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Theory of Motion of Jupiter's Galilean Satellites

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Summary. The final results for the theory enabling one to calculate the positions of the Galilean satellites and their partial derivatives are presented, following the techniques outlined in earlier papers. Extensive use of algebraic manipulation software on a digital computer is employed to generate the final expressions. The new theory is, in effect, a revitalization of Sampson's theory in which we (a) remove algebraic and mathematical errors existing in Sampson's work, (b) introduce some neglected effects due to solar interactions and the 3–7 commensurability, (c) allow for non-zero amplitude and phase of the free libration, (d) express the final results as analytic functions of variations in 49 arbitrary constants of integration and physical parameters, (e) construct the theory in a manner which readily allows for future revision, and (f) provide analytic expressions for the partial derivatives with respect to the 49 parameters.

Key words: Galilean satellites — celestial mechanics — computer manipulation Sampson theory — Jupiter

Introduction

In his monumental work, "Theory of the Four Great Satellites of Jupiter", (hereafter referred to as *Theory*), Sampson (1921) developed trigonometric series with numerical coefficients as expressions for the coordinates of Jupiter's Galilean satellites. His theory remains the most widely used basis for calculating the coordinates of the Galilean satellites today and is a tribute to the great effort expended by Sampson in the early part of this century to develop a theory without benefit of one of today's modern electronic digital computers. Although Sampson did not succeed in attaining his goal of representing the positions to one arc s (Jovicentric) in the coordinates (2 km, 3.3 km, 5.2 km and 9.1 km, respectively, for Satellites I through IV), he did establish a technique which, when combined with the use of a modern digital computer, can enable one to develop highly accurate expressions for the satellite coordinates and to obtain analytic expressions for their partial derivatives as well.

In earlier papers by the present author (1973, *Paper I*; 1974, *Paper II*; 1975, *Paper III*), the general techniques of Sampson were outlined with a view towards revitalizing Sampson's methods by employing a digital computer to perform the millions of routine arithmetic and algebraic operations required in re-developing the theory of the Galilean satellites. The techniques also provide a method for obtaining partial derivatives by means of which one could readily revise the arbitrary constants and physical parameters contained in the theory. The present paper contains the results of the redevelopment and presents final expressions for the coordinates and partial derivatives for the four Galilean satellites.

Because of the manner in which Sampson developed his theory (influenced largely, I suspect, by the lack of a modern-day digital computer available to him), it is not possible to obtain partial derivatives and to alter Sampson's initial numerical values. Hence, modern workers (Peters, 1973; Ferraz-Mello, 1975; Arlot, 1975; Duxbury et al., 1975; Aksnes and Franklin, 1975) are largely constrained to estimate only a "time correction" or longitude offset—rather than estimate a full set of orbital parameters. Such an unrealistic constraint may be the cause of the varying and often inconsistent results obtained by scientists currently working with the Galilean satellites.

In addition to the lack of having a convenient technique for updating and improving Sampson's theory, there are other areas in which improvements are necessary in order to extract the ultimate precision from any Galilean satellite ephemeris. As noted by Vu and Sagnier (1974) and by Lieske (Papers I–III), there are arithmetic errors and neglected physical effects which must be corrected and introduced if one is to improve the mathematical precision of the theory. The present theory represents an effort to remove the arithmetic and algebraic errors in Sampson's development, to introduce the effects of the 3–7 commensurability for Satellites III and IV, (de Haerdtl, 1892; Lieske, 1973), to improve the solar

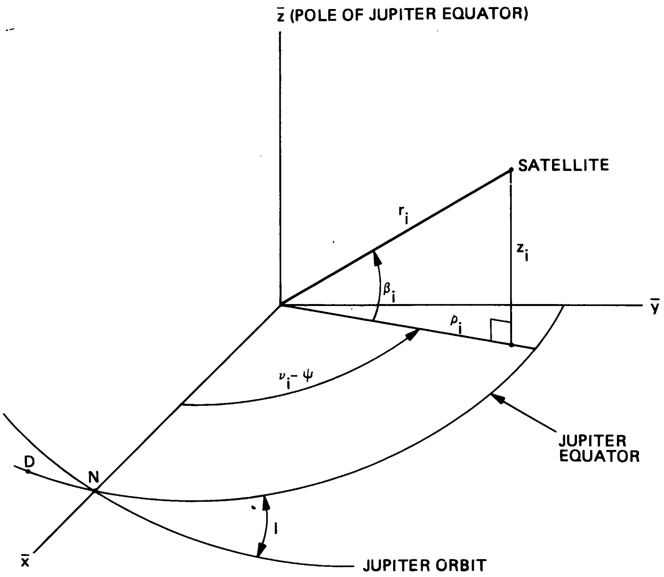

Fig. 1. Satellite geometry in plane of Jupiter's equator ($\bar{x}\bar{y}$), $\bar{x}$ axis pointing to node of Jupiter's equator on its orbit and $\bar{z}$ axis pointing to Jupiter's pole of rotation

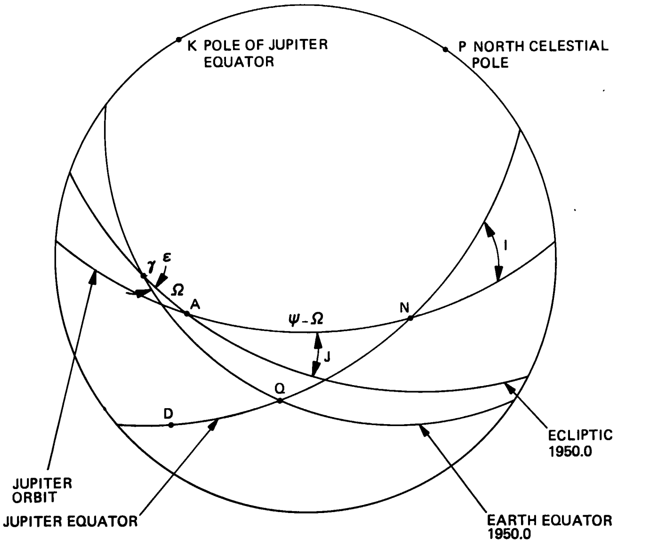

Fig. 2. Global geometry relating Jupiter's equator to fixed equinox and ecliptic of 1950.0. K is the pole of Jupiter, P is the north celestial pole and N corresponds to the direction of the $\bar{x}$ axis of Fig. 1

perturbations (Innes, 1910; Lieske, 1974), to introduce the amplitude and phase of the Laplacian free libration as arbitrary constants, and to provide partial derivatives and means of updating the revised theory.

As outlined in the earlier papers, a set of computer programs has been developed, capable of manipulating Poisson series with up to 73 polynomial parameters and 28 trigonometric variables. The method of stacking the series and performing operations on them follows the techniques developed by Broucke and Garthwaite (1969)—the only modifications being due to the greater number of variables capable of being manipulated in the current work.

The computer software is capable of manipulating Poisson series of the form

$$F(p,\theta)=\sum C_{i;j}p_1^{i_1}\dots p_{73}^{i_{73}}\,{}^{\sin}_{\cos}(j_1\theta_1+\dots+j_{28}\theta_{28}), \tag{1}$$

where $p_k(k=1,73)$ represents the 73 different polynomial parameters and their powers $i_k(0\leqq i_k\leqq 3)$ which are handled as literal algebraic quantities and where $\theta_k(k=1,28)$ represents the 28 trigonometric variables and their coefficients $j_k(-15\leqq j_k\leqq 15)$.

As developed in Papers I and II, the polynomial parameters ε represent relative variations of Sampson's numerical values p_0 of the constants of integration and physical parameters $p=p_0(1+\varepsilon)$. By employing relative variations of the parameters p, as given by the algebraic equations ε, one can considerably shorten the resultant series while still retaining the features of a literal expansion (Lieske, 1973, 1974; Jefferys and Ries, 1975). It is this feature which makes the problem of the Galilean satellites tractable on a digital computer.

In general, Sampson's final theory is employed as the initial approximation in the current work. Each term in Sampson's theory is multiplied by an appropriate algebraic quantity ε_k and, as outlined in Paper III, the preliminary solution is obtained as a function of approximately 125 parameters, of which only 49 are arbitrary. The 76 auxiliary parameters are eliminated and wherever an auxiliary parameter occurs, its series in terms of the arbitrary parameters is substituted. In such a manner, the final solution is obtained as a function of only the 49 arbitrary parameters listed in Tables 2 and 3.

Necessary equations from the earlier papers which are repeated in the present paper for clarity are denoted by roman numerals (indicating the paper) followed by the equation number. For example, Equation (II-7) refers to Equation (7) of Paper II. The reader is referred to the earlier papers and to Sampson's work for a more thorough discussion of the equations where necessary. In general, Paper I introduced the neglected 3–7 commensurability between Satellites III and IV and outlined the ε-parameter approach employed throughout this work. In Paper II, the differential equations were developed in terms of the required series multiplications. That paper developed the approach for solving the latitude equations and much of the material required for longitude and radius. In Paper III, the preliminary latitude solution was presented and the second-degree perturbations developed for longitude and radius. In addition, Paper III presented the means of evaluating the integrals $\{\Omega_{ij}\}$ which resulted from employing the energy integral and which reduced the differential equations to order 5 for each satellite.

In the present paper, the equations will be summarized and the technique outlined for handling the Laplace libration of longitudes for the inner three satellites. The method of obtaining the solution will then be outlined and the final series will be presented in tabular form, as well as on microfiche.[1] Finally, the means of obtaining partial derivatives is given as well as some results derived by adjusting the new theory to Sampson's theory and to a numerical integration. The microfiche which appear as part of this paper contain the complete theory in literal form as well as listings of the FORTRAN computer programs which evaluate the theory and calculate partial derivatives for user-defined values of the arbitrary constants. Hard-copy prints of the contents of the microfiche are available from the author upon request.

Equations of Motion

As developed in Paper II, Sampson's method employs a cylindrical coordinate system as depicted in Figure 1, where the $\bar{x}\bar{y}$ plane is that of Jupiter's equator. The orientation of the $\bar{x}\bar{y}\bar{z}$ system (which is not inertial because of the motion of Jupiter's equator) relative to the earth equator and equinox of 1950.0 is depicted in Figure 2. The basic coordinates for satellite i are the equatorial projection (ϱ_i) of the radial coordinate, the longitude in the plane of Jupiter's equator ($v_i - \psi$) and the height z_i. Ultimately, Sampson employs the variables ξ_i, υ_i and ζ_i (or s_i) where

$$\varrho_i = a_i(1+\xi_i)$$
$$\upsilon_i = v_i - \ell_i$$
$$\zeta_i = z_i/a_i$$

or

$$s_i = z_i/\varrho_i \tag{2}$$

with a_i being a constant and ℓ_i the mean longitude. In his developments, Sampson employs ζ_i, but for evaluation purposes one uses s_i and the "time-completed" feature [see Equation (38)].

As developed in Paper II, the full differential equations of Equation (II-27) are heuristically represented by Equation (II-7) as

$$0 = \xi'' + \xi[1 + 2\varkappa - \gamma + \delta_2 + 3\xi^2 + 2\zeta^2 - \xi'^2 - \zeta'^2 + 2\Sigma\gamma_r \cos rt]$$
$$-\varkappa + \gamma + \delta_1 - 2\xi^2 - \tfrac{1}{2}\zeta^2 + \xi'^2 + \zeta'^2 - \Sigma E_{p'} \cos p't \tag{3}$$
$$0 = -2\upsilon' + \xi[-4 - 6\varkappa + 2\gamma + \delta_4 - 16\xi^2 + 5\zeta^2 + 2\xi'^2 + 2\zeta'^2]$$
$$+ 2\varkappa - \gamma + \delta_3 + 9\xi^2 - \zeta^2 - \xi'^2 - \upsilon'^2 - \zeta'^2 + 2\Sigma B_{l'} \cos l't$$
$$0 = \zeta'' + \zeta[1 + \varkappa + \delta_0 - 3\xi + 6\xi^2 - \tfrac{3}{2}\zeta^2 + 2\Sigma K_1 \cos kt]$$
$$-\Sigma C_{q'} \sin q't, \tag{II-7}$$

where $'$ and $''$ denote $n_i^{-1} d/dt$ and $n_i^{-2} d^2/dt^2$ respectively (the subscripts i for each satellite have been deleted for ease in visualizing the equation), n_i being the mean motion. The fifth-order system of differential equations is augmented by an equation (which is due to employing a first integral)

$$\left\{\frac{\partial}{\partial\tau} {*\Omega_{i,j+s}}\right\}_{\tau=t} \equiv \left[\frac{d\varrho_i}{dt}\frac{\partial}{\partial\varrho_i} + \frac{dv_i}{dt}\frac{\partial}{\partial v_i} + \frac{dz_i}{dt}\frac{\partial}{\partial z_i}\right]\Omega_{i,j+s} \tag{4}$$

$$\{\Omega_{i,j+s}\} = \int \left\{\frac{\partial}{\partial\tau} {*\Omega_{i,j+s}}\right\}_{\tau=t} dt, \tag{II-5), (II-6}$$

where Ω represents the perturbing function [given in Eq. (II-17) to (II-29)]. Equations (3) and (4) represent the sixth-order set of differential equations which must be solved for each satellite. The reader is referred to the preceding papers in this series for a more complete discussion of the variables. As developed by Sampson (Theory, p. 22ff., pp. 5–7), ξ_i, υ_i and ζ_i are purely periodic, with a_i being the constant part of ϱ_i. The constants $\varkappa$ and γ appearing in Equation (3) represent constraints between n, a and the constant of integration C which occurs in the energy integral, so that

$$(1+m_i)/n_i^2 a_i^3 = 1 + \varkappa_i \tag{5}$$

$$2a_i C_i / n_i^2 a_i^3 = 1 + \gamma_i, \tag{II-8}$$

where m_i is the mass of the i^{th} satellite. The auxiliary constants $\delta_j (j = 0, 4)$ contain the constant portions (other than $\varkappa$ and γ) of various portions of the differential equations. For example, by inspection of Equation (3) it is seen that δ_0 represents the constant portion of $2\Sigma\varkappa_1 \cos kt$ inside $\zeta[\;]$ in ζ'', δ_1 represents the constant portion outside $\xi[\;]$ in ξ'', with similar definitions for the other auxiliary constants δ_j. Ultimately, these auxiliary constants are functions of the satellite masses and other arbitrary and physical constants, while the values of δ_j determine the nodal and peri-apse rates under the conditions that ξ, υ and ζ are purely periodic.

In order that the reader may visualize the structure of the generic terms in Equation (3), and how one employs Poisson series in this effort, we list here the detailed nature of the variables of Equation (3) as obtained from inspection of Equation (II-27) [the meaning of all variables is given in Paper II].

