*.

(2) To show that there is a material difference between the "radiative blanket" and the "conductive blanket", which makes change of phase without decay of amplitude possible in the former.

The value of F_1', obtained by inserting the close approximation $T_1/\theta_a = P_1 = -(4+\sigma)\xi_1$ in the rigorous formula for the adiabatic region*, is

$$F_1' = \left\{4 - \frac{1}{\theta_a}\frac{8+\sigma}{4+\sigma} + \lambda\right\} T_1', \tag{5·41}$$

where $\sigma = n^2\xi_0/g_0$, and has nearly the same value 1·5 in all Cepheids on account of the approximate constancy of $\Pi\sqrt{\rho}$; λ is the index in the opacity law $k \propto P/T^\lambda$, and may be taken to be 4·5. Setting $\theta_a = 0{\cdot}4$, corresponding to $\Gamma_a = \frac{5}{3}$, (5·41) gives

$$F_1' = 4{\cdot}18 T_1' = -9{\cdot}2\xi_1'. \tag{5·42}$$

For ξ_1' we may substitute the observed surface value ξ_1^*; this has a fairly constant mean value 0·055 independent of the period. Then $F_1' = 0{\cdot}506$, corresponding to a range of about $1^{\mathrm{m}}{\cdot}1$ bolometric. The mean observed range is about $0^{\mathrm{m}}{\cdot}8$. Thus the loss of amplitude between F_1' and F_1^*, though appreciable, is not very large.

We next examine the equations in the non-adiabatic layer to see why the loss of amplitude is so small. From the equation of radiative equilibrium

$$H = \frac{-ac}{3k\rho}\frac{dT^4}{d\xi} = -\frac{ac\xi^2}{3k\rho_0\xi_0^2}\frac{dT^4}{d\xi_0}$$

we obtain by logarithmic differentiation

$$\begin{aligned} H_1 &= 2\xi_1 - k_1 + 4\frac{d(T_0^4 T_1)}{dT_0^4} \\ &= 2\xi_1 - (P_1 - 4{\cdot}5T_1) + 4T_1 + dT_1/d(\log T_0). \end{aligned} \tag{5·51}$$

Then, using $P_1 = -(4+\sigma)\xi_1 = -5{\cdot}5\xi_1$,

$$F_1 = H_1 + 2\xi_1 = 9{\cdot}5\xi_1 + 8{\cdot}5T_1 + dT_1/d(\log T_0). \tag{5·52}$$

This value is to be inserted in (5.32). Although the ξ_1 term in (5.52) is large, it is not difficult to show that its gradient is small, and it contributes little to dF_1/dz. The important feature is the large term $8{\cdot}5T_1$ which has no counterpart in the conduction problem, where F depends on the gradient of T and not at all on T itself. This is the material difference referred to in (2). If we assume that this term provides the main part of dF_1/dz, (5.32) gives

$$\frac{dT_1}{dz} = \frac{in}{8{\cdot}5F_0}T_1,$$

so that

$$T_1 \propto e^{inz/8{\cdot}5F_0}$$

and the phase of T_1, but not the amplitude, changes with z. The omitted term $dT_1/d(\log T_0)$ will occasion some change of amplitude, but it is not likely to be large.

There is no definite limit to the amount of phase-retardation given by (5.32) and (5.52), and ϕ may even exceed 360° if the region of low θ is sufficiently extensive. The

* *M.N.*, 87, 543, equation (5.25).

fact that the observed values of ϕ are near 90° depends on a quite different consideration, namely the condition for minimum dissipation (5.2). In (5.2) θ_n has been assumed uniform over the whole non-adiabatic region. It must therefore be regarded as a compromise quantity depending on the extent as well as the actual θ of the critical layer. After ϕ reaches 180° both factors of $\theta_n(1-\cos\phi)$ diminish; the maximum therefore occurs between $\phi=0°$ and $\phi=180°$. In a general way we should expect the maximum to be fairly near $\phi=90°$; but it would require a much more elaborate investigation to establish this definitely.

