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HIGH-FREQUENCY OPTICAL VARIABLES. II. LUMINOSITY-VARIABLE WHITE DWARFS AND MAXIMUM ENTROPY SPECTRAL ANALYSIS

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ABSTRACT

In a search for luminosity-variable white dwarf stars, two new periodic variables were found, EG 65 (G117−B15A) and EG 197 (G169−34). The former has a period of 1311 s while the latter varies on a time scale of 465 s. Using EG 65 as a typical variable of this class of stars, we show that the burning of hydrogen accreted from the interstellar medium can account for the energy needed to keep the star in oscillation. Maximum entropy power spectral analysis, which was used to identify the 1311-s period, is discussed in an appendix.

Subject headings: pulsation — variable stars — white dwarf stars

I. INTRODUCTION

This is the second in a series of papers dealing with the equipment, observational techniques, methods of analysis, and theoretical implications resulting from a search for blue stars varying in optical luminosity at high frequency. The main body of the present paper divides itself naturally into two distinct parts. The first main section of this contribution contains a discussion of a continuing observing program which we began early in 1972. The aim of this program is to find single white-dwarf variables in order that we may shed some light on the physical mechanism and distinguishing characteristics of luminosity-variable and nonvariable white-dwarf stars. Preliminary results are presented in this section. Evidence is given which indicates that two new single periodic white-dwarf variables have been discovered: one with a period of 465 s, the other varying on a time scale of 1311 s. In this same section we will discuss a statistical test of whiteness of a power spectrum, and show, to a very high degree of probability, that it is unlikely that the observed power spectrum of either of the above two white dwarfs is the result of a white-noise process.

In the second section of this paper we will show that the energy required to keep a white dwarf pulsating can be derived from the burning of hydrogen accreted from the interstellar medium. This suggestion accounts naturally for the observed fact that the luminosity-variable white dwarfs sometimes do not vary (Hesser and Lasker 1971, hereafter referred to as HL), and also leads to the interesting possibility that variable white dwarfs may be used as very sensitive probes of the density of the interstellar medium.

An appendix has been included in this paper. In it we discuss the maximum entropy method (MEM) of spectral analysis which we briefly alluded to in Paper I (Richer *et al.* 1973). This analysis technique has already found its way into a number of recent papers, and we feel that an astrophysically oriented exposition of this subject is needed.

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II. SEARCH FOR WHITE-DWARF VARIABLES

a) *Philosophy and Previous Efforts*

Simple, zero-temperature pulsation models of white dwarf stars imply that these objects, if variable, should have radial pulsation periods in the range of 2–50 s (Ostriker 1971). Of the four known luminosity-variable white dwarfs outside of binary systems, none has a period shorter than 213 s, while one of them (G44−32) exhibits variation on a time scale of almost half an hour (HL). It seems likely, hence, that we are dealing not with a simple radial pulsation of the star, but with a more complex pulsation mode. Perhaps, as suggested by Chanmugan (1972), the nonradial gravity modes can be excited. These modes lead to periods of about the correct length. In § IV we shall suggest a method of exciting such modes.

The immediate task of any observational program involving white-dwarf variables is to find new variables, as the present sample is extremely small. From the studies of HL, it is clear that the yield of any search will be small: only about one in 10 white dwarfs appears to be fluctuating with an amplitude in excess of 0.004 mag on a time scale of a few seconds through about 1 hour. However, HL have pointed out that three of the four variable white dwarfs have almost identical colors: $(B - V) \simeq +0.25$, $(U - B) \simeq -0.55$. While there are quiescent white dwarfs in the same region of the color-color diagram, the congregation of three of them to this region deserves closer attention.

With these thoughts in mind, we decided to generally concentrate our search in this restrictive color region. Our equipment, observational techniques, and methods of analysis are virtually identical to those discussed in the first paper of this series (Richer *et al.* 1973).

b) *Candidate Objects*

In late March of 1972, and early May of 1973, we had observing runs at Kitt Peak National Observatory

where we used the No. 2 36-inch (91-cm) telescope to monitor the optical output of white dwarfs and white-dwarf suspects selected from the lists of Eggen and Greenstein (1965*a*, *b*, 1967, hereafter referred to as EG) and Giclas, Burnham, and Thomas (1971).

Typically, the stars were observed for 90 minutes on each of two nights with an integration time of 0.010 s. The digital data were recorded on magnetic tape with no time lost between successive integrations. Spectral analysis was performed off-line to search for periodicities in the range 0.020 s to about 1 hour.

The candidate objects were selected according to three main criteria: (1) $(B - V) \simeq +0.25$, $(U - B) \simeq -0.55$. (2) $V < 16.0$. (3) The stars have not previously been observed for variability.

Some of the brightest white dwarfs observed do not satisfy criterion (1). These were observed mainly for reference and standardization purposes. For a star fainter than $V = 16.0$, the number of counts from the moonless sky at Kitt Peak would exceed the number of counts from the star (for a 36-inch telescope and a 15″ diaphragm). Hence, the statistics for such a faint star would be rather poor unless white-dwarf variability were highly modulated—something which does not appear to be the case for the known variables.