From the equation for ξ'' [Eq. (II-27a)] one finds

$$2\Sigma\gamma_r \cos rt$$
$$= \left\{\xi_i^2\left(4\varkappa_i n_i^2 a_i^3 - \gamma_i n_i^2 a_i^3 - \frac{20}{3} j_i - \frac{168}{5} k_i\right)\right.$$
$$+ \zeta_i^2\left(2\varkappa_i n_i^2 a_i^3 + 27 j_i + \frac{175}{2} k_i\right)$$
$$+ \zeta_i^4\left(-\frac{27}{4} n_i^2 a_i^3 - \frac{27}{4}\varkappa_i n_i^2 a_i^3 - 125 j_i\right)$$
$$+ 2a_i\{\Omega_{i,j+s}\}(1 + \xi_i^2 + \xi_i^4)$$
$$+ \sum_{\substack{j=1\\ j\neq i}}^{4} m_j\left(-E_{ij} - \frac{1}{2}\xi_i \mathscr{A}_{ij} - \xi_j \mathscr{B}_{ij} - \upsilon_i \mathscr{E}_{ij} + \upsilon_j \mathscr{E}_{ij}\right)$$
$$+ n_i^2 a_i^3 \left(\frac{N}{n_i}\right)^2 \frac{S}{N^2 R^3}\left(-3\frac{X_i^2}{R^2} + 1 - \frac{9}{2}\frac{a_i^2}{R^2} + \frac{9 a_i X_i}{R^2}\right.$$
$$\left.\left. + \frac{45 a_i^2 X_i^2}{R^4} - \frac{15 a_i X_i^3}{R^4}\right)\right\} / n_i^2 a_i^3, \tag{6a}$$

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[1] Microfiches see third cover page

$$\delta_2 = \mathrm{Const}[2\Sigma\gamma_r \cos rt] + \left[-\frac{4}{3}j_i - \frac{18}{5}k_i\right]/n_i^2 a_i^3 \tag{6b}$$

$$\begin{aligned}
-\Sigma E_{p'}\cos p't
&= \Big\{\xi_i^2\left(-3\varkappa_i n_i^2 a_i^3 + \gamma_i n_i^2 a_i^3 + \frac{10}{3}j_i + \frac{63}{5}k_i\right)\\
&\quad + \zeta_i^2\left(-\frac{1}{2}\varkappa_i n_i^2 a_i^3 - \frac{9}{2}j_i - \frac{25}{2}k_i\right)\\
&\quad + \xi_i^4\left(-4n_i^2 a_i^3 - 5\varkappa_i n_i^2 a_i^3 + \gamma_i n_i^2 a_i^3 + \frac{35}{3}j_i\right)\\
&\quad + \xi_i^2\zeta_i^2\left(-5n_i^2 a_i^3 - 5\varkappa_i n_i^2 a_i^3 - \frac{189}{2}j_i\right)\\
&\quad + \zeta_i^4\left(\frac{9}{8}n_i^2 a_i^3 + \frac{9}{8}\varkappa_i n_i^2 a_i^3 + \frac{125}{8}j_i\right)\\
&\quad + n_i^2 a_i^3 \xi_i^2(\xi_i'^2 + \zeta_i'^2) - 2a_i\{\Omega_{i,j+s}\}(1+\xi_i^2)\\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{4} m_j\Big[-B_{ij} - \xi_j F_{ij} - (\upsilon_i - \upsilon_j)H_{ij} - \frac{1}{2}\xi_j^2\mathscr{C}_{ij}\\
&\quad - \xi_j(\upsilon_i - \upsilon_j)\mathscr{F}_{ij} - \frac{1}{2}(\upsilon_i - \upsilon_j)^2\mathscr{H}_{ij}\Big]\\
&\quad + n_i^2 a_i^3\left(\frac{N}{n_i}\right)^2 \frac{S}{N^2R^3}\Big(-3\frac{X_i^2}{R^2} + 1 - \frac{3\zeta_i X_i Z}{R^2} - \frac{3}{2}\frac{a_i^2}{R^2}\\
&\quad + \frac{9}{2}\frac{a_i X_i}{R^2} + \frac{15a_i^2X_i^2}{R^4} - \frac{15}{2}\frac{a_i X_i^2}{R^4}\Big)\Big\}/n_i^2 a_i^3
\end{aligned} \tag{6c}$$

and

$$\delta_1 = \mathrm{Const}[-\Sigma E_{p'}\cos p't] + \left(\frac{1}{3}j_i + \frac{3}{5}k_i\right)/n_i^2 a_i^3 . \tag{6d}$$

From the equation for υ' [Eq. (II-27b)] one finds from inside $\xi[\;]$

$$\begin{aligned}
\Sigma 2B^*_{\ell'}\cos\ell't
&= \Big\{\xi_i^2\left(-20\varkappa_i n_i^2 a_i^3 + 4\gamma_i n_i^2 a_i^3 - \frac{70}{3}j_i - \frac{168}{5}k_i\right)\\
&\quad + \zeta_i^2(5\varkappa_i n_i^2 a_i^3 + 21j_i + 45k_i)\\
&\quad + \zeta_i^4\left(-\frac{21}{4}n_i^2 a_i^3 - \frac{21}{4}\varkappa_i n_i^2 a_i^3 - \frac{225}{4}j_i\right)\Big\}/n_i^2 a_i^3 ,
\end{aligned} \tag{7a}$$

$$\delta_4 = \mathrm{Const}[2\Sigma B^*_{\ell'}\cos\ell't] + \left(-\frac{10}{3}j_i - \frac{14}{5}k_i\right)/n_i^2 a_i^3 \tag{7b}$$

and finally,

$$\begin{aligned}
\Sigma 2B_{\ell'}\cos\ell't
&= \xi_i \Sigma 2B^*_{\ell'}\cos\ell't + \Big\{\xi_i^2\left(12\varkappa_i n_i^2 a_i^3 - 3\gamma_i n_i^2 a_i^3 + 10j_i - \frac{56}{5}k_i\right)\\
&\quad + \zeta_i^2(-\varkappa_i n_i^2 a_i^3 - 3j_i - 5k_i)\\
&\quad + \xi_i^4\left(25n_i^2 a_i^3 + 30\varkappa_i n_i^2 a_i^3 - 5\gamma_i n_i^2 a_i^3 + \frac{140}{3}j_i\right)\\
&\quad + \xi_i^2\zeta_i^2(-15n_i^2 a_i^3 - 15\varkappa_i n_i^2 a_i^3 - 84j_i)\\
&\quad + \zeta_i^4\left(\frac{3}{4}n_i^2 a_i^3 + \frac{3}{4}\varkappa_i n_i^2 a_i^3 + \frac{25}{4}j_i\right)\\
&\quad - 3\xi_i^2 n_i^2 a_i^3(\xi_i'^2 + \zeta_i'^2)\\
&\quad + 2a_i\{\Omega_{i,j+s}\}(1 - 2\xi_i + 3\xi_i^2 - 4\xi_i^3)\Big\}/n_i^2 a_i^3 ,
\end{aligned} \tag{7c}$$

$$\delta_3 = \mathrm{Const}[\Sigma 2B_{\ell'}\cos\ell't] + \left(\frac{2}{3}j_i + \frac{2}{5}k_i\right)/n_i^2 a_i^3 . \tag{7d}$$

Similarly, from the equation for ζ'' [Eq. (II-27c)] one finds

$$\begin{aligned}
2\Sigma K_1\cos kt
&= \Big\{-\xi_i(3\varkappa_i n_i^2 a_i^3 + 15j_i + 35k_i)\\
&\quad + \xi_i^2(6\varkappa_i n_i^2 a_i^3 + 45j_i + 140k_i)\\
&\quad + \zeta_i^2\left(-\frac{3}{2}\varkappa_i n_i^2 a_i^3 - \frac{25}{2}j_i - \frac{245}{6}k_i\right)\\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{4} m_j(-L_{ij} - \xi_i M_{ij} - \xi_j N_{ij} - \upsilon_i O_{ij} + \upsilon_j O_{ij})\\
&\quad + n_i^2 a_i^3\left(\frac{N}{n_i}\right)^2 \frac{S}{N^2R^3}\Big(-3\frac{Z^2}{R^2} + 1 - \frac{3}{2}\frac{a_i^2}{R^2}\\
&\quad + \frac{3a_i X_i}{R^2}\Big)\Big\}\, n_i^2 a_i^3 ,
\end{aligned} \tag{8a}$$

$$\delta_0 = \mathrm{Const}[2\Sigma K_1\cos kt] + (3j_i + 5k_i)/n_i^2 a_i^3 \tag{8b}$$

and

$$\begin{aligned}
-\Sigma C_{q'}\sin q't
&= \Big\{\sum_{\substack{j=1\\ j\neq i}}^{4} m_j\zeta_j(-P_{ij} - \xi_i Q_{ij} - \xi_j R_{ij} - \upsilon_i S_{ij} + \upsilon_i S_{ij})\\
&\quad + n_i^2 a_i^3\left(\frac{N}{n_i}\right)^2 \frac{S}{N^2R^3}\left[-\frac{3X_i Z}{R^2}(1+\xi_i) + \frac{3}{2}\frac{Za_i}{R^2}\right]\\
&\quad - n_i^2 a_i^3(Z_i/n_i^2 a_i)\Big\}/n_i^2 a_i^3 .
\end{aligned} \tag{8c}$$

Finally, in computing $\{\Omega\}$ from Equation (4) one requires Equation (III-5)

$$\begin{aligned}
a_i\frac{\partial^*\Omega_{ij}}{\partial\tau} &= n_i m_j\{\xi_i' B_{ij} + (1+\upsilon_i')D_{ij} + \xi_i'\xi_i E_{ij} + \xi_i'\xi_j F_{ij}\\
&\quad + [\xi_i'(\upsilon_i - \upsilon_j) + \xi_i(1+\upsilon_i')]H_{ij} + \xi_j(1+\upsilon_i')I_{ij}\\
&\quad + (\upsilon_i - \upsilon_j)(1+\upsilon_i')J_{ij} + \tfrac{1}{2}\xi_i^2\mathscr{E}_{ij} + \xi_i\xi_j\mathscr{F}_{ij} + \tfrac{1}{2}\xi_j^2\mathscr{G}_{ij}\\
&\quad + \xi_i(\upsilon_i - \upsilon_j)\mathscr{H}_{ij} + \xi_j(\upsilon_i - \upsilon_j)\mathscr{I}_{ij} + \tfrac{1}{2}(\upsilon_i - \upsilon_j)^2\mathscr{J}_{ij}\} .
\end{aligned}$$

(9) (III-5)

Each term in Equations (6)–(9) in general represents a series or a product of Poisson series. As noted earlier, each term in Sampson's final solution is put into into the form of a Poisson series by appending an appropriate parameter ε and it is the resultant series which are manipulated according to the preceding equations.

Stage 1: Non-libration Solution

In developing the solution, outlined in Paper II, one first treats the problem as if there were no Laplacian libration among the inner three satellites. Employing the initial approximation series (Sampson's final theory) one forms the series of Equation (3) by evaluating the defining relations given in Equations (6)–(9). In so doing, one develops the auxiliary constants $\delta_j (j=0, 4)$ of Equation (3) which are employed in determining the nodal and peri-apse rates.

In this first stage, the Laplacian constraint is not used and all resultant terms of the form $v_1 - 3v_2 + 2v_3$ are ignored. After forming the series inside the brackets of Equation (3) [viz. $2\Sigma\gamma_r \cos rt$ in ξ'', $\Sigma 2B^*_{\ell'} \cos \ell' t$ in υ' and $2\Sigma K_1 \cos kt$ in ζ''], one multiplies them by the appropriate term ξ or ζ and re-distributes the resultant series into $-\Sigma E_{p'} \cos p't$ for ξ'', $2B_{\ell'} \cos \ell' t$ for υ' and $-\Sigma C_{q'} \sin q't$ for ζ''. Whenever a term of the fundamental period occurs in these latter series [viz. a term for satellite j whose argument is $\ell_j - \pi_j$ for ξ''_j or $\ell_j - \omega_j$ for ζ''_j], that term is replaced as outlined in Equation (II-10) in order to avoid a zero divisor when developing the particular solution to the differential equation.

In all stages it is important to retain only those terms which will affect the final solution at the sought-for level of precision (one arc-second Jovicentric). By re-scaling the ε-parameters employing

$$\varepsilon = \varepsilon_{max} \varepsilon^* ,$$

where ε_{max} is the apriori numerical upper bound established for a particular parameter ε (see Table II of Paper II), ε^* becomes a "new" algebraic parameter whose upper bound is unity. Hence, if the multiplication subroutines account for the maximum of all possible integration factors which might occur, one can truncate the series at the appropriate level. As an example, if p represents the fundamental frequency $(\dot{\ell}_j - \dot{\pi}_j) n_j^{-1}$ for a particular satellite in ξ, then Equation (3) is of the form

$$0 = \xi'' + \xi[p^2 + \ldots + 2\Sigma\gamma_r \cos rt] + \ldots - \Sigma E_{p'} \cos p't . \quad (10)$$

Possible integrating factors (which occur in developing ξ or υ) for a term of the form $\varkappa^{\sin}_{\cos}(kt)$ are then

a) $(p^2 - k^2)^{-1}$ from $\xi'' + p^2 \xi$,
b) $(k^2 - 2|pk|)^{-1}$ from $\xi[\ ]$,
c) $k^{-1}(p^2 - k^2)^{-1}$ from υ' via ξ; also $\{\Omega\}$ and ξ'',
d) $k^{-1}(k^2 - 2|pk|)^{-1}$ from υ' and (b) ,
e) k^{-1} from υ',
f) k^{-2} from $\{\Omega\}$ and υ',
g) $k^{-2}(p^2 - k^2)^{-1}$ from $\{\Omega\}$ and ξ'', υ',
h) $k^{-2}(k^2 - 2|pk|)^{-1}$ from $\{\Omega\}$ and (d).

If the maximum of these integration factors is Q, then the multiplication software keeps the resultant term only when $\varkappa Q$ is less than the tolerance level. In so doing one can delete terms which never will significantly affect the result ($\varkappa$ includes the maximum value of the ε-parameters associated with the term).

After performing the algebraic manipulations outlined, one has the net differential equations and can obtain the solution to Equation (3), ignoring the Laplacian libration among the inner three satellites. In developing the terms in Equation (3) one also has obtained series for the auxiliary constants δ_j.

The peri-apse and nodal rates are then determined, following Sampson (*Theory*, pp. 27–31), from the relations

$$\dot{\pi}_j = n_j(1 - p_j)$$
$$\dot{\omega}_j = n_j(1 - q_j) , \quad (11)$$

where $\dot{\pi}_j$ and $\dot{\omega}_j$ are the peri-apse and nodal rates, respectively. The values of p_j and q_j are related to the aforementioned parameters δ by

$$p^2 = 1 + \delta_2 - \delta_3 + [\iota_\xi + \Sigma \Delta q_u c_u^2]$$
$$q^2 = 1 + \delta_0 - \delta_1 - \delta_3 + [\iota_\zeta - \Sigma \Delta p_r e_r^2] , \quad (12)$$

where the terms in brackets are small correction terms (*Theory*, pp. 27–31) given by

$\iota_\xi =$	$\Sigma \Delta q_u c_u^2 =$
0,06	0
10,83	0,10
1,18	0
0,22	−0,06

$\iota_\zeta =$	$\Sigma \Delta p_r e_r^2 =$
0	0,62
2,85	6,50
0,28	0,09
−0,12	0

(13)

for the four satellites. A camma denotes the seventh decimal, so that 10,83 implies $10.83\,10^{-7}$.

It should be noted in the preceding manipulations and integrations, that $\dot{\pi}$ and $\dot{\omega}$ are not completely known until the whole problem is solved. Hence, several of the auxiliary ε-parameters (see Table II of Paper III) represent the errors in Sampson's values of $\dot{\pi}$ and $\dot{\omega}$ and the integration factors may subsequently be altered. In Paper III, the results at this stage are given for the latitude ζ.

After obtaining the solution outlined above, one must then determine and eliminate the auxiliary parameters in terms of the arbitrary ones, as outlined in Paper III.

Stage 2: Modifications for Libration

At this point one has the solution (including the effects of all the ε-parameters) with the exception of several higher-order two-body effects (Paper III, p. 14) and modifications due to the libration. In order to handle the exact resonance among the inner three satellites, Sampson basically follows the development of Laplace. The true longitudes are represented exactly by

$$v_1 - 3v_2 + 2v_3 = 180° + \lambda \quad (14)\ (\text{II-11})$$

while the mean longitudes and mean motions obey

$$l_1 - 3l_2 + 2l_3 = 180°$$

$$n_1 - 3n_2 + 2n_3 = 0. \quad (15) \; (\text{II-12})$$

In view of Equation (2), Equation (14) may be written as

$$\upsilon_1 - 3\upsilon_2 + 2\upsilon_3 = \lambda, \quad (16) \; (\text{II-13})$$

where λ is periodic and represents the libration in the longitudes of the first three satellites. Following Sampson (*Theory*, pp. 32–35, 217–227) one twice differentiates Equation (16) and obtains a second order equation for λ of the form

$$\lambda'' + L \sin \lambda = F(t). \quad (17)$$

With the exception of the multiplicative constant $(-3 + 4\gamma_0 - 2\lambda_0)$ which differs from Sampson's formulation, the components of λ are obtained, following Sampson (*Theory*, pp. 32–35), from

$$n_i^{-1} d^2\upsilon_i/dt^2 = (-3 + 4\gamma_0 - 2\lambda_0) \left(\frac{\partial^* \Omega_{ij}}{\partial \tau}\right)_{t=\tau} \Big/ n_i^2 a_i^2, \quad (18)$$

where $\partial^* \Omega_{ij}/\partial \tau$ is given in Equations (4) and (9). In calculating the terms in the above equation one employs the results of Stage 1, where no reduction for $l_1 - 3l_2 + 2l_3 = 180°$ has yet been made in the coordinate series. Hence, terms such as $\nu_1 - \nu_2$ in the series for Satellite II (from Satellite I) are kept distinct from ones such as $2(\nu_2 - \nu_3)$ (from Satellite III). In calculating υ_2'' the latter term contributes to the libration while the former does not, when multiplied by the term $\nu_1 - \nu_2$ occurring in the development of $m_1 \Omega$. It is thus seen that in principle L in Equation (17) is of the form

$$L = L_1 m_2 m_3 - 3L_2 m_3 m_1 + 2L_3 m_1 m_2. \quad (19)$$

Following Sampson, retaining $\sin \lambda$ and $\sin 2\lambda$ terms in Equation (17), and then expanding the trigonometric functions in λ one obtains

$$\lambda'' + L\lambda = F(t) \quad (20)$$

whose homogeneous solution is

$$\lambda = \lambda_A \sin(\mu t + \Phi_\lambda), \quad (21)$$

where $\mu = \sqrt{L}$ (measured in deg/day) and where λ_A and Φ_λ represent the amplitude and phase of the free libration, respectively. In Sampson's work, it was assumed that $\lambda_A = \Phi_\lambda = 0$ exactly and Sampson thus removed these degrees of freedom. In the present work, λ_A is one of the polynomial parameters, ε_9, while Φ_λ is an angular variable, β_5 (see Tables 2 and 3 which define the arbitrary constants and parameters of this theory), which may be non-zero. The definitions of all variables are given in Tables 1–3 and are copies of the first three frames on the microfiche. The value of L (the square of the libration frequency) is given in Microfiche Tables A.36–39 and, for Sampson's values of the satellite masses, is

$$L = 295526,6 \; (°/\text{day})^2 \quad (22)$$

implying a free libration period of 2094.1 day which may be compared to Sampson's value of 2041.1 day (*Theory*, p. 264) or to Vu and Sagnier's (1974) value of 2133.8 day. For a given value of the free libration amplitude λ_A, the inner satellites will contain the terms

$$\upsilon_1 = m_2 m_3 L_1 L^{-1} \lambda_A \sin(\mu t + \Phi_\lambda)$$

$$\upsilon_2 = m_3 m_1 L_2 L^{-1} \lambda_A \sin(\mu t + \Phi_\lambda)$$

$$\upsilon_3 = m_1 m_2 L_3 L^{-1} \lambda_A \sin(\mu t + \Phi_\lambda), \quad (23)$$

where λ_A and Φ_λ are arbitrary.