For our numerical calculations we shall take the amplitudes of F_1' and F_1^* equal and $\phi=90°$. Then the semi-amplitude of $F_1'-F_1^*$ is $F_1^*\sqrt{2}$; and the difference between maximum and minimum heat-content of the blanket is $F_0F_1^*\sqrt{2}\Pi/\pi$, where Π is the period. The corresponding quantity per sq. cm. of surface is

$$\frac{\sqrt{2}}{\pi}H_0F_1^*\Pi. \tag{5.6}$$

The blanket has to store this as ionisation energy, its heat capacity being otherwise small. It must therefore contain the right amount of hydrogen per sq. cm. of surface to take up the energy (5.6). In § 7 we shall examine whether this condition can be fulfilled.

The low value of θ makes the critical layer convectively unstable, and it may be suggested that the convection currents provide an alternative mode of energy storage, their speed increasing and decreasing in the course of the pulsation. But since the convective instability is a result of the ionisation, it is scarcely likely that the secondary energy changes of the currents can be as great as the primary energy changes of the ionisation. In any case upper limits to the possible speed of the convection currents can be found, which make it clear that their energy storage is unimportant.

6. *The Period-Luminosity Relation.*—We shall try to determine the form of the period-luminosity relation by the principle of homology; that is to say, we assume that the Cepheids form a series of stars whose outer layers are homologous with respect to those characteristics on which (according to the foregoing theory) pulsatory instability depends. We employ as variates the (steady) radiation flux H and surface-gravity g; a relation between these is equivalent to a mass-density relation, and hence to a period-luminosity relation.

By (5.6) the quantity of hydrogen needed in the critical layer is proportional to $H\Pi$, and its weight to $H\Pi g$. We have therefore to secure that in different Cepheids the pressure P_x at points with the same degree of hydrogen ionisation x is proportional to $H\Pi g$. In pulsation the equilibrium value x will be replaced by an oscillation between limits x_1 and x_2. We are assuming that these limits depend only on the amplitude F_1 or T_1 of the pulsation, and not on the temperature and pressure at the level x which will be different in different stars; it is found that this assumption is approximately true.

Taking the approximate mass-luminosity law $L \propto M^3$, and the theoretical period-density law $\Pi \propto \rho^{-\frac{1}{2}}$, we easily find that $H\Pi g \propto H^{\frac{9}{8}}g^{\frac{1}{8}}$, so that the homology condition is

$$P_x \propto H^{\frac{9}{8}}g^{\frac{1}{8}}. \tag{6.1}$$

Assuming radiative equilibrium with k varying as $\rho/T^{\frac{7}{2}}$, we have (at not too small optical depth) $P \propto H^{-\frac{1}{2}}g^{\frac{1}{2}}T^{4\cdot25}$, so that

$$P_x \propto H^{-\frac{1}{2}}g^{\frac{1}{2}}T_x^{4\cdot25}. \tag{6.2}$$

By the ionisation formula P_x varies very rapidly with T_x, *e.g.* as the 15th power at 12,000°, and as the 10th power at 20,000°. We are concerned with temperatures between these limits (§ 7), and may conveniently adopt

$$P_x \propto T_x^{12\cdot75}. \tag{6.3}$$

Eliminating P_x and T_x between the relations (6.1), (6.2) and (6.3), we obtain

$$H \propto g^{\frac{1}{3}}. \tag{6.4}$$

For comparison we give the corresponding result when k is simply a constant, as when the absorption is due entirely to free electron scattering. Then (6.2) is replaced by

$$P_x \propto H^{-1} g T_x^{\,4}. \tag{6.2a}$$

Combined with (6.1) and (6.3), this gives

$$H \propto g^{0\cdot 52}. \tag{6.4a}$$

From calculations made in an earlier paper * for different points on the observed period-luminosity and period-spectrum curves, the empirical relation between H and g can be obtained. For the range of period from 1·4 to 39 days, I find $H \propto g^{0\cdot 45}$, which may be considered to be in reasonably good agreement with either (6.4) or (6.4a).

7. *Energy Storage in the Critical Layer.*—The ionisation energy of a hydrogen atom is equal to the mean translatory energy of a particle at 105,000°. In typical conditions two-thirds of the ionisation occurs in a temperature range of 20 per cent., say from 10,000° to 12,000°. Thus the specific heat of hydrogen at the critical stage of ionisation may well be 30 or 40 times its ordinary value.