Table 1 contains a list of, and data pertinent to, the 16 stars observed for variability. Columns (1) and (2) contain the EG and Giclas designations, respectively; columns (3), (4), (5), and (6) the V-magnitude, $B - V$, $U - B$, and spectral type from EG; column (7) the date of the observation; column (8) the Universal Time of the beginning of the observation; and column (9) the number of 0.010-s integrations performed. The stars are listed in order of increasing right ascension.

DQ Herculis, with its 71-s period, was also observed on most nights as a check both on the frequency calibration of the system and on the amplitude of any other suspected variable.

III. RESULTS OF ANALYSIS

a) *Nonvariable White Dwarfs*

Of the 16 white dwarfs observed, 14 showed no clear-cut evidence of variability of any sort. To be specific, periods from 0.020 s through about 1 hour were searched for, using standard Fourier and maximum-entropy spectral techniques. The upper limit to any variability was about 0.003 mag. The 14 non-variable stars are all those of table 1 excluding EG 65 and EG 197.

Figure 1 illustrates the power spectrum of the white dwarf EG 117 (G138−8) calculated from data obtained on 1973 May 7/8. This spectrum is shown to illustrate the power spectrum of a typical nonvariable white dwarf. Only the low-frequency end of the power spectrum is shown. While no peaks in this spectrum *appear* to be statistically significant, implying that there is no detectable periodic variability, we wish to be more quantitative about such statements.

TABLE 1

WHITE DWARFS OBSERVED FOR VARIABILITY

Star Name								
EG (1)	*Giclas* (2)	*V* (3)	$(B - V)$ (4)	$(U - B)$ (5)	Spectral Type (6)	Date (7)	UT (8)	*N* (9)
64	G116−16	15.3	0.24	−0.53	DAs	1973 5/1–5/2	3:42	535,552
						1973 5/6–5/7	3:26	541,696
65	G117−B15A	15.3	0.20	−0.56	DA	1973 5/3–5/4	3:25	542,720
						1973 5/7–5/8	3:24	536,576
67	G116−52	13.3	0.07	−0.54	DA	1972 3/15–3/16	4:46	162,877
	G116−57	13.8	...	...	...	1972 3/16–3/17	4:27	206,030
70	G162−66	13.0	−0.15	−1.02	DAn	1972 3/15–3/16	2:58	288,744
						1972 3/19–3/20	5:21	451,328
184	GD 140	12.5	−0.06	−0.98	DAwk	1972 3/15–3/16	6:09	352,419
						1973 3/21–3/22	4:32	372,736
	G149−28	15.4	0.28	−0.58	DAs	1973 5/1–5/2	6:23	544,768
						1973 5/7–5/8	5:17	493,568
188	G177−31	14.1	0.03	−0.56	DAs	1972 3/16–3/17	10:07	690,688
						1972 3/12–3/18	8:37	344,064
111	G66−32	15.8	0.02	−0.67	DA	1973 5/3–5/4	5:25	540,672
						1973 5/7–5/8	6:55	539,648
191	GD 178	14.1	0.09	−0.65	DAn	1972 3/19–3/20	8:06	291,584
192	GD 185	14.0	0.00	−0.81	DAn	1972 3/19–3/20	9:22	417,280
117	G138−8	15.1	0.23	−0.65	DAs	1973 5/2–5/3	8:28	450,560
						1973 5/7–5/8	9:08	418,816
196	G138−56	15.7	0.35	−0.57	DA	1973 5/3–5/4	7:24	577,536
						1973 5/5–5/6	8:02	361,472
						1973 5/8–5/9	7:28	359,424
197	G169−34	14.1	0.24	−0.62	DAss	1973 5/1–5/2	8:23	539,648
						1973 5/6–5/7	7:48	493,568
175	G154−B5B	14.2	0.32	−0.45	DA	1973 5/3–5/4	9:26	527,360
						1973 5/8–5/9	9:05	332,800
126	G21−16	14.5	0.24	−0.56	DAs	1973 5/2–5/3	10:12	406,528
						1973 5/8–5/9	10:07	365,568

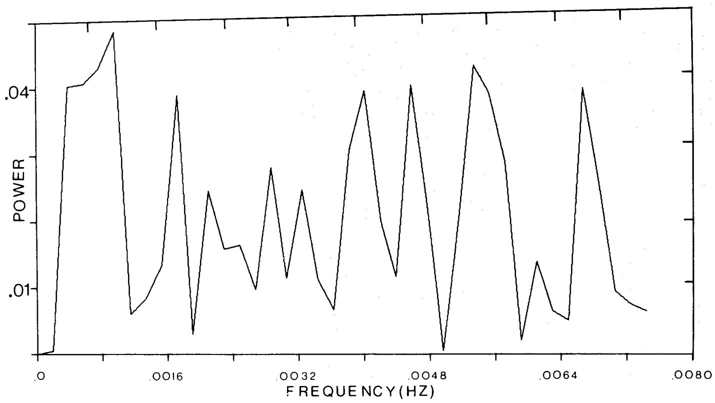


FIG. 1.—Low-frequency end of the power spectrum of the white dwarf EG 117 (G138−8). The units of power are arbitrary.