Insofar as the particular solution to the libration is concerned, the results are handled in exactly the same manner as Sampson (*Theory*, p. 35, pp. 217–227).

As shown by Sampson, the particular solution to Equation (20) will yield terms with exactly the same periods as some previously obtained in the steps of Stage 1. Hence, the particular solution from the λ'' equation is re-distributed among the longitudes of the first three satellites as modifications of the non-libration solution of Stage 1. The corrective terms are (*Theory*, p. 35)

$$\Delta \upsilon_i = -\frac{m_j m_k L_i}{(L - l^2)} [B_l^{(1)} - 3B_l^{(2)} + 2B_l^{(3)}] \sin lt, \quad (24)$$

where $B_l^{(i)}$ represents the coefficient of $\sin lt$ obtained from Stage 1 (the non-libration solution) for satellite i and where $i \neq j \neq k$ with $i = 1, 2, 3$. Up until this latter step, it was possible to obtain all integration factors without resort to special techniques (i.e. the possible variations in integration factors were within the range of a Taylor series expansion). However, in solving Equation (24) it was found that because of the possible variations in L (see Microfiche A.36–39 and A.30–35 for the full series for L and $\sqrt{L}$ in terms of the polynomial parameters), there were several terms with period approximately $360°/\sqrt{L}$ days which would not permit an expansion of $(L - l^2)^{-1}$. For example, the terms involving twice the mean anomaly of Jupiter, 2 G, (see Table 3 for a definition of the angles and the terms on Microfiche A.41–43) have a period of 2166 day which is very close to the free libration period of 2094 day. With the possible variations in $\sqrt{L}$ (see Microfiche A.30–35) it is seen that very small changes in satellite masses could alter the libration period considerably and, in fact, move the libration period to the opposite side of the 2 G frequency. Hence such terms are handled separately from the normal series of υ_j given in Microfiche Tables A.52–106. Such special terms (called "BADTRM" in Microfiche Tables A.41–43) are to be calculated from a reciprocal expansion

$$\delta \upsilon_j = \Delta \upsilon_j^* (R^{-1} - 1) \quad (j = 1, 3), \quad (25)$$

where $\Delta \upsilon_j^*$ represents the "BADTRM" series and where

$$R = 1 + Fl^2/(L - l^2) \quad (26)$$

Table 1

Table of Contents A.1			Polynomial Parameters A.2		Trigonometric Parameters A.3	
			I	II	III	IV
n^2a^3	(N2A3)		A.4	A.5	A.6	A.7
a	(AXIS-A)		A.8, C.1	A.9, C.1	A.10, C.1	A.11, C.1
$\dot{\pi}_j$	(PIDOT)		A.12-16, C.1	A.17-19, C.1	A.20-21, C.1	A.22, C.1
$\dot{\omega}_j$	(OMDOT)		A.23-24, C.2	A.25, C.2	A.26-27, C.2	A.28, C.2
$\dot{\psi}$	(PSIDOT)	A.29, C.2				
$\sqrt{L}$	(SQRTLB)	A.30-35, C.2				
L	(LIBNET)	A.36-39				
	(BADFAC)	A.40				
	(BADTRM)		A.41	A.42	A.43	
ξ_j	(XISOL)		A.44-45, B.1, C.3	A.46-47, B.2, C.3	A.48-49, B.3, C.4	A.50-51, B.4, C.5
ν_j	(V-SOL)		A.52-69, B.5, C.6	A.70-88, B.6-7, C.7-8	A.89-106, B.8-9, C.9-10	A.107-116, B.10-11, C.11-12
$\zeta_j = z_j/a_j$	(LATSOL)		A.117, B.12, C.13	A.118-119, B.13, C.13	A.120-121, B.14, C.14	A.122-123, B.15, C.14
Δr_j	(Z2-DEG)		A.124, B.16, C.15	A.125, B.16, C.15	A.126, B.17, C.16	A.127, B.17, C.16

Tables A.1 - A.80 are on Microfiche 1, and Tables A.81 - A.127, B.1-17, C.1-16 are on Microfiche 2. Table A contains the complete theory and partial derivatives; Table B contains only the terms from Table A where $\epsilon = 0$; Table 3 contains an approximation to Sampson's theory with $\epsilon_1 = 0.053211$, $\epsilon_2 = -0.002373$, $\epsilon_3 = -0.017883$, $\epsilon_4 = -0.019815$, $\epsilon_{11} = -0.004002$ with all other $\epsilon = 0$.

The series BADFAC and BADTRM are employed in obtaining corrections to ν_j (j = 1, 3) in the form $\Delta\nu_j = \mathrm{BADTRM}_j\,(1/R - 1)$ where $R = 1+\mathrm{BADFAC}*\ell^2/(L_o - \ell^2)$ where $L_o = 0.029552665$ and ℓ is the rate of a term in BADTRM measured in deg/day of the form sin ℓt.

At the top of each page is a header containing the name (NAME) of the series (e.g. XISOL ⇒ ξ), and index (INDEX) usually denoting a satellite, the number of terms and a scale factor [SCALE = 10*EXP(k)] which indicates the power of 10 by which the printed series has been multiplied. Following the header are the actual terms in each series, beginning with an identifying line number, an integer coefficient, possibly a term $+k_x$ which is an auxiliary scale factor, the powers of the 25 polynomial indices $i_1 \rightarrow i_{25}$, the trigonometric function sine or cosine, and the 21 angular variable indices $j_1 \rightarrow j_{21}$. The meaning of polynomial and trigonometric indices are listed in Tables A.1 and A.2. The general form is

$$10^k * \mathrm{NAME} = \mathrm{COEF} * 10^{k_x}\, p_1^{i_1} \cdots p_{25}^{i_{25}} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (j_1\,\ell_1 + j_2\,\ell_2 + \cdots + j_{21}\,\phi_{21})$$

As examples, term number 40 of $\sqrt{L}$ (Table A.30) is given as

```
 LINE                     I1  I3  I5   I7  I9  I11 I13 I15  I17 I19  I21 I23 I25        J1  J3  J5   J7  J9  J11 J13 J15  J17 J19  J21
          COEFFICIENT       I2  I4   I6  I8 I10  I12 I14  I16 I18 I20  I22 I24     TRIG    J2  J4   J6  J8 J10  J12 J14  J16 J18 J20
-------** SCALE = 10*EXP(8) **-------------- NAME ** SQRTLB **------------- INDEX   ** 123 **---------- NO. TERMS **  298 **------LINE
   40  4549967084 +3 0 0 0 0 0  1 1 0 0 0  0 0 0 0 0  0 0 0 0 0  0 0 0 0 0  COS  0 0 0 0 0  0 0 0 0 0  0 0 0 0 0  0 0 0 0 0  0   40
```

which is interpreted as

$$\sqrt{L} = \cdots + 4549967084 \times 10^{-8+3}\,\epsilon_6\,\epsilon_7 \cos 0 \cdots = \cdots + 45499.67084\,\epsilon_6\,\epsilon_7 \cdots$$

Similarly, term number 171 of ν_1 (Table A.55) contains

```
 LINE                     I1  I3  I5   I7  I9  I11 I13 I15  I17 I19  I21 I23 I25        J1  J3  J5   J7  J9  J11 J13 J15  J17 J19  J21
          COEFFICIENT       I2  I4   I6  I8 I10  I12 I14  I16 I18 I20  I22 I24     TRIG    J2  J4   J6  J8 J10  J12 J14  J16 J18 J20
-------** SCALE = 10*EXP(7) **-------------- NAME ** V-SOL  **------------- INDEX   **   1 **---------- NO. TERMS **  867 **------LINE
  171       121006    2 0 0 1 0  0 0 0 0 0  0 0 1 0 0  0 0 0 0 0  0 0 0 0 0  SIN  0 0 0 0 0  0 0 1-1 0  0 0 0 0 0  0 0 0 0 0  0  171
```

which is interpreted as

$$\nu_1 = \cdots + 0.0121006\,\epsilon_1^2\,\epsilon_4\,\epsilon_{13} \sin(\pi_3 - \pi_4)$$

Table 2

Index	Symbol $\epsilon(\ )$	Generating Value A_o* $\epsilon = (A - A_o)/A_o$	Name
1	m_1	449,7	Mass of Satellite I relative to Jupiter
2	m_2	252,9	Mass of Satellite II relative to Jupiter
3	m_3	798,8	Mass of Satellite III relative to Jupiter
4	m_4	450,4	Mass of Satellite IV relative to Jupiter
5	S/J	1047.355	Mass of Sun relative to Jupiter system
6	n_1	203.48895 4208	Mean motion of Satellite I, deg/day
7	n_2	101.37472 3445	Mean motion of Satellite II, deg/day
8	n_4	21.57107 1403	Mean motion of Satellite IV, deg/day
9	λ_A	0†	Libration phase angle amplitude
10	n_J	$8.30912\ 15712 \times 10^{-2}$	Mean motion of Jupiter, deg/day
11	J_2	0.01484 85	Jupiter J_2
12	J_4	-8.107×10^{-4}	Jupiter J_4
13	R_J	71420	Radius of Jupiter, km
14	P_J	9.924825	Jupiter period of rotation, hr
15	3(C-A)/2C	0.111	Ratio Jupiter moments of inertia
16	e_{11}	465,	Primary eccentricity of Satellite I, rad
17	e_{22}	825,	Primary eccentricity of Satellite II, rad
18	e_{33}	15164,	Primary eccentricity of Satellite III, rad
19	e_{44}	73725,	Primary eccentricity of Satellite IV, rad
20	e_J	0.04846 02472	Eccentricity of Jupiter
21	c_{11}	4756,	Primary sine inclination of Satellite I
22	c_{22}	81490,	Primary sine inclination of Satellite II
23	c_{33}	31108,	Primary sine inclination of Satellite III
24	c_{44}	47460,	Primary sine inclination of Satellite IV
25	I_J	3.10401	Inclination of Jupiter orbit to equator, deg
26	J	1.30691	Inclination of Jupiter orbit to ecliptic, deg
27	ϵ	23° 26' 44.84"	Obliquity of ecliptic at 1950.0
28	n_S	$3.34597\ 33896 \times 10^{-2}$	Mean motion of Saturn, deg/day

Notes: *Nominal value are $\epsilon = 0$
A comma denotes the seventh decimal (e.g. $449,7 = 449.7 \times 10^{-7}$)
†This phase angle amplitude for libration is an absolute quantity (i.e. $\epsilon_9 = \lambda_A$), measured in radians. Sampson value: $\lambda_A = 0$.
Form of free libration is $\lambda_A \sin(\mu t + \phi_\lambda)$ where $\mu = \sqrt{L}$ and $\phi_\lambda = \beta_5$

Note: If, in evaluating the latitude series ζ_j (= Z/a), the "time-completed" is employed by means of

$$t_j = t_o + \Delta t_j$$
$$\Delta t_j = \upsilon_j/n_j$$
$$t_o = \text{ephemeris time}$$

then the resultant series represents $s_j = Z_j/a_j\,(1 + \xi_j) = Z_j/\rho_j$ complete through second order and the second order terms in ζ_j (Z2-DEG) should not be applied. The second-order terms in ζ_j should be employed only if the time-completed feature is not utilized.

Table 3

Index	Symbol	Epoch Value (JD 2443000.5)*	Rate (deg/day)	Name
1	ℓ_1	$106^\circ.03042 + \beta_1$	$203.48895\,4208\,(1 + \epsilon_6)$	Mean longitude of Satellite I
2	ℓ_2	$175^\circ.74748 + \beta_2$	$101.37472\,3445\,(1 + \epsilon_7)$	Mean longitude of Satellite II
3	ℓ_3	$120^\circ.60601 - 1/2\,\beta_1 + 3/2\,\beta_2$	$50.31760\,80635\left[1 - 2\epsilon_6 + 3\epsilon_7 - 2.20451\,8497 \times 10^{-2}(\epsilon_6 - \epsilon_7)\right]$	Mean longitude of Satellite III
4	ℓ_4	$84^\circ.51861 + \beta_4$	$21.57107\,1403\,(1 + \epsilon_8)$	Mean longitude of Satellite IV
5	ϕ_λ	$0 + \beta_5$	$\sqrt{L}$ (= 0.1719 0889 + ···)	See Table A.30 Libration phase angle
6	π_1	$4^\circ.51172 + \beta_6$	$\dot{\pi}_1$ (= 0.1612 2004 + ···)	See Table A.12 Proper periapse of Satellite I
7	π_2	$74^\circ.53051 + \beta_7$	$\dot{\pi}_2$ (= 0.0476 3124 + ···)	See Table A.17 Proper periapse of Satellite II
8	π_3	$174^\circ.85831 + \beta_8$	$\dot{\pi}_3$ (= 0.0069 7450 + ···)	See Table A.20 Proper periapse of Satellite III
9	π_4	$336^\circ.02667 + \beta_9$	$\dot{\pi}_4$ (= 0.0018 7351 + ···)	See Table A.22 Proper periapse of Satellite IV
10	Π	$13^\circ.30364 + \beta_{10}$	0	Longitude of perihelion of Jupiter
11	ω_1	$242^\circ.73706 + \beta_{11}$	$\dot{\omega}_1$ (= -0.1340 1884 + ···)	See Table A.23 Proper node of Satellite I
12	ω_2	$95^\circ.28556 + \beta_{12}$	$\dot{\omega}_2$ (= -0.0327 4001 + ···)	See Table A.25 Proper node of Satellite II
13	ω_3	$125^\circ.14673 + \beta_{13}$	$\dot{\omega}_3$ (= -0.0070 1735 + ···)	See Table A.26 Proper node of Satellite III
14	ω_4	$317^\circ.89250 + \beta_{14}$	$\dot{\omega}_4$ (= -0.0018 0387 + ···)	See Table A.28 Proper node of Satellite IV
15	ψ	$316^\circ.73369 + \beta_{15}$	$\dot{\psi}$ (= -0.0000 0229 + ···)	See Table A.29 Longitude of origin of coordinates (Jupiter's pole)
16	G'	$31^\circ.97852\,80244 + \beta_{16}$	$3.34597\,33896 \times 10^{-2}\,(1 + \epsilon_{28})$	Mean anomaly of Saturn
17	G	$30^\circ.37841\,20168 + \beta_{17} + \delta G^\dagger$	$8.30912\,15712 \times 10^{-2}\,(1 + \epsilon_{10})$	Mean anomaly of Jupiter
18	ϕ_1	$172^\circ.84\,(1 - 0.014\,\epsilon_{20}) + \beta_{18}$	0	Phase angle occurring in solar $(A/R)^3$ with angle 2G' - G
19	ϕ_2	$47^\circ.03\,(1 - 0.156\,\epsilon_{20}) + \beta_{19}$	0	Phase angle occurring in solar $(A/R)^3$ with angle 5G' - 2G
20	ϕ_3	$259^\circ.18 + \beta_{20}$	0	Phase angle occurring in solar $(A/R)^3$ with angle G' - G
21	ϕ_4	$157^\circ.12\,(1 + 0.0014\,\epsilon_{20}) + \beta_{21}$	0	Phase angle occurring in solar $(A/R)^3$ with angle 2G' - 2G
22	Ω_J	$99^\circ.95326 + \beta_{22}$	0	Longitude of ascending node of Jupiter's orbit

Notes: *All parameters β are measured in degrees. Nominal values are $\beta = 0$

$^\dagger\delta G = 0^\circ.03439 \sin(2G' - G + 0^\circ.76699) + 0^\circ.33033 \sin(5G' - 2G - 0.0227694694\,t_y + 64^\circ.26288)$ where $t_y = (JD - 2443000.5)/365.25$

Table 4. Additional second-degree terms $\delta\xi$, δv[a]

$$10^7\delta\xi_4 = -9\cos(2l_3-3l_4+\pi_4)+9\cos(2l_3-l_4-\pi_4)-6\cos(3l_3-4l_4+\pi_4)$$
$$+6\cos(3l_3-2l_4-\pi_4)-11\cos(l_2-2l_4+\pi_4)+11\cos(l_2-\pi_4)$$
$$-(86+87\varepsilon_1+86\varepsilon_{19})\cos(l_1-2l_4+\pi_4)$$
$$+(86+87\varepsilon_1+86\varepsilon_{19})\cos(l_1-\pi_4)$$
$$10^7\delta v_4 = -6\sin(l_4+\pi_4-2\Pi-2G)+13\sin(l_3-2l_4+\pi_4)$$
$$+11\sin(2l_3-3l_4+\pi_4)+8\sin(l_2-2l_4+\pi_4)$$
$$+(23+23\varepsilon_1)\sin(l_1-2l_4+\pi_4)-(18+18\varepsilon_1)\sin(l_1-\pi_4)$$

[a] These terms are to be added to those given in Microfiche tables A.50, B.4 and C.5 for ξ_4 and Tables A.107, B.10 and C.11 for v_4

with the series F (called "BADFAC") appearing in Microfiche Table A.40.