The ionisation formula for hydrogen, with numerical values inserted, is conveniently written

$$\log \frac{P}{1+f} = 11\cdot 602 - \frac{68000}{T} - \frac{5}{2} \log \frac{68000}{T} + \log \frac{1-x}{x}, \tag{7.1}$$

where f is the proportion of ions (of all elements) to electrons, so that $P/(1+f)$ is the electron pressure P_e. The logarithms are to the base 10.

We define (conventionally) the inner and outer boundaries of the critical layer to be the depths at which $x = \frac{5}{6}, \frac{1}{6}$ in the undisturbed star, and denote the corresponding pressures and temperatures by P_i, P_o, T_i, T_o. The mass of material per sq. cm. in the critical layer is then $(P_i - P_o)/g$. On any reasonable estimate of the law of connection of P and T in this part of the star, P_i is several times greater than P_o; and the mass may with sufficient approximation be taken to be P_i/g. If X is the abundance of hydrogen, the mass of hydrogen is XP_i/g.

We consider an oscillation of amplitude $T_1 = 0\cdot 1$, so that $F_1 = H_1 = 0\cdot 4$.† The 20 per cent. range of temperature ionises and de-ionises $\frac{2}{3}$ of the hydrogen at the middle of the critical layer. We therefore equate the ionisation energy of $\frac{2}{3}XP_i/g$ grams of hydrogen to the energy storage given by (5.6).

TABLE I

	A	B
Mass	5	18 × ☉
Period	1·42	38·8 days
Mean Eff. Temp.	6500	4030°
g	1280	18·6 cm. sec.$^{-2}$
$H_0 H_1$	$4\cdot 05 \,.\, 10^{10}$	$6\cdot 04 \,.\, 10^{9}$ erg cm.$^{-2}$ sec.$^{-1}$
Energy stored	$2\cdot 23 \,.\, 10^{15}$	$9\cdot 13 \,.\, 10^{15}$ erg cm.$^{-2}$
Mass of hydrogen	257	1050 gm. cm.$^{-2}$
Weight of hydrogen	$3\cdot 30 \,.\, 10^{5}$	$1\cdot 95 \,.\, 10^{4}$ dyne cm.$^{-2}$
P_i	$6\cdot 6 \,.\, 10^{5}$	$3\cdot 9 \,.\, 10^{4}$ dyne cm.$^{-2}$
T_i	17,700	14,050°
$\bar{k}_i$	0·142	0·093

* *M.N.*, **92**, 480, Table III (1932).

† The approximation $F_1 = H_1 = 4T_1$ is rough, but is adequate for our purpose. We can justify $F_1 = 4T_1$ at the inner boundary of the layer by (5.42). Strictly $F_1 = H_1 + 2\xi_1$; but, since H_1 and ξ_1 differ 90° in phase, the amplitudes of F_1 and H_1 are not much different.

The above results (Table I) are found for a short-period and long-period Cepheid for which the principal data were computed in an earlier paper.*

In the following explanation the numbers refer to star A. The energy stored, $2{\cdot}23 \cdot 10^{15}$ ergs per sq. cm. of surface, is calculated by (5.6). This is equal to $\frac{2}{3}$ of the ionisation energy of 257 gm. of hydrogen. Assuming a hydrogen abundance of 50 per cent., the corresponding mass of material is 514 gm., and the pressure P_i is taken to be the weight of this (P_0 being relatively small). The electron pressure has been taken to be $\frac{1}{2}P_i$, corresponding to one ion per electron, since $\frac{5}{6}$ of the hydrogen is ionised at the level i. Then T_i is determined from (7.1) with $x=\frac{5}{6}$. The meaning of $\bar{k}_i$ will be explained in § 8.

If the existing equilibrium theory of the outer layers were adequate to determine the pressure-temperature relation at this depth in the star, we should know independently the pressure at the depth corresponding to the temperature 17,700°, and could test whether the above calculated value $6{\cdot}6 \cdot 10^5$ agreed. In particular, if the equilibrium theory showed conclusively that the pressure at that temperature was substantially less than $6{\cdot}6 \cdot 10^5$, it would mean that there is not enough hydrogen at a critical stage of ionisation to perform the task allotted to it in our theory.