Jenkins and Watts (1968, p. 234) have outlined a simple statistical test that can be done on a power spectrum which answers the question of whether or not the observed spectrum can be regarded as a realization of a white-noise process. If the power $P(f_k)$ has been calculated at the harmonic frequencies $f_k = k/N\Delta T$, $k = 0, 1, 2, \ldots, N/2$, where N is the number of data points transformed and ΔT the integration time for each point, then $I(f_k)$ can be calculated from the integrated spectrum

$$I(f_k) = \frac{1}{N\Delta T}\sum_{q=1}^{k} P(f_q)\,. \tag{1}$$

Since the mean is always removed before we transform our data, $P(0)$ is zero. Jenkins and Watts (1968) then show that if the estimate

$$\frac{I(f_k)}{S^2} = \frac{1}{N\Delta T S^2}\sum_{q=1}^{k} P(f_q) \tag{2}$$

obtained from an observed time series is plotted against $2\Delta T f_k$, the points should scatter about the straight line through the points (0, 0), (1, 1). In equation (2) above, S^2 is the variance in the data set or, equivalently, the average value of the power in the power spectrum. In the plot of $I(f_k)/S^2$ against $2\Delta T f_k$, since $I(f_k)$ is the sum of random variables, the Kolmogorov-Smirnov (see Hald 1952) probability limits can be used to assess when a significant departure from linearity occurs. This consists of constructing a band $\pm\lambda/[(N/2) - 1]^{1/2}$ about the theoretical line. For a significance level of 0.95, λ is equal to 1.36.

The above test was calculated and plotted for all the power spectra obtained, and the one for EG 117 is illustrated in figure 2. In this sample $\Delta T = 5.12$ s, $N = 1024$, and thus the 95 percent limits are ± 0.06. These limits are shown as broken lines in figure 2, and it is seen that $I(f_k)/S^2$ falls well within them. Hence, there is no evidence that the sample does not come from a white-noise source. The interpretation of the 95 percent limits is that on one out of 20 plots, on average, the maximum deviation from the theoretical line will lie outside the limits if the process is in fact white noise.

Results similar to the above were obtained for the 14 white dwarfs mentioned, and hence for these objects we have no evidence whatsoever of variability.

b) *Variable White Dwarfs*

Two of the 16 stars observed showed strong evidence of periodic variability. These white dwarfs are EG 65 (G117−B15A) and EG 197 (G169−34).

i) *EG 197*

The data set which we obtained for EG 197 on 1973 May 6/7 is shown in figure 3. Each plotted point is the sum of 512 0.010-s integrations, and the individual points have been joined. The zeroth first and second moments have been removed from this data set, but no other filtering has been done. An apparent sinusoidal variation can be seen directly in the data.

The power spectrum of the data of figure 3 is shown in the plot of figure 4, where only the low-frequency

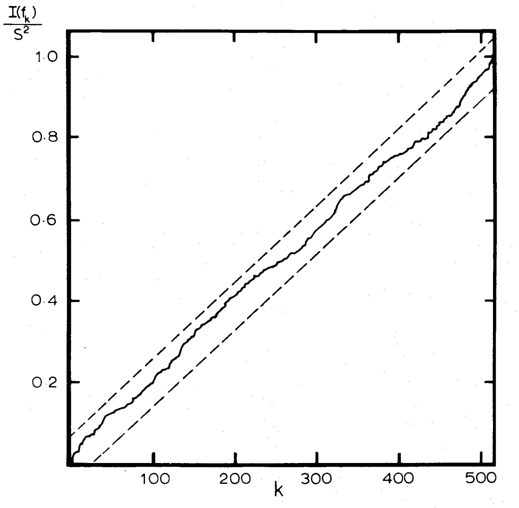


FIG. 2.—Whiteness test of the power spectrum of EG 117. The broken lines represent the 95% confidence limits. No significant deviation from whiteness is seen in this power spectrum.

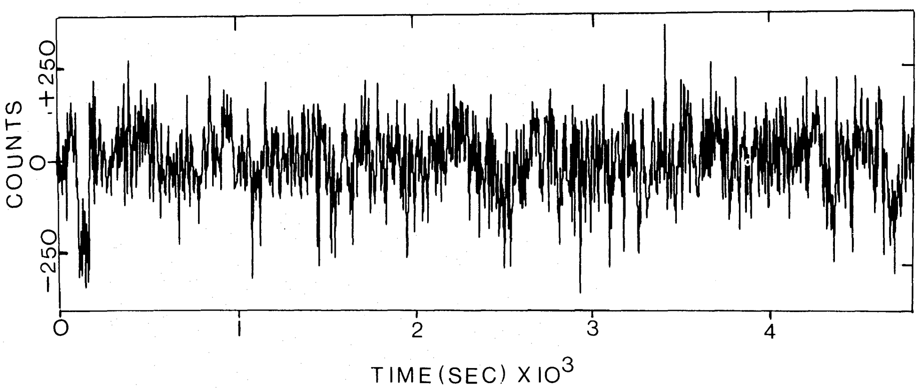


FIG. 3.—Data set for EG 197 (G169−34) obtained on 1973 May 6/7. Each point represents a 5.12-s integration.

end of the power spectrum is illustrated. Only one peak in the spectrum seems significant, and it corresponds to a period of 465 s. No harmonics are seen in the spectrum, implying that the variation is indeed a pure sinusoid. The power spectrum calculated from the data obtained on 1973 May 1/2 for this object again shows the strongest peak at 465 s, providing strong evidence for the reality of this periodicity.