The formulation of the terms Δv_i^* and F are as follows. The theory given in Microfiche Tables A.52–106 already contains *some* of the corrective terms in Equation (24) for the problem periods close to $360°/\sqrt{L}$. The tables actually contain, for these troublesome terms,

$$\Delta v_i^* = -\frac{L_i m_j m_k}{L}\left[B_i^{(1)}-3B_i^{(2)}+2B_i^{(3)}\right]\frac{L_0}{L_0-l^2}, \qquad (27)$$

where L_0 is a pure number whose fixed value has arbitrarily been selected as that value in Equation (22) corresponding to Sampson's nominal masses. As noted earlier, it is the possible variations in L (depending upon the arbitrary parameters corresponding to satellite masses) which cause difficulty.

By comparing Equation (27) with Equation (24) it is seen that an additional correction δv_i is required, namely

$$\delta v_i = \Delta v_i^*\left[\frac{L}{L_0}\cdot\frac{L_0-l^2}{L-l^2}-1\right] \quad (i=1,3) \qquad (28)$$

and by comparison of Equation (28) with Equations (25) and (26) one finds the series for R is

$$R=\frac{L_0}{L}\cdot\frac{L-l^2}{L_0-l^2} \qquad (29)$$

and it is this series for R which is developed in Equation (26) and given in Microfiche A.40. The difficulty with algebraic convergence has thus been avoided, since R is calculated as a pure number from user-defined numerical values of the arbitrary constants of Table 2 and the numerical value R^{-1} is employed in calculating δv. It is seen that if Sampson's masses are employed, $L=L_0$ and so $R=1$ which yields $\delta v=0$.

Final Solution

At this point one has calculated the non-libration perturbations, the modifications due to the Laplace libration, those from the 3–7 commensurability (Paper I) and the additional corrections due to higher-degree two-body effects [the quadratic terms in Equation (3)] and thus has the final solution given in Microfiche Tables A. There is one slight correction still required for Satellite IV, which will be developed in Table 4 subsequently. The final solutions are presented in the first two microfiche and Tables 1–3 are copies of the initial microfiche frames. Microfiche Tables A.1 to A.127 contain the complete theory while Microfiche Tables B.1–B.17 contains only the theory for the case when all ε-parameters of Table 2 are zero (viz. the "Sampson" theory with errors removed). Microfiche Tables C.1 to C.16 contain the theory for the case in which the four satellite masses (ε_1 through ε_4) and $J_2(\varepsilon_{11})$ are adjusted, following the manner of Sampson (*Theory*, pp. 176–178), by adjusting the masses and J_2 to fit certain specific periodic terms and peri-apse rates adopted as "observable" quantities by Sampson. The values computed in this way for the ε-parameters are given in Table 1 and Microfiche A.1, while the resultant theory is in Microfiche Tables C.1 to C.16. It should be noted that the value $\varepsilon_1 = +0.053211$ of Table 1 is correct, the value (0.053711) given in Microfiche A.1 being a typographical error. The fact that the "derived" masses in the above procedure are not Sampson's values (viz. $\varepsilon=0$) indicates that the present theory differs from Sampson's in the structure of these coefficients of the selected periodic terms and peri-apse rates. As indicated in Tables 2 and 3, the final theory is a function of 49 arbitrary constants and parameters. Each of these variables may be assigned a numerical value and the theory can be regenerated from those specific constants easily.

The final corrections, alluded to earlier, are presented in Table 4, which contains some additional two-body corrections required for Satellite IV. These corrections are *not* contained in the Microfiche and should be added. The corrections $\delta\xi_4$ and δv_4 of Table 4 result from the manner in which the second-degree two-body effects are calculated in Equation (III-6) and by Sampson (*Theory*, pp. 25 and 26). The basic method originally employed to calculate the second-degree terms involved an expansion of the frequency f of a specific term about the mean motion of a satellite n_i in the form $f=n_i(1+\Delta p)$. In comparing the theory with a numerical integration performed by Peters (1975), it was found that Satellite IV exhibited high-frequency errors of approximately 20 km amplitude. Subsequently it was found that the source of these high-frequency errors was due to the expansion of the reciprocal of $f=n_i(1+\Delta p)$ in the two-body corrections of Equation (III-6). Although Sampson's theory, in principle, has the same problem, it is not manifested there because Sampson's initial approximation did not include any high-frequency terms.

It can be shown that the additional corrections to Equation (III-6) and to Sampson's terms (*Theory*, pp. 25 and 26) are

$$\delta\xi = \sum_r\sum_s x'_r x'_s\left[-5+4p_r+\frac{-5+p_r^2+p_s^2}{(1-p_r^2-p_s^2)^2-4p_r^2p_s^2}\right]$$
$$+\sum_r\sum_s x_r x_s\left[\frac{2(1-p_r^2-p_s^2+p_r^2p_s^2)}{(1-p_r^2-p_s^2)^2-4p_r^2p_s^2}\right], \qquad (30)$$

$$\delta v' = -2\delta\xi\,,$$

Table 5. Basic theory with $\varepsilon=\beta=0$

----------- SERIES FOR XI-1 (COS) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+170	+193	+170	L1 -L2	.50181707
2	+109	+109	+110	L1 -L3	.75272560
3	-465	-465	-465	L1 -PI1	.99920772
4	-406	-372	-355	L1 -PI3	.99996573
5	-179	-166	-169	L1 -PI4	.99999079
6	-75			L1 +PI3 -2*PIJ -2*G	.99921761
7	-41312	-41240	-41267	2*L1 -2*L2	1.00363414
8	-131	-130	-130	4*L1 -4*L2	2.00726828

----------- SERIES FOR V-1 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	-37			-2*PIJ+2*PSI -2*G	-.00081669
2	-499			-2*PIJ+2*PSI	-.00000002
3	+81			-2*PIJ +OM3 +PSI -2*G	-.00085116
4	+98	+84	+70	-OM2 +PSI	.00016088
5	-72	+134	-76	-OM3 +PSI	.00003447
6	-49	-61	-47	-OM4 +PSI	.00000885
7	-310	-270	-419	G	.00040833
8	+99	+61	+248	2*G	.00081667
9	-33	-30	-33	5*G' -2*G +PHI2	.00000549
10	-27	-39	-32	OM3 -OM4	-.00002562
11	+156	+168	+131	OM2 -OM3	-.00012641
12	+30	+26	+22	OM2 -OM4	-.00015203
13	-38			PI4 -PIJ	.00000921
14	-556		+2740	PI3 -PI4	.00002507
15	+314			PI2 -PI3	.00019980
16	+119			PI2 -PI4	.00022487
17	-40			PI1 -PI3	.00075800
18	-26			PI1 -PI4	.00078307
19	-31			PI1 +PI4 -2*PIJ -2*G	-.00001518
20	-6139			PI1 +PI3 -2*PIJ -2*G	.00000989
21	+39			3*L3 -7*L4+4*PI4	-.00018272
22	-32			3*L3 -7*L4 +PI3 +3*PI4	-.00015765
23	-905	-616/-618	-783	L1 -2*L2 +PI4	.00364335
24	-2066	-1578/-1581	-1607	L1 -2*L2 +PI3	.00366841
25	-700	-687	-799	L1 -2*L2 +PI2	.00386821
26	+1286	+1259/+1256	+1176	L1 -2*L2 +PI1	.00442642
27	-618	-690	-618	L1 -L2	.50181707
28	-276	-276	-278	L1 -L3	.75272560
29	-26	-25	-26	L1 -L4	.89399390
30	+931	+929	+929	L1 -PI1	.99920772
31	+810	+743	+709	L1 -PI3	.99996573
32	+370	+332	+337	L1 -PI4	.99999079
33	+152			L1 +PI3 -2*PIJ -2*G	.99921761
34	+41			2*L1 -4*L2 +OM2 +OM3	.00707290
35	+306			2*L1 -4*L2+2*OM2	.00694649
36	+82429	+82295	+82344	2*L1 -2*L2	1.00363414
37	-30	-44	-28	2*L1 -2*L3	1.50545121
38	+276	+270	+273	4*L1 -4*L2	2.00726828

----------- SERIES FOR LAT-1 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+46	+44	+44	L1 -2*PIJ +PSI -2*G	.99918332
2	+4756	+4756	+4756	L1 -OM1	1.00065860
3	+1822	+1766	+1758	L1 -OM2	1.00016089
4	+320	+318/+364	+365	L1 -OM3	1.00003449
5	+93	+94	+94	L1 -OM4	1.00000886
6	-310	-307	-306	L1 -PSI	1.00000001
7	+75	-75	-72	3*L1 -4*L2 +OM2	1.00710739

----------- SERIES FOR XI-2 (COS) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	-18	-23	-20	OM2 -OM3	-.00025374
2	-27	-30	-31	2*L3-2*PIJ -2*G	.99106592
3	+565	+574	+574	L2 -L3	.50364739
4	+45	+46	+45	L2 -L4	.78721450
5	-101	-80	-78	L2 -PI1	.99840966
6	-825	-825	-825	L2 -PI2	.99953015
7	-3341	-3077	-3135	L2 -PI3	.99993120
8	-1445	-1294	-1395	L2 -PI4	.99998152
9	-15			L2 -PIJ -G	.99918036
10	-51	-51	-51	2*L2 -2*L4	1.57442900
11	+166	+167	+166	2*L2-2*OM2	2.00064592
12	+18	+23	+20	2*L2 -OM2 -OM3	2.00039218
13	-55	-53	-58	5*L2 -5*L3	2.51823697
14	-30	-20	-23	L1 -2*L2 +PI4	.00731327
15	-68	-49	-54	L1 -2*L2 +PI3	.00736359
16	+94558	+94296	+94239	L1 -L2	1.00729479
17	+49	+73	+56	L1 -2*L3 +PI4	1.01460806
18	+112	+176	+133	L1 -2*L3 +PI3	1.01468838
19	-19	-24	-18	L1 -2*L3 +PI2	1.01505943
20	+535	+540	+535	L1 -L3	1.51094218
21	+31	+22	+29	L1 -PI3	2.00722599
22	-314	-310	-319	2*L1 -2*L2	2.01458958
23	-88	-87	-85	2*L1 -2*L3	3.02188437
24	+22			4*L1 -4*L2	4.02917916

----------- SERIES FOR V-2 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+125			-2*PIJ+2*PSI -2*G	-.00163933
2	-1322			-2*PIJ+2*PSI	-.00000005
3	-186			-2*PIJ +OM3 +PSI -2*G	-.00170853
4	+24			-2*PIJ +OM2 +PSI -2*G	-.00196227
5	+32	+27	+22	-OM2 +PSI	.00032294
6	+229	-182/-208	+223	-OM3 +PSI	.00006920
7	+172	+160	+148	-OM4 +PSI	.00001777
8	-1842	-1828	-1596	G	.00081964
9	-321	-205	-655	2*G	.00163929
10	+18			2*G' -2*G +PHI4	-.00097917
11	+19	+19	+19	2*G' -G +PHI1	-.00015952
12	-15			5*G' -3*G +PHI1	-.00080863
13	-150	-148	-148	5*G' -2*G +PHI2	.00001101
14	+81	+98	+94	OM3 -OM4	-.00005143
15	+57	+57	+45	OM2 -OM3	-.00025374
16	+56			PI4 -PIJ	.00001848
17	+2720	+81	+1904	PI3 -PI4	.00005032
18	+35			PI3 -PI4 +OM3 -OM4	-.00000111
19	-74			PI2 -PI3	.00040105
20	-26			PI2 -PI4	.00045137
21	-31			PI1 -PI2	.00112048
22	+140			PI1 -PI3	.00152154
23	+92			PI1 -PI4	.00157186
24	-807			PI1 +PI3 -2*PIJ -2*G	.00001985
25	+55	+59	+62	2*L3-2*PIJ -2*G	.99106592
26	-89			3*L3 -7*L4+4*PI4	-.00036677
27	+73			3*L3 -7*L4 +PI3 +3*PI4	-.00031646
28	-25			3*L3 -7*L4+2*PI3+2*PI4	-.00026614
29	-2036	-2063	-2055	L2 -L3	.50364739
30	-109	-113	-109	L2 -L4	.78721450
31	+191	+161	+156	L2 -PI1	.99840966
32	+1651	+1650	+1650	L2 -PI2	.99953015
33	+6682	+6154	+6269	L2 -PI3	.99993120
34	+2879	+2588	+2789	L2 -PI4	.99998152
35	+30			L2 -PIJ -G	.99918036
36	-18			2*L2 -3*L3 +PI4	.51096067
37	-39			2*L2 -3*L3 +PI3	.51101098
38	+78	+78	+78	2*L2 -2*L4	1.57442900
39	-166	-167	-166	2*L2-2*OM2	2.00064592
40	-18	-23	-20	2*L2 -OM2 -OM3	2.00039218
41	+74	+66	+77	5*L2 -5*L3	2.51823697
42	+30			L1 -2*L2 -PI3 +2*PIJ +2*G	.00886528
43	+3414	3096/3138	+3228	L1 -2*L2 +PI4	.00731327
44	+7705	7676/7775	+7116	L1 -2*L2 +PI3	.00736359
45	-1435	-1310/-1336	-1317	L1 -2*L2 +PI2	.00776464
46	-2154	-1990/-1992	-1802	L1 -2*L2 +PI1	.00888513
47	-187293	-186778	-186827	L1 -L2	1.00729479
48	-94	-144	-110	L1 -2*L3 +PI4	1.01460806
49	-222	-347	-263	L1 -2*L3 +PI3	1.01465838
50	+39	+47	+35	L1 -2*L3 +PI2	1.01505943
51	-16	-4	-15	L1 -2*L3 +PI1	1.01617992
52	-821	-838	-823	L1 -L3	1.51094218
53	-19	-20	-19	L1 -PI2	2.00682494
54	-79	-54	-72	L1 -PI3	2.00722599
55	-32	-24	-30	L1 -PI4	2.00727631
56	-36			2*L1 -4*L2 +OM3 +PSI	.01452034
57	+44			2*L1 -4*L2+2*OM3	.01445114
58	-72			2*L1 -4*L2 +OM2 +OM3	.01419740
59	-539			2*L1 -4*L2+2*OM2	.01394366
60	-27			2*L1 -4*L2+2*PIJ +2*G	.01622887
61	-66			2*L1 -4*L2 +PI3 +PI4	.01467686
62	-76			2*L1 -4*L2+2*PI3	.01472718
63	+953	+945	+955	2*L1 -2*L2	2.01458958
64	+93	+95	+95	2*L1 -2*L3	3.02188437
65	-18			4*L1 -4*L2	4.02917916