The most suitable material for comparison is Unsöld's computation for a model atmosphere on the Sun.† This gives a pressure $5{\cdot}10^3$ in the Sun's photosphere; for the star A it would be rather less. To agree with our result P must vary as T^6, or thereabouts, between the photosphere and the level i. At first sight this variation does not seem extravagant; for example, in the standard model of the deep interior $P \propto T^4$. But there are circumstances which make the radiative gradient $d(\log P)/d(\log T)$ abnormally low in the region below the photosphere; and, in fact, Unsöld's table gives $P=4{\cdot}10^4$ at our temperature T_i. Since the result for star A would be lower, this is a wide discrepancy.

The assumption of adiabatic equilibrium would be more favourable, since it gives a steeper ascent of P. It appears that between 7800° and 14,000° the adiabatic gradient $d(\log P)/d(\log T)$ is greater than 5 and has a maximum of nearly 10.‡ This would suit very well our values of P_i and T_i. But adiabatic equilibrium implies that most of the outflowing heat is transported across the critical layer by convection, and involves velocities of the currents which are quite out of the question.

The large discrepancy has led me to re-examine the conditions in a convective layer. The results of the discussion are given in an accompanying paper, and will be applied in the next section to the problem here raised.

8. *Convection in the Critical Layer.*—The difficulty encountered in § 7 can be put most forcibly in the following way. Assuming radiative equilibrium, the optical depth τ_i of the level i is given by

$$T_i^4=\tfrac{1}{2}T_e^4(1+\tfrac{3}{2}\tau_i).$$

With $T_i=17{,}700°$, $T_e=6500°$, we obtain $\tau_i=72{\cdot}7$. Since the mass of material to this depth is 514 gm. cm.$^{-2}$, the mean opacity $\bar{k}_i$ is $72{\cdot}7/514=0{\cdot}142$. For the star B we find similarly $\bar{k}_i=0{\cdot}093$.

These values of $\bar{k}_i$ are scarcely more than would result from free-electron scattering alone. The scattering coefficient for material containing one free electron for every two units of atomic weight is 0·2; so that 0·1 would correspond to a column which is half hydrogen on the average half ionised. To obtain our value of P_i it would be necessary to assume the absence of all other contributions to the opacity.

In the accompanying paper § I have found that the convection leads to a complete break-down of ionisation equilibrium, and have traced some of the consequences of this.

* *M.N.*, **92**, 480, 1932. These are not actual stars, but mean objects which precisely obey Shapley's period-luminosity and period-spectrum curves.

† *Physik der Sternatmosphären*, p. 144, Table XXXVI.

‡ *Ibid.*, p. 382, fig. 125.

§ Quoted as "I", *M.N.*, **101**, 177, 1941.

One result is a substantial reduction of opacity in the convective region (I, § 4). The reduction is insufficient; but it helps to close the gap between the Cepheid calculation and Unsöld's calculation of P_i. There remains a somewhat reduced discrepancy which, I think, is accounted for as follows:—

In Table I we found the value of T_i corresponding to P_i by the equilibrium ionisation formula (7.1). The result of convection is that this formula no longer applies. The neutral hydrogen convected by the downward current reaches the level 17,700° too quickly to be ionised—the proportion ionised being perhaps no more than $\frac{1}{15}$ (I, § 4). The limit $x_i = \frac{5}{6}$ is therefore depressed to a much higher temperature level. There seems to be no great difficulty in supposing it to fall near 30,000°.

I think therefore that, partly by the reduced opacity and partly by the deeper extension of the critical layer, an adequate amount of hydrogen is obtained. At any rate the close consideration of the effects of convection reveals such considerable modifications that we need not condemn the present theory of the cause of Cepheid pulsation on account of disagreement with the crude equilibrium calculations.

In § 5 we assumed θ to be very small in the blanketing layer; if it is not small the efficiency of the blanket is much reduced. But we have now introduced a flattening of the ionisation gradient by convection, which would increase θ perhaps to the value $\frac{1}{4}$, corresponding to $\Gamma = \frac{4}{3}$. It may be objected that we cannot have it both ways. But, as a matter of fact, we can have it both ways. Γ, being a function of the speed of change, is not the same constant in the two investigations; and θ may well be $\frac{1}{4}$ for the purpose of convection and $\frac{1}{12}$ for the purpose of pulsation. The pulsation is a relatively slow variation which allows time for the ionisation to adjust itself to the altered temperature and pressure. The combination of convection with pulsation is a somewhat complicated problem; but in principle each stage of the pulsation can be treated as quasi-static, the speed and extent of the convection currents and the consequent distribution of pressure, temperature and non-equilibrium ionisation being calculated for successive stages independently.