There is no apparent high-frequency optical variability associated with EG 197. Power spectra were calculated down to the Nyquist frequency of 50 Hz, and no statistically significant deviation from whiteness was seen. These power spectra are shown in figure 5.

The test for whiteness used in the previous section on EG 117 was applied to the power spectrum of EG 197. The corresponding plot is seen in figure 6, and it is clearly obvious that the power spectrum deviates from that of a white-noise process. We conclude, thus, that the peak in the power spectrum of EG 197 corresponding to a periodic variation of 465 s is statistically significant.

In order to construct a light curve for this variable, the data of figure 3 have been folded modulo the period of 465 s. Slightly more than 10 periods are present. The result of this manipulation is seen in figure 7, wherein the excess number of counts above or below the mean is plotted against phase. An error of plus or minus one standard deviation is indicated. Since the mean that was removed is about 77,000 counts, the amplitude of the sinusoid is quite small—about 0.02 mag.

ii) *EG 65*

EG 65 is the white-dwarf primary of a wide binary system, the secondary being an M2 dwarf (EG 1965*a*). The separation between the two components is large enough so that the evolution of the white dwarf is unlikely to have been seriously affected by its companion, and thus we can consider it as a single star.

The low-frequency end of the Fourier power spectrum of the data set which we obtained for this star on 1973 May 7/8 is illustrated in figure 8*a*, where a large peak at the period of 1311 s is seen. Since the frequency of this peak is so low, the actual value of the period is quite uncertain. However, the MEM spectrum of the same data (fig. 8*b*) shows a peak at the identical frequency. Before calculating the spectrum seen in figure 8*b*, we added a 1 percent tracer signal to the data at a frequency of 0.004 Hz in order to properly estimate the amplitude of variability. Since

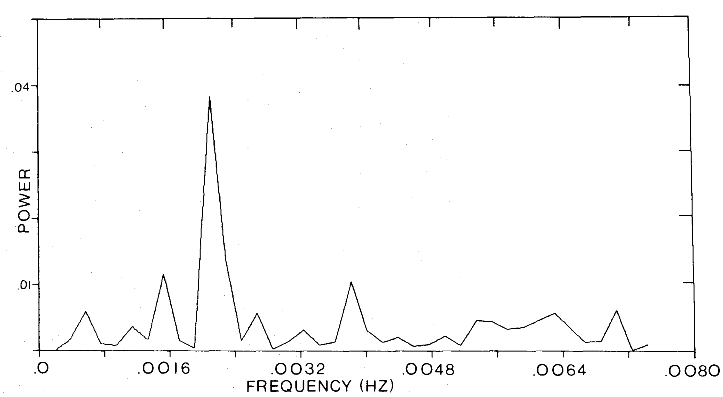
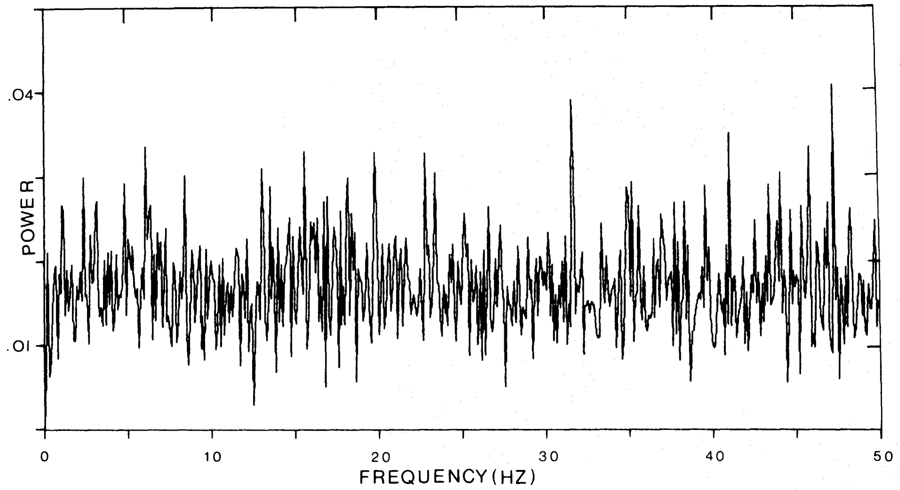


FIG. 4.—Low-frequency end of the power spectrum of EG 197 using the data of fig. 3. The peak at 465 s is clearly seen. The units of power are arbitrary.