----------- SERIES FOR LAT-2 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+17	+17	+17	L2 -2*PIJ +PSI -3*G	.99754104
2	+143	+136	+137	L2 -2*PIJ +PSI -2*G	.99836069
3	-136	-138	-132	L2 -OM1	1.00132201
4	+81490	+81490	+81490	L2 -OM2	1.00032296
5	+4397	+4368/4995	+5107	L2 -OM3	1.00006922
6	+1026	+1027	+1044	L2 -OM4	1.00001779
7	-19	-17	-17	L2 -PSI -G	.99918038
8	-2988	-2966	-3027	L2 -PSI	1.00000002
9	+35	+40	+40	L2 -PSI +G	1.00081967
10	-28	-47	-34	L1 -2*L3 +OM3	1.01452036
11	+264	+245	+256	L1 -2*L3 +OM2	1.01426662
(12	+11	+19	+12	L1 -2*L3 +PSI	1.0145896)

----------- SERIES FOR XI-3 (COS) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+24	+27	+27	-OM3 +PSI	.00013942
2	-9	-11	-11	OM3 -OM4	-.00010361
3	+10	+10	+10	PI3 -PI4	.00010138
4	+234	+234	+233	L3 -L4	.57130173
5	+18	+27	+17	L3 -PI2	.99905339
6	-15164	-15164	-15164	L3 -PI3	.99986139
7	-6568	-6435	-6656	L3 -PI4	.99996277
8	-23			L3 -PIJ -G	.99834867
9	-20	-21	-23	L3 +PI4 -2*PIJ -2*G	.99673456
10	-51	-47	-56	L3 +PI3 -2*PIJ -2*G	.99683594
11	+31			2*L3 -3*L4 +PI4	.71394243
12	-1399	-1399	-1396	2*L3 -2*L4	1.14260346
13	-11	-11	-12	2*L3-2*PI3	1.99972278
14	-10	-10	-10	2*L3 -PI3 -PI4	1.99982416
15	-27	-28	-27	2*L3-2*PIJ -2*G	1.99669733
16	+24	+32	+32	2*L3-2*OM3	2.00027892
17	+9	+11	+11	2*L3 -OM3 -OM4	2.00017531
18	-24	-27	-27	2*L3 -OM3 -PSI	2.00013951
19	-16			3*L3 -4*L4 +PI4	1.28524416
20	-124	-124	-124	3*L3 -3*L4	1.71390520
21	-33	-33	-33	4*L3 -4*L4	2.28520693
22	-11	-11	-11	5*L3 -5*L4	2.85650866
23	+6341	+6335	+6329	L2 -L3	1.01469679
24	+9	+10	+10	L2 -PI3	2.01455818
25	+31	+31	+30	2*L2 -3*L3 +PI4	1.02943081
26	+70	+71	+70	2*L2 -3*L3 +PI3	1.02953219
27	+10	+8	+8	L1 -2*L2 +PI4	.01473402
28	+20	+18	+18	L1 -2*L2 +PI3	.01483540
29	-153	-153	-152	L1 -L2	2.02939358
30	+147	+147	+141	L1 -L3	3.04409037
31	+11	+12	+13	2*L1 -2*L2	4.05878716
(32	+7	+77	+7	L3 -2*L4 +PI4	.1416404)

----------- SERIES FOR V-3 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+10			-PI3 +PI4 -OM3 +PSI	.00003804
2	+25			-2*PIJ+2*PSI -2*G	-.00330276
3	-1733			-2*PIJ+2*PSI	-.00000009
4	+24			-2*PIJ +OM3 +PSI -2*G	-.00344218
5	+14	+9	+9	-OM2 +PSI	.00065062
6	+379	-330/-377	+372	-OM3 +PSI	.00013942
7	+282	+270	+246	-OM4 +PSI	.00003580
8	-2332	-2348	-2008	G	.00165133
9	-60	-72	-27	2*G	.00330267
10	+10			G' -G +PHI3	-.00098636
11	+22			2*G' -2*G +PHI4	-.00197273
12	+26	+26	+26	2*G' -G +PHI1	-.00032139
13	+11			3*G' -2*G +PHI2 +PHI3	-.00130776
14	+9			3*G' -G +PHI1 -PHI2	.00034358
15	-19			5*G' -3*G +PHI1	-.00162915
16	-208	-207	-206	5*G' -2*G +PHI2	.00002218
17	+136	+167	+158	OM3 -OM4	-.00010361

Table 5 (continued)

----------- SERIES FOR V-3 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
18	+21	+13	+13	OM2 -OM3	-.00051121
19	+98			PI4 -PIJ	.00003723
20	+4391	+121	+1487	PI3 -PI4	.00010138
21	+63			PI3 -PI4 +OM3 -OM4	-.00000224
22	-93			PI2 -PI3	.00080800
23	-39			PI2 -PI4	.00090938
24	-26			PI1 -PI3	.00306544
25	-11			PI1 -PI4	.00316681
26	+17			PI1 +PI4 -2*PIJ -2*G	-.00006139
27	+1917			PI1 +PI3 -2*PIJ -2*G	.00003999
28	-10			L4 -PI4	.42866103
29	-85	-82/-1054	-83	L3 -2*L4 +PI4	.14264070
30	+66	+2	+67	L3 -2*L4 +PI3	.14274207
31	-750	-749	-748	L3 -L4	.57130173
32	-37	-53	-33	L3 -PI2	.99905339
33	+30332	+30341	+30341	L3 -PI3	.99986139
34	+13136	+12876	+12876	L3 -PI4	.99996277
35	+46			L3 -PIJ -G	.99834867
36	+40	+42	+46	L3 +PI4 -2*PIJ -2*G	.99673456
37	+11			L3 +PI3 -2*PIJ -3*G	.99518461
38	+102	+94	+93	L3 +PI3 -2*PIJ -2*G	.99683594
39	-80			2*L3 -3*L4 +PI4	.71394243
40	+13			2*L3 -3*L4 +PI3	.71404381
41	+2560	+2560	+2554	2*L3 -2*L4	1.14260346
42	+29	+29	+29	2*L3-2*PI3	1.99972278
43	+25	+25	+24	2*L3 -PI3 -PI4	1.99982416
44	+37	+38	+38	2*L3-2*PIJ -2*G	1.99669733
45	-24	-32	-32	2*L3-2*OM3	2.00027892
46	-9	-11	-11	2*L3 -OM3 -OM4	2.00017531
47	+24	+25/+28	+27	2*L3 -OM3 -PSI	2.00013951
48	-145			3*L3 -7*L4+4*PI4	-.00073894
49	+121			3*L3 -7*L4 +PI3 +3*PI4	-.00063756
50	-44			3*L3 -7*L4+2*PI3+2*PI4	-.00053619
51	+27			3*L3 -4*L4 +PI4	1.28524416
52	+180	+180	+180	3*L3 -3*L4	1.71390520
53	+42	+42	+42	4*L3 -4*L4	2.28520693
54	+13	+13	+13	5*L3 -5*L4	2.85650866
55	+34	-10	+33	L2 -3*L3 +2*L4	-.12790667
56	-12050	-12074/-12039	-12029	L2 -L3	1.01469679
57	-24	-25	-24	L2 -PI3	2.01455818
58	-10	-10	-10	L2 -PI4	2.01465956
59	-62	-61	-58	2*L2 -3*L3 +PI4	1.02943081
60	-137	-137	-137	2*L2 -3*L3 +PI3	1.02953219
61	-552	-592/-512	-538	L1 -2*L2 +PI4	.01473402
62	-1236	-1365/-1175	-1228	L1 -2*L2 +PI3	.01483540
63	+630	+579	+597	L1 -2*L2 +PI2	.01564340
64	+89	+62	+49	L1 -2*L2 +PI1	.01790084
65	+190	+191	+188	L1 -L2	2.02939358
66	+210	+209	+204	L1 -L3	3.04409037
67	+9			2*L1 -4*L2 +OM3 +PSI	.02925407
68	-12			2*L1 -4*L2+2*OM3	.02911466
69	+28			2*L1 -4*L2+2*OM2	.02809225
70	+18			2*L1 -4*L2 +PI3 +PI4	.02956942
71	+21			2*L1 -4*L2+2*PI3	.02967080
72	-13	-12	-12	2*L1 -2*L2	4.05878716

----------- SERIES FOR LAT-3 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+37	+37	+37	L3 -2*PIJ +PSI -3*G	.99504595
2	+321	+317	+321	L3 -2*PIJ +PSI -2*G	.99669728
3	-15	-15	-15	L3 -2*PIJ +PSI -G	.99834862
4	-36	-42	-35	L3 -2*PIJ +PSI	.99999995
5	-2770	-2796	-2777	L3 -OM2	1.00065067
6	+31108	+31108/35572	+35572	L3 -OM3	1.00013946
7	+6080	+6090	+6067	L3 -OM4	1.00003585
8	-45	-45	-45	L3 -PSI -G	.99834871
9	-15146	-15179	-15064	L3 -PSI	1.00000005
10	+51	+53	+53	L3 -PSI +G	1.00165138
11	+10	+10	+10	2*L2 -3*L3 +PSI	1.02939353
12	-21	-25	-24	2*L2 -3*L3 +OM3	1.02925412
13	+30	+27	+30	2*L2 -3*L3 +OM2	1.02874291

----------- SERIES FOR XI-4 (COS) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	-19	-24	-22	-OM3 +PSI	.00032521
2	+177	+167	+178	-OM4 +PSI	.00008352
3	+11			G	.00385197
4	+12	+15	+14	OM3 -OM4	-.00024169
5	-13	-14	-14	PI3 -PI4	.00023647
6	+1812	+1814	+1784	L4 -PI3	.99967667
7	-24			L4 -PI4 -2*PIJ+2*PSI	.99991293
8	-17	-16	-16	L4 -PI4 -G	.99606117
9	-73725	-73725	-73725	L4 -PI4	.99991315
10	+15	+16	+16	L4 -PI4 +G	1.00376512
11	+30			L4 -PI4 +2*PIJ-2*PSI	.99991336
12	-5			L4 -PIJ -2*G	.99229605
13	-89			L4 -PIJ -G	.99614803
14	+180			L4 -PIJ	1.00000000
15	-6			L4 +PI4 -2*PIJ -4*G	.98467896
16	-62			L4 +PI4 -2*PIJ -3*G	.98853093
17	-544	-550	-603	L4 +PI4 -2*PIJ -2*G	.99238290
18	+27			L4 +PI4 -2*PIJ -G	.99623488
19	+6			L4 +PI4 -2*PIJ	1.00008685
20	+6			L4 +PI4 -OM4 -PSI	1.00017058
21	-9			L4 +PI3 -2*PI4	1.00014962
22	+14			L4 +PI3 -2*PIJ -2*G	.99261938
23	+13	+14	+13	2*L4 -PI3 -PI4	1.99958982
24	-272	-271	-272	2*L4-2*PI4	1.99982629
25	-25			2*L4-2*PIJ -3*G	1.98844408
26	-155	-151	-150	2*L4-2*PIJ -2*G	1.99229605
27	-12	-15	-14	2*L4 -OM3 -OM4	2.00040894
28	+19	+24	+22	2*L4 -OM3 -PSI	2.00032542
29	+56	+53	+56	2*L4-2*OM4	2.00016725
30	-177	-167	-178	2*L4 -OM4 -PSI	2.00008373
31	+139	+132	+140	2*L4-2*PSI	2.00000021
32	-23	-36	-27	L3 -2*L4 +PI4	.33272982
33	+20	+1	+20	L3 -2*L4 +PI3	.33296630
34	+997	+997	+1002	L3 -L4	1.33264297
35	+25			2*L3 -3*L4 +PI4	1.66537279
36	+181	+181	+182	2*L3 -2*L4	2.66528594
37	+4			3*L3 -4*L4 +PI4	2.99801576
38	+43	+43	+43	3*L3 -3*L4	3.99792891
39	+14	+14	+14	4*L3 -4*L4	5.33057188
40	+5			5*L3 -5*L4	6.66321485

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT

----------- SERIES FOR XI-4 (COS) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
41	-8			L2 -3*L3 +2*L4	-.29836061
42	+92	+92	+92	L2 -L4	3.69956830
43	+100	+101	+98	L1 -L4	8.43341897
(44	-2	-19	-2	L1 -2*L2 +L4	1.0342823)

----------- SERIES FOR V-4 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+8			-PI3 -PI4 +2*PSI	-.00041039
2	-9			-PI3 -PI4 +OM4 +PSI	-.0004939
3	+27			-PI3 +PI4 -OM4 +PSI	-.00015296
4	-381			-2*PI4+2*PSI	-.00017392
5	+330			-2*PI4 +OM4 +PSI	-.00025744
6	-19			-2*PI4 +OM3 +PSI	-.00049912
7	+8			-PI4 -PIJ +2*PSI	-.00008707
8	-5			-PI4 -PIJ +OM4 +PSI	-.00017058
9	+111			-PI4 +PIJ -OM4 +PSI	-.00000333
10	+8			-2*PIJ+2*PSI -3*G	-.01155614
11	+73			-2*PIJ+2*PSI -2*G	-.00770416
12	-5349			-2*PIJ+2*PSI	-.00000021
13	+16			-2*PIJ +OM4 +PSI -2*G	-.00778768
14	-100	0	+32	-OM3 +PSI	.00032521
15	+150			-2*OM4+2*PSI	.00016704
16	+2236	+937	+916	-OM4 +PSI	.00008352
17	-5589	-5613	-5608	G	.00385197
18	-203	-203	-203	2*G	.00770395
19	-10			3*G	.01155592
20	+24			G' -G +PHI3	-.00230084
21	+11			G' +PHI1-2*PH2	.00155114
22	+52			2*G' -2*G +PHI4	-.00460167
23	+61	+62	+62	2*G' -G +PHI1	-.00074970
24	+25			3*G' -2*G +PHI2 +PHI3	-.00305053
25	+21			3*G' -G +PHI1 -PHI2	.00080144
26	-45			5*G' -3*G +PHI1	-.00380023
27	-495	-491	-491	5*G' -2*G +PHI2	.00005175
28	-45		-1	OM3 -OM4	-.00024169
29	+5			PI4 -PIJ -G	-.00376512
30	+231			PI4 -PIJ	.00008685
31	+11			2*PI4-2*PIJ -2*G	-.00753024
32	-10			2*PI4 -OM3 -OM4	.00058264
33	+79			2*PI4-2*OM4	.00034095
34	-13			PI3 -PI4 -OM4 +PSI	.00031999
35	-6966	+30	-4032	PI3 -PI4	.00023647
36	+48			PI3 -PI4 +OM3 -OM4	-.00000521
37	-3632	-3628	-3567	L4 -PI3	.99967667
38	+48			L4 -PI4 -2*PIJ+2*PSI	.99991293
39	+10	+6	+6	L4 -PI4 -OM4 +PSI	.99999667
40	+33	+31	+31	L4 -PI4 -G	.99606117
41	+147467	+147464	+147464	L4 -PI4	.99991315
42	-31	-31	-31	L4 -PI4 +G	1.00376512
43	-6	-6	-12	L4 -PI4 +OM4 -PSI	.99982963
44	-61			L4 -PI4 +2*PIJ-2*PSI	.99991336
45	+10			L4 -PIJ -2*G	.99229605
46	+178			L4 -PIJ -G	.99614803
47	-360			L4 -PIJ	1.00000000
48	+5			L4 +PI4 -2*PIJ -5*G' +2*G -PHI1	1.00003511
49	+12			L4 +PI4 -2*PIJ -4*G	.98467896
50	+124			L4 +PI4 -2*PIJ -3*G	.98853093
51	+1091	+1108	+1212	L4 +PI4 -2*PIJ -2*G	.99238290
52	-55			L4 +PI4 -2*PIJ -G	.99623488
53	-12			L4 +PI4 -2*PIJ	1.00008685
54	-13			L4 +PI4 -OM4 -PSI	1.00017058
55	+6			L4 +PI4 -2*PSI	1.00008707
56	+17			L4 +PI3 -2*PI4	1.00014962
57	-28			L4 +PI3 -2*PIJ -2*G	.99261938
58	-34	-35	-33	2*L4 -PI3 -PI4	1.99958982
59	+679	+678	+680	2*L4-2*PI4	1.99982629
60	+36			2*L4-2*PIJ -3*G	1.98844408
61	+218	+207	+220	2*L4-2*PIJ -2*G	1.99229605
62	-5			2*L4-2*PIJ -G	1.99614803
63	+12	+15	+14	2*L4 -OM3 -OM4	2.00040894
64	-19	-21/-24	-22	2*L4 -OM3 -PSI	2.00032542
65	-56	-53	-56	2*L4-2*OM4	2.00016725
66	+177	+167	+178	2*L4 -OM4 -PSI	2.00008373
67	-139	-132	-140	2*L4-2*PSI	2.00000021
68	+163	+175	+205	L3 -2*L4 +PI4	.33272982
69	-101	-8	-102	L3 -2*L4 +PI3	.33296630
70	-399	-400	-401	L3 -L4	1.33264297
71	+9			2*L3 -4*L4+2*PI4	.66545965
72	-38			2*L3 -3*L4 +PI4	1.66537279
73	+6			2*L3 -3*L4 +PI3	1.66560927
74	-200	-200	-201	2*L3 -2*L4	2.66528594
75	+6			3*L3 -7*L4+2*PI4 +OM4 +PSI	-.00198111
76	+205			3*L3 -7*L4+4*PI4	-.00172368
77	-171			3*L3 -7*L4 +PI3 +3*PI4	-.00148720
78	+59			3*L3 -7*L4+2*PI3+2*PI4	-.00125073
79	-10			3*L3 -7*L4+3*PI3 +PI4	-.00101426
80	+6			3*L3 -6*L4+3*PI4	.99818947
81	-8			3*L3 -4*L4 +PI4	2.99801576
82	-42	-43	-42	3*L3 -3*L4	3.99792891
83	-13	-13	-13	4*L3 -4*L4	5.33057188
84	-45			L2 -3*L3 +2*L4	-.29836061
85	+89	+89	+89	L2 -L4	3.69956830
86	+101	+101	+97	L1 -L4	8.43341897
(87	+4	+59	+5	L1 -2*L2 +L4	1.0324282)
(88	-4	-12	-14	L3 -PI4	2.3325568)