9. *Limit to the Amplitude.*—The amplitude $T_1 = 0{\cdot}1$ was chosen in § 7 because a temperature range of 20 per cent. covers most of the hydrogen ionisation; for example, two-thirds of the ionisation occurs between 10,900° and 13,400° at pressure 10^4, and between 12,600° and 15,900° at pressure 10^5. If we take a smaller amplitude, the required mass of hydrogen is not much altered; there is less radiation to be stored, and the range of x is smaller in much the same proportion. But if we take a larger amplitude, there is more radiation to be stored, and the range of x which is already near its limit is not proportionately increased. Thus a layer, which is just thick enough to blanket oscillations of any amplitude up to $T_1 = 0{\cdot}1$, is not thick enough to blanket stronger oscillations; and beyond $T_1 = 0{\cdot}1$ the dissipation increases rapidly. Accordingly the pulsation, due to the exceptionally low dissipation in a star which has reached the Cepheid stage, will not increase much beyond this limit.

There is thus a natural limit to the pulsation at about $T_1 = 0{\cdot}1$, or $H_1 = 0{\cdot}4$. This gives a range of $0^{\mathrm{m}}{\cdot}8$ bolometric.

It has been pointed out that another natural limit is imposed by the condition that the pressure must not become negative in the course of the pulsation (*I.C.S.*, § 131); but the actual pulsations fall short of this limit. It is forestalled by the new limit, which agrees reasonably well with observation.

10. *Comparison with other Investigations.*—The retardation of phase has recently been treated by M. Schwarzschild.* I think his work is not in conflict with the present conclusions. He shows that the change of phase cannot occur without a modification of hitherto accepted boundary conditions, but does not suggest any particular cause for the modification. Our investigation indicates the low value of θ in the critical layer as the cause. To make a nearer comparison we must distinguish between (*a*) the change of

* *Zeits. f. Astrophysik*, 15, 14, 1938.

phase which occurs in the modified region, and (*b*) the change of phase imposed on the interior solution by the modification of its boundary conditions. Schwarzschild gives a specimen solution which takes it for granted that (*b*) is the main change. I think the dissipation formula (4.3) is decisive against this; unless a considerable part of the change of phase is actually within the region where θ is small, we obtain negative dissipation for the whole star. Doubtless the critical layer disturbs the conditions for a short distance interior to it; but mathematical analysis indicates that such a disturbance dies out rapidly as we go inwards, and it could scarcely reach the adiabatic region. I conclude that the effect (*b*) is negligible compared with (*a*).

All former calculations of the dissipation have been restricted to D_a, and it might seem that these require revision in view of the present results relating to D_n. But the purpose of the calculations has usually been to examine the stability of ordinary stars rather than the instability of Cepheids. The large negative D_n is a transient condition, of which the phase-retardation is a symptom; as soon as we get away from the Cepheids, $-D_n$ is small. Thus the former calculations are valid for the purpose for which they are intended. In regard to the calculations of D_a, I agree with Cowling * that my own and other early calculations underestimated the dissipation, owing to the integrations being carried not far enough towards the outside of the star, where most of the dissipation (positive and negative) occurs.

Summary

A study of the dissipation of the energy of pulsation leads to the conclusion that the quarter-period retardation of the outflow of heat is necessarily associated with an abnormally low adiabatic constant in the non-adiabatic part of the star. This points unmistakably to the critical layer, where hydrogen is in the mid-stage of ionisation, as responsible for the phase-retardation. It is verified that phase change can occur in this region without the rapid decay of amplitude which might have been expected from analogy with conduction problems. For stars in general no fixed amount of phase change is indicated; but in Cepheids it will correspond to an optimum extension of the critical layer which makes the dissipation a minimum, the sharply reduced dissipation being the cause of the pulsatory instability. A certain amount of confirmatory evidence is found. The theory requires an amount of hydrogen in the critical layer which at first sight seems impossibly large; but it is believed that the discrepancy is removed by the study of the effects of convection contained in an accompanying paper.

* *M.N.*, 94, 768, 1934.