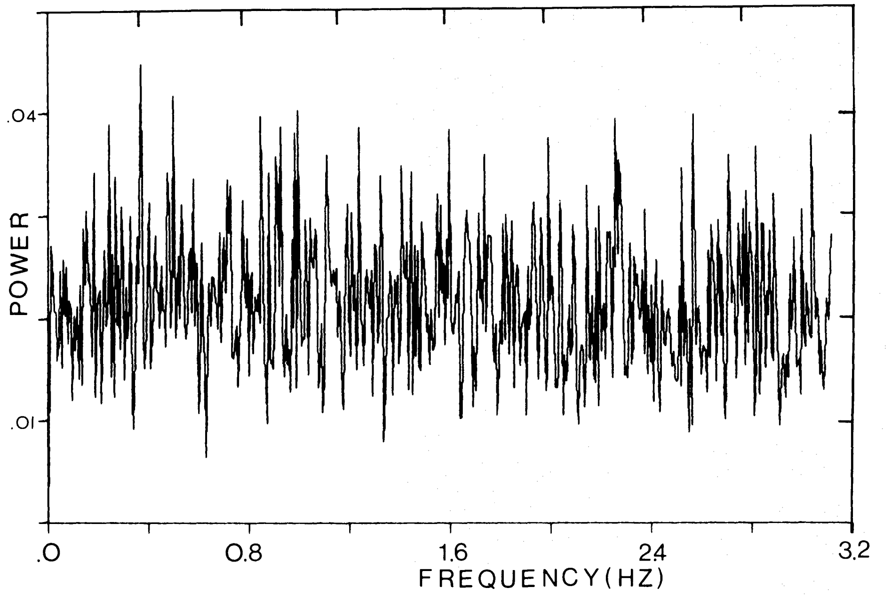


FIG. 5.—High-frequency end of the power spectrum of EG 197. In fig. 5*a* (*top*) the frequency range 0–50 Hz is covered while in fig. 5*b* (*bottom*) the range 0–3 Hz is illustrated. No statistically significant peaks are seen. The units of power are arbitrary.

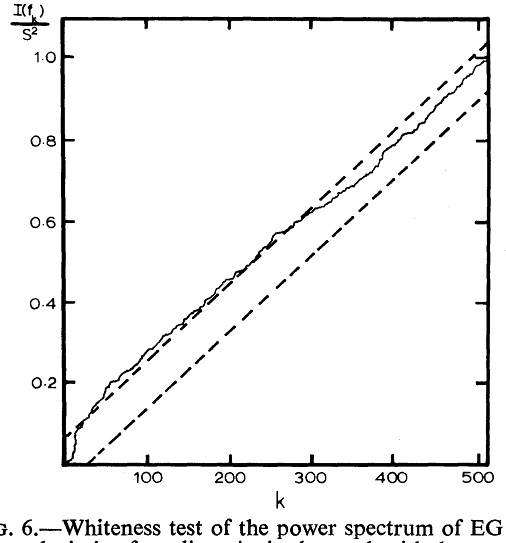


FIG. 6.—Whiteness test of the power spectrum of EG 197. A strong deviation from linearity is observed, with the resulting conclusion that the peak seen at the period of 465 seconds is significant to well above the 95% level.

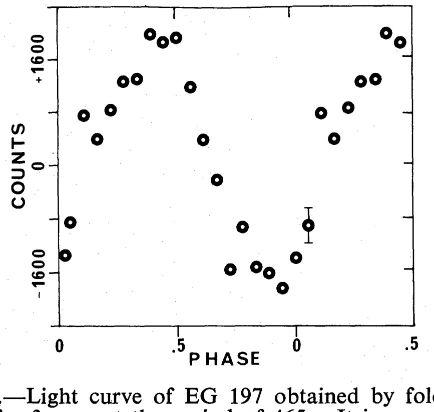


FIG. 7.—Light curve of EG 197 obtained by folding the data of fig. 3 over at the period of 465 s. It is seen that the points are well fitted by a sinusoid of amplitude 0.02 mag.

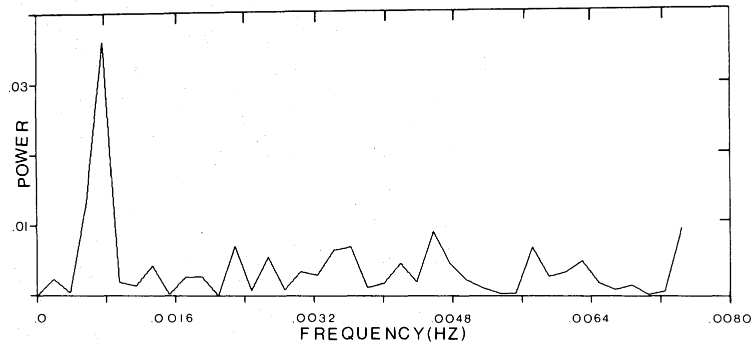


FIG. 8*a*.—Low-frequency end of the power spectrum of EG 65 calculated by using standard Fourier techniques. The peak corresponding to a period of 1311 s is seen. The units of power are arbitrary.

the MEM resolves periodic variability when only a few cycles are present in the data, we conclude that the peak seen in figures 8*a* and 8*b* is due to genuine harmonic activity and not to low-frequency flickering. Again, no harmonics are observed, and the implication is that the variation is purely sinusoidal. From figures 8*a* and 8*b* we conclude that the amplitude of variation is about 0.01 mag.