----------- SERIES FOR LAT-4 (SIN) -----------

INDEX	LIESKE	SAMPSON	VU-SAGNIER	ANGLE	RATIO N/NSAT
1	+8			L4 -2*PIJ -OM4 +2*PSI	1.00008341
2	+8	+6	+8	L4 -2*PIJ +PSI -4*G	.98459200
3	+88	+88	+88	L4 -2*PIJ +PSI -3*G	.98844397
4	+769	+768	+769	L4 -2*PIJ +PSI -2*G	.99229594
5	-38	-38	-37	L4 -2*PIJ +PSI -G	.99614792
6	+9	+6	+6	L4 -OM1	1.00621290
7	-5178	-5177/-5920	-5990	L4 -OM3	1.00032531
8	-7	-6	-7	L4 -OM4 -G	.99623165
9	+47460	+47460	+47460	L4 -OM4	1.00008362
10	+7	+6	+7	L4 -OM4 +G	1.00393560
11	-102	-102	-102	L4 -PSI -G	.99614813
12	-74602	-74902	-72860	L4 -PSI	1.00000011
13	+104	+106	+106	L4 -PSI +G	1.00385208
14	-10			L4 -PSI +5*G' -2*G +PHI2	1.00005185
15	-11			L3 -2*L4 +PSI	.33264286
16	+7			L3 -2*L4 +OM4	.33255935
(17	+2	-383	-4	L4 -2*PIJ +PSI	.9999998)

Table 6. Parameters for adjusting present theory to Sampson

ε	A	B	β	A	B
1	+0.053211	+0.0925451	1	−0°.01702668	− 0°.0212216
2	−0.002373	−0.0038329	2	−0.04669171	− 0.0467981
3	−0.017883	−0.0289340	3	—	—
4	−0.019815	−0.0032786	4	−0.08903931	− 0.0939410
5		+5.52 $E-6$	5	0.	0.
6		+1.97 $E-9$	6	+0.91058989	+ 2.7273485
7		+1.568 $E-8$	7	−5.27767720	− 4.9601321
8		−1.521 $E-7$	8	−0.06507313	− 0.1748600
9		0.	9	−0.02035130	− 0.0365290
10		0.	10	0.	0.
11	−0.004002	−0.0098435	11	+0.52898522	+ 0.2415856
12		+2.8721829	12	+0.11419524	+ 0.0226533
13		−0.0032266	13	+0.32356643	− 0.1460358
14		−0.1713952	14	−0.76503236	− 0.4133256
15		+0.1091876	15	−0.01575083	− 0.0058319
16		−0.1470758	16	0.	0.
17		+0.0380236	17	+3.7680334	+ 1.1150624
18		+0.0098544	18	+0.33	+ 0.33
19		+0.0020228	19	+1.61	+ 1.61
20		0.	20	0.	−259.18
21		+0.0036081	21	0.	−157.12
22		+0.0017704	22	−0.12025885	+ 0.0020596
23		+0.0025175			
24		+0.0208736			
25		+0.0002797			
26		+0.0001224			
27		0.			
28		0.			

Table 7. Derived rate terms for various theories

	Sampson	Lieske (ε=0)	Lieske A	Lieske B
n_1	203.488954208	203.488954208	Same as ε=0	203.488954609
n_2	101.374723445	101.374723445	Same as ε=0	101.374725034
n_3	50.317608063	50.3176080635	Same as ε=0	50.317610247
n_4	21.571071403	21.571071403	Same as ε=0	21.571068122
$\dot{\pi}_1$	+ 0.1578973	+ 0.16122004	+ 0.16136240	+ 0.16176842
$\dot{\pi}_2$	+ 0.0470774	+ 0.04763124	+ 0.04652139	+ 0.04572976
$\dot{\pi}_3$	+ 0.0069131	+ 0.00697450	+ 0.00696079	+ 0.00693422
$\dot{\pi}_4$	+ 0.0018592	+ 0.00187351	+ 0.00185923	+ 0.00184228
$\dot{\omega}_1$	− 0.1340687	− 0.13401884	− 0.13349115	− 0.13334541
$\dot{\omega}_2$	− 0.0327375	− 0.03274001	− 0.03273748	− 0.03264050
$\dot{\omega}_3$	− 0.0070158	− 0.00701735	− 0.00699982	− 0.00697295
$\dot{\omega}_4$	− 0.0017936	− 0.00180387	− 0.00179150	− 0.00177488
$\sqrt{L}$	+ 0.1763438	+ 0.17190889	+ 0.17460381	+ 0.17662634
$\dot{\psi}$	− 0.00000231	− 0.00000229	− 0.00000226	− 0.00000213
$10^7 m_1$	449.7	449.7	473.6	491.3
$10^7 m_2$	252.9	252.9	252.3	251.9
$10^7 m_3$	798.8	798.8	784.5	775.7
$10^7 m_4$	450.4	450.4	441.5	448.9

where

$$\xi = \sum_r x_r,$$

$$x_r = \varkappa_r \cos p_r t.$$

Such terms are only appreciable for Satellite IV, in which case they generally cancel the terms calculated from the standard formulation of Sampson and Equation (III-6). The required corrections are given in Table 4. Although similar effects would develop in the second-degree terms $\Delta\zeta$, they are not derived here since it is expected that users will employ the "time-completed" feature of the theory, in which case $\Delta\zeta$ is not employed.

Microfiche Tables A.1–A.127, coupled with Table 4, enable one to calculate the theory of motion of the

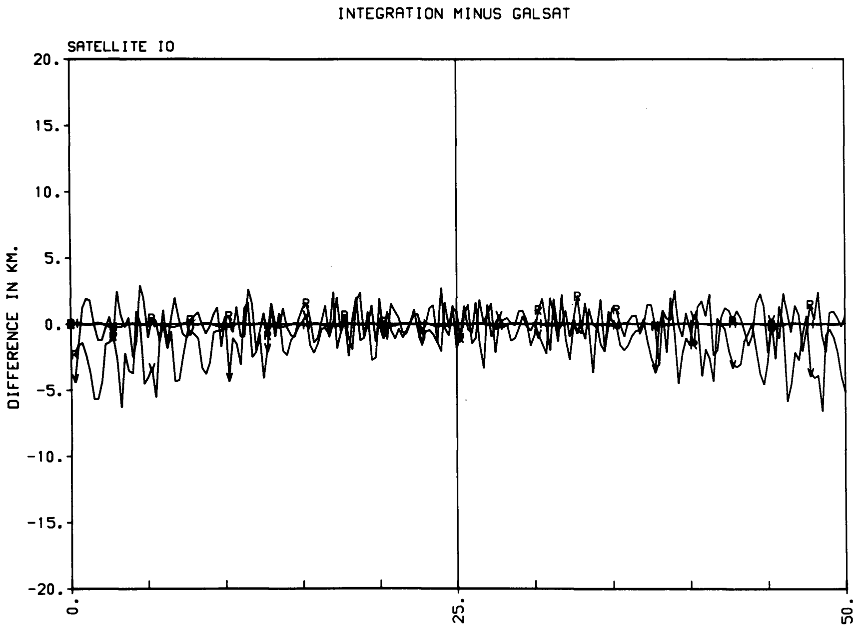

Fig. 3

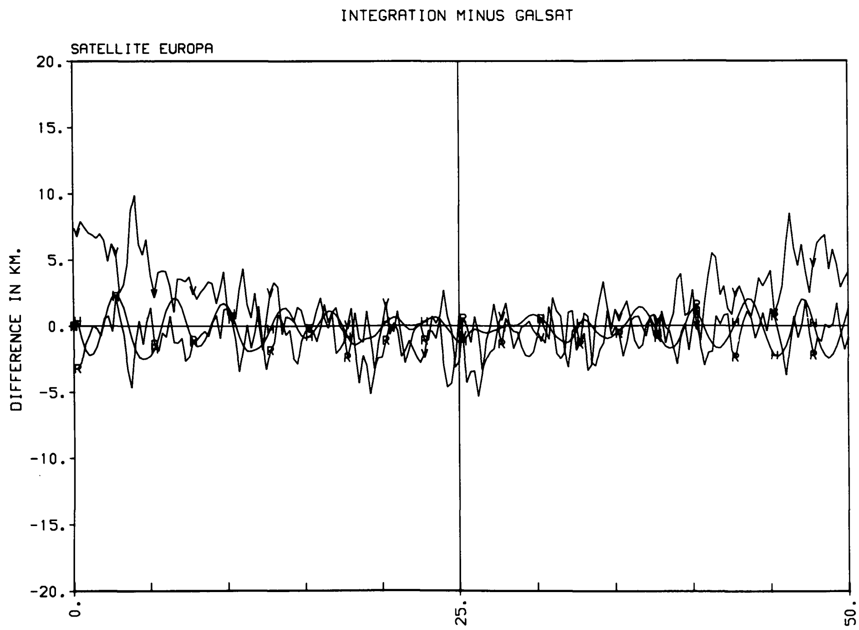

Fig. 4

Figs. 3—6. Differences between numerical integration and the present theory in Jupiter-centered radial (*R*), in-track (*V*), and out-of-plane (*H*) components for the four satellites

Galilean satellites and the partial derivatives for any user-defined set of numerical values for the 49 input constants.

The resultant theory (with all parameters ε and β being set equal to zero) as well as those of Sampson (1921) and the formulation Vu and Sagnier (1974) are presented in Table 5. In Column 3 of the table, Sampson's results are given, sometimes of the form A/B. Whenever two entries occur, A is the coefficient adapted as the preliminary solution for the present theory, usually resulting from revisions by Sampson (*Theory*, pp. 263–270) while B is the coefficient originally given by Sampson (*Theory*, Table LI, pp. 236–246).

If one compares the $\varepsilon=\beta=0$ theory with Sampson's series, the resultant coordinates differ on the order of 500 km, the difference primarily being due to the fact that

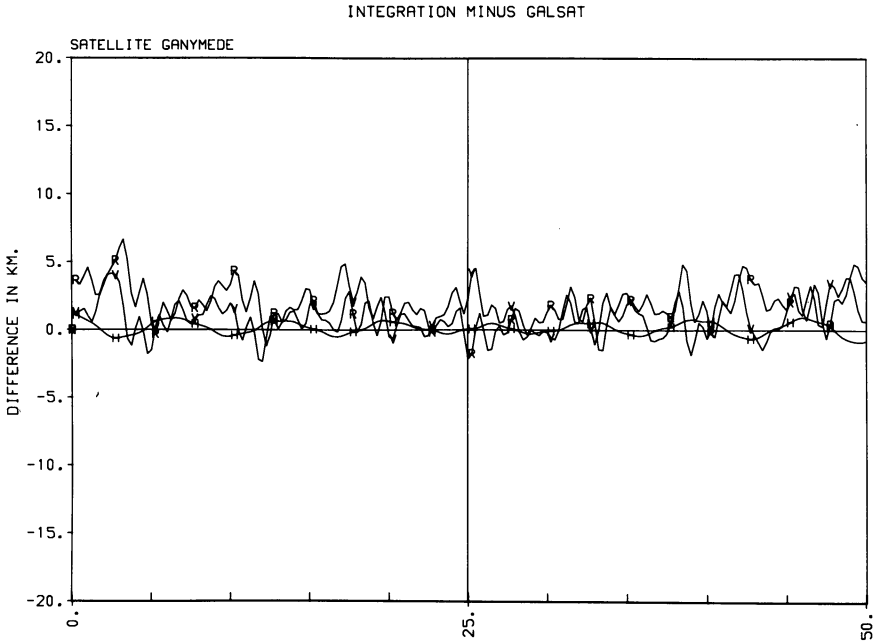

Fig. 5

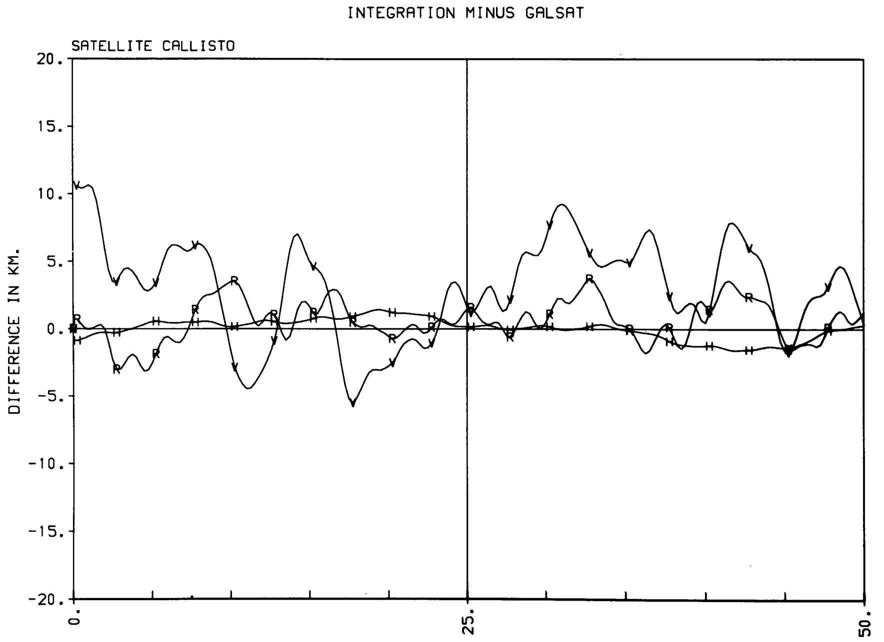

Fig. 6

the additional long-period terms contained in the present theory effectively alter the mean longitudes. In Table 6, two sets of $\varepsilon-\beta$ values are presented, the values being derived by fitting the present theory to Sampson's. Column A of Table 6 contains $\varepsilon-\beta$ values obtained by adjusting the masses and J_2 according to the manner of Sampson outlined earlier, coupled with changes in the angles (β-parameters).

The resultant theory and sample computations are contained in Microfiche Tables C.1–C.16 and on Microfiche 3 (where they are called "SAMCARD/BETA"). The positions differ from Sampson's with a standard deviation of 38 km. In Column B of Table 6, both ε and β parameters are adjusted with a resulting standard deviation of 16 km. The resultant theory and sample computations are obtained in Microfiche 3 under the heading "SAMCARD/EPSBETA". In Table 7 the derived rates (in deg/day) for four theories are given for comparison purposes. The second column contains the values employed by Sampson, while the remaining three columns correspond to the three sets mentioned above. It should be noted that none of these sets of $\varepsilon-\beta$ values have been adjusted to fit modern observations, but are comparisons only with Sampson's theory. Adjustments of the $\varepsilon-\beta$ values to real data are currently in progress. As noted by Aksnes and Franklin (1976) the rates of Sampson and those given by de Sitter (1931) in his Darwin lecture lead to differing values of nodes and peri-apses at current epochs. Hence it will not be surprising if some of the β variables are as large as 100° when derived from analysis of current observations.

In order to provide some feeling for the mathematical integrity of the present theory, Peters (1975) has provided a numerical integration of the Galilean satellites and the theory has been adjusted to the integration. The results for Satellites I–IV are presented in Figures 3–6 for a 50-day interval. The three components plotted are the differences in radial position (R), intrack (V) and out-of-plane (H). Components V and H correspond closely to longitude and latitude, respectively. Comparisons over longer periods suggest a term of the present theory is not complete and will require some modification (see concluding comments at end of paper). In general, it appears that the mathematical integrity of the present theory is better than 10 km. After the next section concerning the calculation of positions and partial derivatives, the paragraph under Concluding Remarks will suggest how this goal is obtained.