As with all the other white dwarfs observed, power spectra were calculated down to the Nyquist frequency of 50 Hz. No evidence of periodic variability was discerned aside from the 1311-s period. The whiteness test for EG 65, discussed in the previous sections, is shown in figure 9. As with EG 197, we conclude that the power spectrum of EG 65 has a probability of well less than 5 percent of being the realization of a white-noise process.

IV. THE CAUSES OF VARIABILITY

If the two new white-dwarf variables discussed in the previous section are confirmed by further observations, this will bring to six the total number of known variables of this type outside of close binary systems. Five of these have colors given approximately by $(B - V) \simeq +0.25$, $(U - B) \simeq -0.55$. However, this statistic should be regarded with some caution, as the stars which were observed were certainly not chosen at random.

Clearly, the most important question to consider is why some white dwarfs are variable and others are not. To go further, why is it that two white dwarfs can lie in the same region of the color-color diagram, have the same spectral characteristics, yet one will vary with an observable amplitude and the other not? Let us make use of the data presently available to see if we can shed some light on this question.

Table 2 contains data on those white dwarfs of spectral type DA, of known absolute magnitude, which have been examined for variability either by ourselves or by HL. All the column headings are self-explanatory. The absolute magnitudes were derived by EG, as all the white dwarfs of table 2 are members of wide binary systems. The absolute magnitude of EG 65 is uncertain, as its red companion (M2 V) was observed only in *UBV* by EG, and not in *R* and *I*. Blanketing effects are severe in the *UBV* region for such a red star, making the calibration of the $(M_v, B - V)$-relation uncertain. However, the derived M_v of EG 65 is unlikely to be in error by as much as 2 mag, and we thus conclude that it is, in fact, significantly less luminous than the other stars of table 2 which have the same temperature. The effective temperatures were

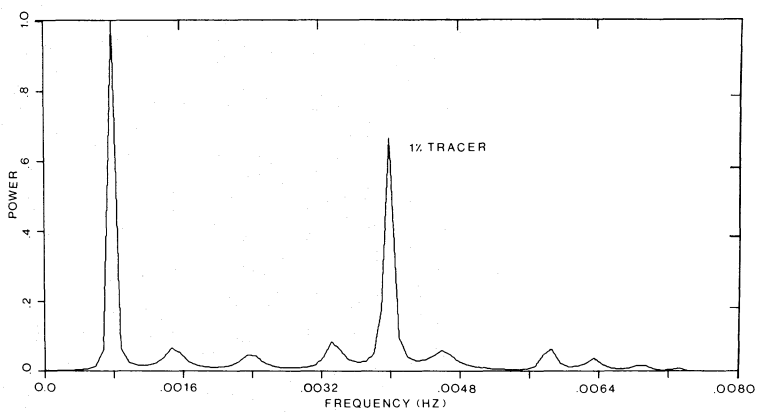


FIG. 8*b*.—Low-frequency end of the power spectrum of EG 65 calculated by using MEM. A 1% tracer signal was added to the data before transformation, and it is indicated in the figure. A spectral peak is seen at a period of 1311 s as in fig. 8*a*. The units of power are arbitrary.

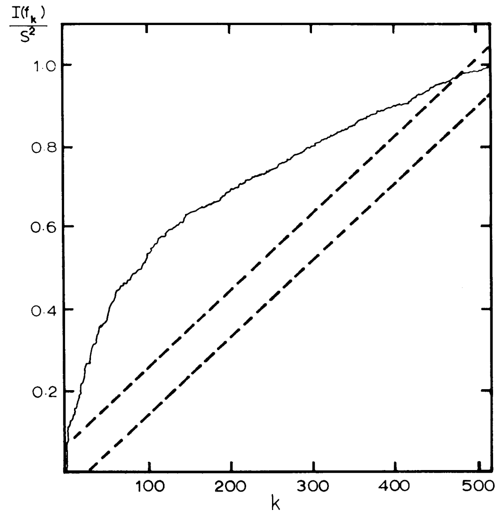


FIG. 9.—Whiteness test calculated from the power spectrum of EG 65. A large deviation from linearity is seen, implying that the probability that this spectrum was produced by a white noise process is much less than 5%.

derived from the colors of EG together with their calibration of these colors in terms of surface temperatures (see, in particular, fig. 6 in EG 1965*a*). These should be considered only as fairly rough values but will suffice for our purposes. The masses have been estimated by placing the stars in the theoretical Hertzsprung-Russell diagram given by Schwarzschild (1958, p. 236). Finally, the space velocities were calculated from the *UVW* velocities given by EG. The important points to note about table 2 are that EG 65 (the only variable star in the table) is the most massive object and also the star with the second lowest space velocity among the seven stars listed.