Calculation of Positions and Velocities

Given a series for ξ (Microfiche Tables A.44–A.51), υ (Microfiche Tables A.52–A.116 and A.40–A.43) or ζ (Microfiche Tables A.117–A.123), then a single term is represented by $C^{\sin}_{\cos}\,\Theta$ where the coefficient is

$$C=C_{i;j}\prod_{k=1}^{28}\varepsilon_k^{i_k}$$
$$=C_{i_1,i_2,\ldots,i_{28};j_1,j_2,\ldots,j_{22}}\varepsilon_1^{i_1}\varepsilon_2^{i_2}\ldots\varepsilon_{28}^{i_{28}} \quad (31)$$

with $0\leqq i_k\leqq 3$ and where the angle Θ is defined as

$$\Theta=\sum_{l=1}^{22} j_l\theta_l=j_1\theta_1+j_2\theta_2+\ldots+j_{22}\theta_{22}. \quad (32)$$

The meaning of parameters ε_k is given in Table 2 and θ_l is of the form (see Table 3)

$$\theta_l=\theta_l^{(0)}+\beta_l+\mu_l(t-t_0) \quad (33)$$

with $\theta_l^{(0)}$, β_l defined in Table 3 (Column 3) and μ_l in Table 3 (Column 4) and in Microfiche Tables A.12–A.35 [for $\dot{\pi}_j$, $\dot{\omega}_j$ $(j=1,4)$, $\dot{\psi}$ and $\sqrt{L}$]. The angles $\theta_l^{(0)}$ are fixed numbers and are Sampson's values at epoch $t_0=$J.E.D. 2443000.5 (1976, Aug. 10). The parameters β_l $(l=1,22)$ are arbitrary numbers specified by the user and represent the constants of integration of the theory. (Only 21 of the β_l are arbitrary, β_3 being specified by the libration constraint as $\beta_3=-\frac{1}{2}\beta_1+\frac{3}{2}\beta_2$.)

In general, the values μ_l $(l=1,22)$ are functions of the ε_k $(k=1,28)$,

$$\mu_l=\sum_n M_{l;0}^{(n)}\varepsilon_1^{i_1}\varepsilon_2^{i_2}\ldots\varepsilon_{28}^{i_{28}} \quad (34)$$

and are given in Table 3 and Microfiche Tables A.12–A.35, although the first four μ_l's (the mean motions of Satellites I–IV) are rather simple (see Table 3):

$$\mu_1=203.48895\,4208\ \ (1+\varepsilon_6)$$
$$\mu_2=101.37472\,3445\ \ (1+\varepsilon_7)$$
$$\mu_3=\ 50.31760\,80635$$
$$[1-2\varepsilon_6+3\varepsilon_7-2.20451\,8497\,10^{-2}(\varepsilon_6-\varepsilon_7)]$$
$$\mu_4=\ 21.57107\,1403\ \ (1+\varepsilon_8), \quad (35)$$

the motions being expressed in deg/day. The expression for μ_3 is a result of the libration constraint $n_1-3n_2+2n_3=0$. The arbitrary constants which replace the mean longitude and mean motion for one of the satellites (arbitrarily chosen to be Satellite III) are the amplitude (ε_9) and phase (β_5) of the free libration. All parameters β are expressed in degrees (see Table 3, Column 3), while the ε_k are dimensionless (see Table 2) with the exception of the amplitude of the free libration, ε_9, which is expressed in radians. Inspection of the microfiche tables shows that the series for ξ, υ, ζ are functions of only the first 25 ε_k and the first 21 β_k, the remaining 3 ε_k and β_{22} being required to obtain the coordinates relative to the earth equator and equinox of 1950.0 and for calculation of partial derivatives.

With specified values of the arbitrary parameters ε_k, β_l one evaluates the series, ξ, for example, as

$$\xi=\sum_m C_{i;j}^{(m)}\prod_{k=1}^{28}\varepsilon_k^{i_k}\cos\Theta^{(m)} \quad (36)$$

and after inserting numerical values for ε and β obtains a much shorter series

$$\xi=\sum_n K^{(n)}\cos\Theta^{(n)} \quad (37)$$

which represents a new theory based upon the specified values of ε and β, and which then can be evaluated for each date at which coordinates and velocities are required. The complete series for ξ, υ, ζ of the type indicated by Equation (36) are given in Microfiche Tables A, while the reduced series of the form Equation (37) for $\varepsilon_k=0$ are given in Table 5 and Microfiche Tables B. Similar reduced tables for the theory specified in Column A of Table 6 are given in Microfiche Tables C.

If one has available the reduced series for ξ, υ and ζ for a satellite, then the coordinates and velocities relative to the earth equator and equinox of 1950.0 are calculated as follows:

Given the time t (Julian Date in ephemeris time) one computes (where $K_j{}^{\sin}_{\cos}\,\Theta_j$ means $\sum_n K_j^{(n)}{}^{\sin}_{\cos}\,\Theta_j^{(n)}$)

$$\xi(t)=K_1\cos\Theta_1$$
$$\upsilon(t)=K_2\sin\Theta_2$$
$$s(t)=\zeta(\tau)=K_3\sin\Theta_3, \quad (38)$$

where $\Theta_j = \Theta_j^{(0)} + \dot{\Theta}_j(t-t_0)$ for $j=1,2$ and $\Theta_3 = \Theta_3^{(0)} + \dot{\Theta}_3(t+\Delta t - t_0)$ for ζ, with $\tau = t + \Delta t$, $\Delta t = \upsilon n^{-1}$ being the so-called "time-completed" (*Theory*, pp. 229 and 230) which enables the latitude series to be shorter than otherwise possible. The "time-completed" aspect may be interpreted as viewing the latitude series as a function of true longitude rather than mean longitude. By employing the "time-completed" one does not require the use of the second-degree series $\Delta\zeta$ given in Microfiche Tables A.124–A.127, in calculating $s(t) = z/\varrho$ from $\zeta(t) = z/a$ [see Fig. 1 and Eq. (2)].

One then calculates

$$\left.\begin{aligned} \dot{\xi} &= -K_1 \dot{\Theta}_1 \sin\Theta_1 \\ \dot{\upsilon} &= \quad K_2 \dot{\Theta}_2 \cos\Theta_2 \\ \dot{s} &= \quad K_3 \Theta_3'(1+\dot{\upsilon}/n)\cos\Theta_3 \end{aligned}\right\} \tag{39}$$

where $\dot{\Theta} = \dfrac{d\Theta}{dt}$, $\Theta' = \dfrac{d\Theta}{d\tau}$.

From Figure 1 and Equation (2) one can calculate the coordinates $(\bar{x}, \bar{y}, \bar{z})$ and velocities $(\dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{z}})$ in the plane of Jupiter's (moving) equator as follows:

$$\left.\begin{aligned} \bar{x} &= a(1+\xi)\cos(l-\psi+\upsilon) \\ \bar{y} &= a(1+\xi)\sin(l-\psi-\upsilon) \\ \bar{z} &= a(1+\xi)s \\ \dot{\bar{x}} &= a\dot{\xi}\cos(l-\psi+\upsilon) - \bar{y}(\dot{l}-\dot{\psi}+\dot{\upsilon}) \\ \dot{\bar{y}} &= a\dot{\xi}\sin(l-\psi+\upsilon) + \bar{x}(\dot{l}-\dot{\psi}+\dot{\upsilon}) \\ \dot{\bar{z}} &= a\dot{\xi} + a(1+\xi)\dot{s} \end{aligned}\right\}, \tag{40}$$

where the scale factor a (originally a series in ε—see Microfiche Tables A.8–A.11) is expressed in a.u. Values of 10^9 a_i (for all $\varepsilon = 0$) are 2819545, 4486187, 7155856 and 12586256 for Satellites I–IV respectively.

Finally, the coordinates $\boldsymbol{r}$ and velocities $\dot{\boldsymbol{r}}$ in the system of the earth equator and equinox of 1950.0 are (see Fig. 2)

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{BA}\bar{\boldsymbol{r}} \\ \dot{\boldsymbol{r}} &= \boldsymbol{B}\dot{\boldsymbol{A}}\boldsymbol{r} + \boldsymbol{BA}\dot{\bar{\boldsymbol{r}}}, \end{aligned} \tag{41}$$

where

$$\bar{\boldsymbol{r}} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \quad \dot{\bar{\boldsymbol{r}}} = \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\bar{z}} \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \dot{\boldsymbol{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{P}(-\varepsilon) \\ \boldsymbol{A} &= \boldsymbol{R}(-\Omega)\boldsymbol{P}(-J)\boldsymbol{R}(-\Phi)\boldsymbol{P}(-I) \end{aligned} \tag{42}$$

with

$$\boldsymbol{P}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\boldsymbol{R}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in which $\Phi = \psi - \Omega$ and where the angles (and their values) ε_0, Ω, J, ψ, I are defined in Tables 2 and 3 (Parameters $\varepsilon_{27}, \beta_{22}, \varepsilon_{26}, \beta_{15}, \varepsilon_{25}$). Coordinates are given in a.u. and velocities in a.u./day. Computer programs (named KODLOD, KODQIK and GALSAT) have been developed and are available for the preceding evaluations. On the Univac 1108 it requires 7 s to regenerate the theories of all four satellites [viz. develop the results of Equation (37) from Equation (36)] from arbitrary values of ε and β. The program calculates coordinates and velocities at the rate of twenty per second. The computer programs and sample output are given on Microfiche 3.

Calculation of Partial Derivatives

The parameters $\varepsilon_k (k=1, 28)$ and $\beta_\ell (\ell = 1, 22)$ represent the constants of the theory and may readily be adjusted. To obtain partial derivatives of the coordinates and velocities with respect to the ε_k and β_ℓ one requires the partial derivatives of C, Θ and μ of Equations (31)–(34). These partials may readily be obtained and are

$$\left.\begin{aligned} &\frac{\partial C}{\partial \beta_\ell} = 0 \qquad (\ell = 1, 22) \\ &\frac{\partial \Theta}{\partial \beta_1} = j_1 - \frac{1}{2} j_3 \\ &\frac{\partial \Theta}{\partial \beta_2} = j_2 + \frac{3}{2} j_3 \\ &\frac{\partial \Theta}{\partial \beta_3} = 0 \\ &\frac{\partial \Theta}{\partial \beta_\ell} = j_\ell \quad \text{for} \quad \ell \geqq 4 \end{aligned}\right\} \tag{43}$$

and

$$\left.\begin{aligned} &\frac{\partial C}{\partial \varepsilon_k} = i_k \varepsilon_k^{i_k - 1} C_{i;j} \varepsilon_1^{i_1} \ldots \varepsilon_{k-1}^{i_{k-1}} \cdot \varepsilon_{k+1}^{i_{k+1}} \ldots \varepsilon_{28}^{i_{28}} \\ &\frac{\partial \Theta}{\partial \varepsilon_k} = \sum_{\ell=1}^{22} j_\ell (t - t_0) \frac{\partial \mu_\ell}{\partial \varepsilon_k} \\ &\text{where} \\ &\frac{\partial \mu_\ell}{\partial \varepsilon_k} = i_k \varepsilon_k^{i_k - 1} M_{i;0} \varepsilon_1^{i_1} \ldots \varepsilon_{k-1}^{i_{k-1}} \cdot \varepsilon_{k+1}^{i_{k+1}} \ldots \varepsilon_{28}^{i_{28}} \end{aligned}\right\} \tag{44}$$

for $k = 1, 28$.

The results for $\frac{\partial \Theta}{\partial \beta_k}(k=1, 3)$ are different because of the libration constraint which requires that only two of the first three angles are independent.

By employing the series used earlier to calculate the coordinates in Equation (38), together with those immediately preceding one can evaluate for any satellite

$$\left.\begin{aligned}
\frac{\partial \xi}{\partial \beta_\ell} &= -K_1 \sin \Theta_1 \frac{\partial \Theta_1}{\partial \beta_\ell}\\
\frac{\partial \upsilon}{\partial \beta_\ell} &= K_2 \cos \Theta_2 \frac{\partial \Theta_2}{\partial \beta_\ell}\\
\frac{\partial s}{\partial \beta_\ell} &= K_3 \cos \Theta_3 \frac{\partial \Theta_3}{\partial \beta_\ell}\\
&\text{and}\\
\frac{\partial \dot{\xi}}{\partial \beta_\ell} &= -K_1 \dot{\Theta}_1 \cos \Theta_1 \frac{\partial \Theta_1}{\partial \beta_\ell}\\
\frac{\partial \dot{\upsilon}}{\partial \beta_\ell} &= -K_2 \dot{\Theta}_2 \sin \Theta_2 \frac{\partial \Theta_2}{\partial \beta_\ell}\\
\frac{\partial \dot{s}}{\partial \beta_\ell} &= -K_3 \dot{\Theta}_3 (1+\dot{\upsilon}/n) \sin \Theta_3 \frac{\partial \Theta_3}{\partial \beta_\ell}
\end{aligned}\right\} \qquad (45)$$

for $\ell = 1, 22$.

The partial derivatives of ξ, υ and s with respect to ε_k are

$$\left.\begin{aligned}
\frac{\partial \xi}{\partial \varepsilon_k} &= \frac{\partial K_1}{\partial \varepsilon_k} \cos \Theta_1 - K_1 \sin \Theta_1 \frac{\partial \Theta_1}{\partial \varepsilon_k}\\
\frac{\partial \upsilon}{\partial \varepsilon_k} &= \frac{\partial K_2}{\partial \varepsilon_k} \sin \Theta_2 + K_2 \cos \Theta_2 \frac{\partial \Theta_2}{\partial \varepsilon_k}\\
\frac{\partial s}{\partial \varepsilon_k} &= \frac{\partial K_3}{\partial \varepsilon_k} \sin \Theta_3 + K_3 \cos \Theta_3 \frac{\partial \Theta_3}{\partial \varepsilon_k}\\
&\text{and}\\
\frac{\partial \dot{\xi}}{\partial \varepsilon_k} &= -\frac{\partial K_1}{\partial \varepsilon_k} \dot{\Theta}_1 \sin \Theta_1 - K_1 \frac{\partial \dot{\Theta}_1}{\partial \varepsilon_k} \sin \Theta_1\\
&\quad - K_1 \dot{\Theta}_1 \cos \Theta_1 \frac{\partial \Theta_1}{\partial \varepsilon_k}\\
\frac{\partial \dot{\upsilon}}{\partial \varepsilon_k} &= \frac{\partial K_2}{\partial \varepsilon_k} \dot{\Theta}_2 \cos \Theta_2 + K_2 \frac{\partial \dot{\Theta}_2}{\partial \varepsilon_k} \cos \Theta_2\\
&\quad - K_2 \dot{\Theta}_2 \sin \Theta_2 \frac{\partial \Theta_2}{\partial \varepsilon_k}\\
\frac{\partial \dot{s}}{\partial \varepsilon_k} &= (1+\dot{\upsilon}/n)\left[\frac{\partial K_3}{\partial \varepsilon_k} \dot{\Theta}_3 \cos \Theta_3 + K_3 \frac{\partial \dot{\Theta}_3}{\partial \varepsilon_k} \cos \Theta_3\right.\\
&\quad \left. - K_3 \dot{\Theta}_3 \sin \Theta_3 \frac{\partial \Theta_3}{\partial \varepsilon_k}\right]
\end{aligned}\right\} \qquad (46)$$

for $k = 1, 28$.

If one employs [see Eq. (40)]

$$\left.\begin{aligned}
Q_1 &= a \cos(\ell - \psi + \upsilon)\\
Q_2 &= a \sin(\ell - \psi + \upsilon)\\
Q_3 &= as\\
Q_4 &= 1 + \xi\\
\bar{x} &= Q_1 Q_4\\
\bar{y} &= Q_2 Q_4\\
\bar{z} &= Q_3 Q_4\\
\dot{\bar{x}} &= Q_1 \dot{\xi} - \bar{y}(\dot{\ell} - \dot{\psi} + \dot{\upsilon})\\
\dot{\bar{y}} &= Q_2 \dot{\xi} + \bar{x}(\dot{\ell} - \dot{\psi} + \dot{\upsilon})\\
\dot{\bar{z}} &= Q_3 \dot{\xi} + Q_1 Q_4 \dot{s}
\end{aligned}\right\}$$

then the partial derivatives of $\bar{\boldsymbol{r}}$ and $\dot{\bar{\boldsymbol{r}}}$ in the plane of Jupiter's equator are $\left(\text{ignoring, for the time being, } \frac{\partial a}{\partial \varepsilon_k}\right)$

$$\left.\begin{aligned}
\frac{\partial \bar{x}}{\partial \beta_k} &= Q_1 \frac{\partial \xi}{\partial \beta_k} - \bar{y} \frac{\partial}{\partial \beta_k}(\ell - \psi + \upsilon)\\
\frac{\partial \bar{y}}{\partial \beta_k} &= Q_2 \frac{\partial \xi}{\partial \beta_k} + \bar{x} \frac{\partial}{\partial \beta_k}(\ell - \psi + \upsilon)\\
\frac{\partial \bar{z}}{\partial \beta_k} &= Q_3 \frac{\partial \xi}{\partial \beta_k} + a Q_4 \frac{\partial s}{\partial \beta_k}\\
&\text{and}\\
\frac{\partial \dot{\bar{x}}}{\partial \beta_k} &= -Q_2(\dot{\ell} - \dot{\psi} + \dot{\upsilon}) \frac{\partial \xi}{\partial \beta_k} + Q_1 \frac{\partial \dot{\xi}}{\partial \beta_k}\\
&\quad - \bar{y} \frac{\partial \dot{\upsilon}}{\partial \beta_k} - \dot{\bar{y}} \frac{\partial}{\partial \beta_k}(\ell - \psi + \upsilon)\\
\frac{\partial \dot{\bar{y}}}{\partial \beta_k} &= Q_1(\dot{\ell} - \dot{\psi} + \dot{\upsilon}) \frac{\partial \xi}{\partial \beta_k} + Q_2 \frac{\partial \dot{\xi}}{\partial \beta_k}\\
&\quad + \bar{x} \frac{\partial \dot{\upsilon}}{\partial \beta_k} + \dot{\bar{x}} \frac{\partial}{\partial \beta_k}(\ell - \psi + \upsilon)\\
\frac{\partial \dot{\bar{z}}}{\partial \beta_k} &= a\dot{s} \frac{\partial \xi}{\partial \beta_k} + Q_3 \frac{\partial \dot{\xi}}{\partial \beta_k} + a\dot{\xi} \frac{\partial s}{\partial \beta_k}\\
&\quad + a Q_4 \frac{\partial \dot{s}}{\partial \beta_k}
\end{aligned}\right\} \qquad (48)$$

for $k = 1, 22$.

In the above equation, ℓ represents the mean longitude of a particular satellite $\boldsymbol{i}$. Consequently

$$\frac{\partial \ell_i}{\partial \beta_k} = \delta_{ik} = \begin{matrix} 0 & \text{if} & k \neq i \\ 1 & \text{if} & k = i \end{matrix}$$

with the modifications due to the libration contraint in the case $i = 3$

$$\frac{\partial \ell_i}{\partial \beta_1} = -1/2$$

$$\frac{\partial \ell_i}{\partial \beta_2} = +3/2$$

$$\frac{\partial \ell_i}{\partial \beta_3} = 0.$$

The ε-partial derivatives are

$$\left.\begin{aligned}
\frac{\partial \bar{x}}{\partial \varepsilon_k} &= Q_1 \frac{\partial \xi}{\partial \varepsilon_k} - \bar{y} \frac{\partial}{\partial \varepsilon_k}(\ell - \psi + \upsilon) \\
\frac{\partial \bar{y}}{\partial \varepsilon_k} &= Q_2 \frac{\partial \xi}{\partial \varepsilon_k} + \bar{x} \frac{\partial}{\partial \varepsilon_k}(\ell - \psi + \upsilon) \\
\frac{\partial \bar{z}}{\partial \varepsilon_k} &= Q_3 \frac{\partial \xi}{\partial \varepsilon_k} + a Q_4 \frac{\partial s}{\partial \varepsilon_k} \\
&\text{and} \\
\frac{\partial \dot{\bar{x}}}{\partial \varepsilon_k} &= -Q_2(\dot{\ell} - \dot{\psi} + \dot{\upsilon}) \frac{\partial \xi}{\partial \varepsilon_k} + Q_1 \frac{\partial \dot{\xi}}{\partial \varepsilon_k} \\
&\quad - \bar{y} \frac{\partial}{\partial \varepsilon_k}(\dot{\ell} - \dot{\psi} + \dot{\upsilon}) - \dot{\bar{y}} \frac{\partial}{\partial \varepsilon_k}(\ell - \psi + \upsilon) \\
\frac{\partial \dot{\bar{y}}}{\partial \varepsilon_k} &= Q_1(\dot{\ell} - \dot{\psi} + \dot{\upsilon}) \frac{\partial \xi}{\partial \varepsilon_k} + Q_2 \frac{\partial \dot{\xi}}{\partial \varepsilon_k} \\
&\quad + \bar{x} \frac{\partial}{\partial \varepsilon_k}(\dot{\ell} - \dot{\psi} + \dot{\upsilon}) + \dot{\bar{x}} \frac{\partial}{\partial \varepsilon_k}(\ell - \psi + \upsilon) \\
\frac{\partial \dot{\bar{z}}}{\partial \varepsilon_k} &= a\dot{s} \frac{\partial \xi}{\partial \varepsilon_k} + Q_3 \frac{\partial \dot{\xi}}{\partial \varepsilon_k} + a\dot{\xi} \frac{\partial s}{\partial \varepsilon_k} + a Q_4 \frac{\partial \dot{s}}{\partial \varepsilon_k}
\end{aligned}\right\} \quad (49)$$

for $k=1, 28$.

Finally, in the 1950.0 earth equator and equinox system, the β-partials are from Equations (48), (41) and (42)

$$\left.\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \beta_k} &= \boldsymbol{BA} \frac{\partial \bar{\boldsymbol{r}}}{\partial \beta_k} + \left[\boldsymbol{B} \frac{\partial \boldsymbol{A}}{\partial \beta_k} \bar{\boldsymbol{r}}\right] \\
\frac{\partial \dot{\boldsymbol{r}}}{\partial \beta_k} &= \boldsymbol{B\dot{A}} \frac{\partial \bar{\boldsymbol{r}}}{\partial \beta_k} + \boldsymbol{BA} \frac{\partial \dot{\bar{\boldsymbol{r}}}}{\partial \beta_k} + \left[\boldsymbol{B} \frac{\partial \dot{\boldsymbol{A}}}{\partial \beta_k} \bar{\boldsymbol{r}} + \boldsymbol{B} \frac{\partial \boldsymbol{A}}{\partial \beta_k} \dot{\bar{\boldsymbol{r}}}\right]
\end{aligned}\right\} \quad (50)$$

for $k=1, 22$, where the terms in brackets are only present for $\beta_{15}(\psi)$ and $\beta_{22}(\Omega)$.

The ε-partials in the 1950.0 coordinate system are from Equations (49), (41) and (42)

$$\left.\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \varepsilon_k} &= \boldsymbol{BA} \frac{\partial \bar{\boldsymbol{r}}}{\partial \varepsilon_k} + \left[\boldsymbol{B} \frac{\partial \boldsymbol{A}}{\partial \varepsilon_k} \bar{\boldsymbol{r}}\right] \\
\frac{\partial \dot{\boldsymbol{r}}}{\partial \varepsilon_k} &= \boldsymbol{B\dot{A}} \frac{\partial \bar{\boldsymbol{r}}}{\partial \varepsilon_k} + \boldsymbol{BA} \frac{\partial \dot{\bar{\boldsymbol{r}}}}{\partial \varepsilon_k} + \left[\boldsymbol{B} \frac{\partial \dot{\boldsymbol{A}}}{\partial \varepsilon_k} \bar{\boldsymbol{r}} + \boldsymbol{B} \frac{\partial \boldsymbol{A}}{\partial \varepsilon_k} \dot{\bar{\boldsymbol{r}}}\right]
\end{aligned}\right\} \quad (51)$$

for $k=1, 28$, where the terms in brackets are required only for $\varepsilon_{25}(I_J)$ and $\varepsilon_{26}(J)$. In the case of ε_{27} (obliquity) the partials are more readily calculated from

$$\frac{\partial \boldsymbol{r}}{\partial \varepsilon_{27}} = \varepsilon_0 \begin{pmatrix} 0 \\ -z \\ +y \end{pmatrix}$$

and $$\frac{\partial \dot{\boldsymbol{r}}}{\partial \varepsilon_{27}} = \varepsilon_0 \begin{pmatrix} 0 \\ -\dot{z} \\ +\dot{y} \end{pmatrix}$$

where $\varepsilon_0 = 23^\circ\!.445788$ expressed in radians.

The effects (previously bypassed) of the scale factor $(\partial a/\partial \varepsilon_k)$ are finally calculated as increments to Equation (51) as

$$\left.\begin{aligned}
&+\frac{\boldsymbol{r}}{a}\frac{\partial a}{\partial \varepsilon_k} \quad \text{in} \quad \frac{\partial \boldsymbol{r}}{\partial \varepsilon_k} \\
&\text{and} \\
&+\frac{\dot{\boldsymbol{r}}}{a}\frac{\partial a}{\partial \varepsilon_k} \quad \text{in} \quad \frac{\partial \dot{\boldsymbol{r}}}{\partial \varepsilon_k}.
\end{aligned}\right. \quad (52)$$

The computer programs KODLOP, KODQIP and GALSAP have been developed to regenerate the theory and calculate the state and its partial derivatives. It requires 70 s to completely regenerate the theory for arbitrary $\varepsilon-\beta$ values and two sets per second can be calculated for state and partial derivatives. The FORTRAN programs are listed on Microfiche 3. Sample theories and output are also given there for the nominal theory ($\varepsilon=\beta=0$) as well as those listed in Table 6. The computer programs are designed to be readily transportable from one computer to another and the series coefficients, partials, angles and rates are optimally packed in a manner suggested by Hahn (1974).

A modification is made in the case of the partial derivatives with respect to β_5 (the phase of the free libration). In Sampson's work, the amplitude (ε_9) of the free libration

$$\nu_1 - 3\nu_2 + 2\nu_3 = 180^\circ + \varepsilon_9 \sin[\beta_5 + L^{1/2}(t-t_0)] \quad (53)$$

was taken as identically zero. Consequently, since the first three satellites exhibit β_5 only in the form

$$\upsilon_j = \ldots + K_j \varepsilon_9 \sin[\beta_5 + L^{1/2}(t-t_0)] + \ldots (j=1,3)$$

it is apparent that one cannot obtain $\partial/\partial\beta_5$ if $\varepsilon_9=0$. Therefore, the computer programs actually calculate $\varepsilon_9^{-1}\partial/\partial\beta_5$ rather than only $\partial/\partial\beta_5$ (the input parameter to the theory, however, remains β_5 and not $\varepsilon_9^{-1}\beta_5$). In this way one can more readily solve for the amplitude and phase of the free libration.

There are no constraints whatsoever on permissible values of $\beta_k (k=1, 22)$ in evaluating the theory although if initially the value of β_ℓ is far away from its final value, then the partial derivatives may at first be rather poor. For fullest accuracy, the values of $\varepsilon_k (k=1, 25)$ should lie within the bounds given in Paper 3. If any ε_ℓ lies outside its presumed bound, the precision of the theory will degrade somewhat from its best level. It is not anticipated that scientists will ever solve for all of the 49 parameters of the theory. Many of them (e.g. ε_{27} = obliquity) are retained simply in order that the theory can be revised when the system of astronomical constants is changed.

Concluding Remarks

As outlined in Papers I-III, a method has been devised to enable one to calculate the positions and partial derivatives of the Galilean satellites. The results obtained from comparison of the theory with a numerical integration suggests the new theory has a mathematical precision on the order of 10 km. It appears from adjustments of the theory to fit longer integrations, that some terms of the form $2\ell_1 - 4\ell_2 + \alpha$ still require some additional effort. It is known, for example, that neglected terms in $\partial^*\Omega/\partial\tau$ of Equation (4) as used in Equation (9) will yield such angles. Sampson (and the present author) neglected the terms in Equation (4) of the form $\dot{z}\partial\Omega/\partial z$ since they already contain three small quantities (viz. the perturbing mass and quadratics in ζ). However, such additional terms, of the form

$$a_i \Delta \frac{\partial^* \Omega_{ij}}{\partial \tau} = m_j \dot{\zeta}_i (L_{ij}\zeta_i + P_{ij}\zeta_j)\,, \qquad (54)$$

where L_{ij} and P_{ij} are defined in Table (II-1), will produce terms such as $2\ell_1 - 4\ell_2 + 2\omega_2$ (period ≈ 250 day) which will tend to improve the comparison with integration.

The major advantage of the new theory is the ability to calculate and readily adjust the basic parameters. Computer programs and listings are available from the author and listings are given in the microfiche attached to this paper. Preliminary evaluation of the new theory using radar data obtained at Arecibo (Duxbury and Lieske, 1976) suggests that the theory is indeed accurate and that the possibility of solving for the free libration may prove to be very useful.

Finally, known typographical errors of the earlier papers in this series are as follows:

Paper II: p. 140, Equation (7) $2v'$ should read $-2v$; p. 143, Equation (24) $2G' - G + 336.^\circ903$ should read $2G' - 2G + 336.^\circ903$; p. 145, Equation (28) $1/2\xi_j(v_i - v_j)^2 \mathscr{I}_{ij}$ should read $1/2\xi_j(v_i - v_j)^2 \mathscr{I}_{ij}$.

Paper III: p. 12, Equation (5) $n_i m_i$ should read $n_i m_j$.

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Note. In addition to the neglected effects mentioned in Equation (54), a problem arises from the term in the latitude of Satellite I with argument $3\ell_1 - 4\ell_2 + \omega_2$ (Term No. 7 in Table 5). As is evident from Table 5, Sampson's term has the wrong sign, due to use of an erroneous integrating factor. However, since Sampson's final theory was the initial approximation employed in the present theory, Sampson's erroneous term in ζ_1 will propagate and produce fictitious terms in longitude, among which is one with argument $2\ell_1 - 4\ell_2 + 2\omega_2$.

Note added in proof. In the definition of Parameter ε_5, the description in Table 2 should read "Mass of Sun relative to Jupiter" rather than relative to Jupiter system.

In addition, as outlined in Equation (54), terms arising from

$$a_i \frac{\partial^* \Omega_{ij}}{\partial \tau} = m_j \dot{\zeta}_i [\zeta_i L_{ij} + \zeta_i \xi_i M_{ij} + \zeta_i \xi_j N_{ij} + \zeta_i (v_i - v_j) O_{ij} + \zeta_j P_{ij} + \zeta_j \xi_i Q_{ij} + \zeta_j \xi_j R_{ij} + \zeta_j (v_i - v_j) S_{ij}]$$

have been neglected. Such terms involve the third power of small quantities and may safely be ignored except in cases where small divisors occur.

Consideration of the above two effects yields the following corrections which should be added to the microfiche tables and to Table 5:

$$\begin{aligned}
10^7 v_1 = &-20\sin(2\ell_1 - 4\ell_2 + \omega_2 + \omega_3)\\
&+(-506 - 311\varepsilon_2)\sin(2\ell_1 - 4\ell_2 + 2\omega_2)\\
10^7 v_2 = &+27\sin(2\ell_1 - 4\ell_2 + \omega_3 + \psi)\\
&-40\sin(2\ell_1 - 4\ell_2 + 2\omega_3)\\
&+(58 + 49\varepsilon_1)\sin(2\ell_1 - 4\ell_2 + \omega_2 + \omega_3)\\
&+(689 + 424\varepsilon_1 + 93\varepsilon_3)\sin(2\ell_1 - 4\ell_2 + 2\omega_2)\\
&+16\sin(2\ell_1 - 4\ell_2 + 2\Pi + 2G)\\
&+57\sin(2\ell_1 - 4\ell_2 + \pi_3 + \pi_4)\\
&+68\sin(2\ell_1 - 4\ell_2 + 2\pi_3)\\
10^7 v_3 = &-7\sin(2\ell_1 - 4\ell_2 + \omega_3 + \psi)\\
&+8\sin(2\ell_1 - 4\ell_2 + 2\omega_3)\\
&-25\sin(2\ell_1 - 4\ell_2 + 2\omega_2)\\
&-16\sin(2\ell_1 - 4\ell_2 + \pi_3 + \pi_4)\\
&-19\sin(2\ell_1 - 4\ell_2 + 2\pi_3)\,.
\end{aligned}$$

The original longitude coefficients, which are to be modified by the addition of the above expressions, may be found as Terms 843–845 on Microfiche A.68 for Satellite I, Terms 884–893 on Microfiche A.87 for Satellite II and Terms 855–859 on Microfiche A.106 for Satellite III.

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