Let us now consider the rate at which the stars of table 2 will accrete mass from the interstellar medium which we can consider to be 100 percent hydrogen. For a star of mass $\tilde{M}$ (in $M_\odot$) traveling through a medium of particle density n (cm^{-3}), the rate of accretion of mass is given by Bondi and Hoyle (1944) as

$$\frac{dM}{dt} \simeq 10^{14}\,\frac{\tilde{M}^2 n}{v^3}\,\mathrm{g\,s^{-1}}\,, \qquad (3)$$

where v is the velocity ($\mathrm{km\,s^{-1}}$) of the star through the medium. In order that this accretion mechanism work, it is necessary that a tail shock form to destroy momentum at right angles to the direction of motion of the star. Greenstein (1951) has questioned whether such a shock can form in the low-density interstellar medium. Strittmatter and Wickramasinghe (1971) have, however, suggested that only for low-mass stars ($\tilde{M} < 0.1$) moving at high velocity ($v > 100$) will the shock not form. In our case, we are not dealing with these extreme cases and therefore consider equation (3) to be appropriate to the present discussion.

If we now take $n = 3$, we can estimate the accretion rate for the stars of table 2. We shall consider the two extreme cases of dM/dt, namely, EG 65 with the highest rate and EG 64 with the lowest rate. For EG 64, $dM/dt = 3 \times 10^7\ \mathrm{g\,s^{-1}}$, while $dM/dt = 8 \times 10^9$ for EG 65. It is important to realize that this accretion will not be symmetric over the surface area of the star. Any rotation will cause nonuniform accretion (as long as the rotation axis is not perpendicular to the direction of motion through the medium), as will the mere fact that the star is moving through the medium from which it is accreting hydrogen.

Van Horn (1970) has shown that the entire nondegenerate region of low-luminosity white dwarfs ($L \simeq 10^{-3.5} L_\odot$) may be convective. Here, then, is an effective method of transporting the accreted hydrogen down into the interior of the star where it may be burned. For EG 65, the temperature and density at the interface between the degenerate and nondegenerate regions may be calculated from expressions given in Schwarzschild (1958, p. 237) and yield a temperature of 6×10^6 °K and a density of 800 g cm^{-3}. The thickness of this nondegenerate region is about 15 km. Hydrogen can easily burn via the *p-p* chain under these conditions.

The luminosity of EG 65 is $\simeq 10^{-3.5} L_\odot$. Its amplitude of variation is ~ 0.01 mag or ~ 1 percent. Hence, we look for a driving force for the oscillation which must be able to provide the $\sim 10^{-5.5} L_\odot$ or 10^{28} ergs s^{-1} needed. If the accreted hydrogen were convected down to depths where it could burn, it could supply up to $0.007(dM/dt)c^2 \simeq 6 \times 10^{28}\ \mathrm{ergs\,s^{-1}}$, easily enough to drive the observed 1 percent luminosity variation. Since, as previously mentioned, the accretion is expected to be asymmetric, we can imagine that the nuclear burning also will not have radial symmetry,

TABLE 2

DA WHITE DWARFS OF KNOWN M_v WHICH HAVE BEEN EXAMINED FOR VARIABILITY

Star (1)	M_v (2)	T_e (° K) (3)	Mass ($M_\odot$) (4)	Space Velocity ($\mathrm{km\,s^{-1}}$) (5)	Variable? (6)
EG 30......	11.1	8,000	0.20	59	No
EG 31......	10.9	12,000	0.40	60	No
EG 42......	10.8	16,000	0.40	59	No
EG 64......	11.7	8,000	0.25	90	No
EG 65......	13.4:	8,000	1.00	34	Yes
EG 165......	13.4	10,000	0.80	50	No
EG 175......	11.2	7,500	0.25	15	No

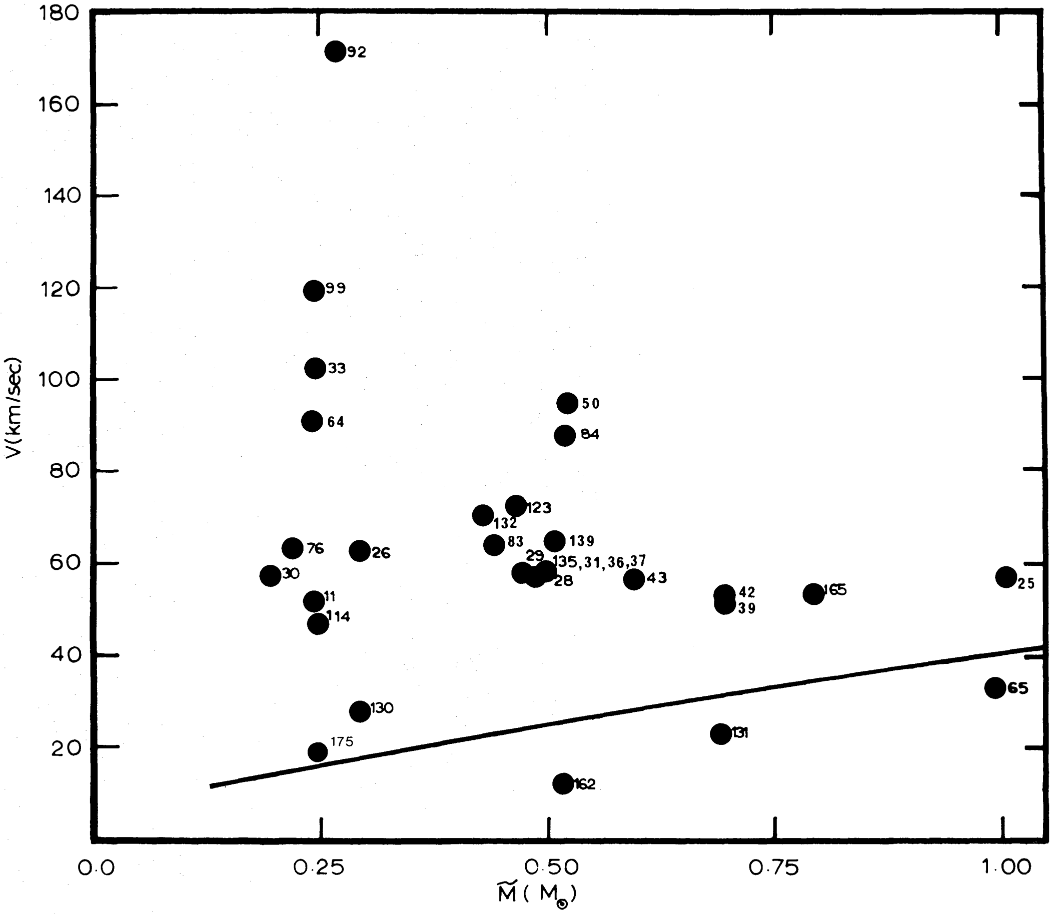


FIG. 10.—Plot of the mass versus space velocity of DA white dwarfs. The number beside each point is its EG designation. According to the ideas discussed in § IV the three stars lying below the line have an accretion rate large enough to drive observable oscillations.

and thus may be capable of exciting the nonradial gravity modes which Chanmugan (1972) has suggested as the mode of variability for white dwarfs.

Several previously unexplained facets of white-dwarf variability are very simply understood within the framework of the above ideas. (1) We can now perceive why two white dwarfs, having the same colors, can be so different as regards their variability. Because of the needed high accretion rate of hydrogen, we would expect that, all other things being equal, the more massive star (or less luminous star at a given color) would be more likely to be variable. As can be calculated for EG 64, in the framework of the above ideas, its maximum amplitude of variability could only be about 0.0003 mag and hence would never be detected. (2) It has been observed by HL that the amplitude of variation of the white-dwarf variables is quite erratic, sometimes disappearing entirely. This is understood if the rate of accretion of interstellar hydrogen is allowed to fluctuate. If the star is moving through a region of high density, dM/dt is appreciable and the stellar oscillations are large. If n is small, the oscillations may drop to an unobservable level. This suggests the possibility that the white-dwarf variables may be used as extremely sensitive probes of the density of the interstellar medium.

V. CONCLUSIONS AND SUGGESTED OBSERVATIONS

The ideas presented in the previous section are capable of being tested. Future observations should concentrate on the massive white dwarfs with low space velocity. If our ideas are correct, numerous new variables should be found among this group of stars.

As a practical application, consider the 31 stars of spectral type DA that are contained in tables 2, 3, and 4 of EG 1965*a*. EG have made reasonably accurate estimates of M_V for all of these stars as either they are in wide binary systems or in open clusters, or they are close enough to have measurable trigonometric parallaxes. In figure 10 we plot the estimated mass $\widetilde{M}$ ($M_{\odot}$) of these stars against their space velocity V (km s^{-1}). The masses were estimated via the same procedure discussed in § IV. The solid curve included in figure 10

is the locus along which the accretion rate of interstellar material is such that $0.007(dM/dt)c^2 = 1$ percent of the luminosity of the star. For stars lying above this line $0.007(dM/dt)c^2$ is less than 1 percent, while for stars below the line it is greater.

Thus, if our ideas are correct, we would expect to find many more variables below this line than above it. Of the 28 stars lying above the line, only six (EG 30, 31, 42, 64, 165, 175) have been examined for high-speed variability either by ourselves or by HL. None are variable. Of the three stars lying below this line, only one (EG 65) has thus far been examined for variability with the positive results discussed in this paper. *And you EG 131 and 162?*

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APPENDIX

MAXIMUM-ENTROPY POWER SPECTRUM

We are interested here in the analysis of short records—records whose lengths are of the order of the periods of interest. It is well known that in such cases the common methods of power spectral analysis suffer considerably from inadequate resolution. This fact is due primarily to the unrealistic assumption which these methods require concerning the behavior of the data outside the known interval. Thus, the periodogram (Jones 1965) assumes a periodic extension to the data, whereas the autocorrelation approach (Blackman and Tukey 1958) assumes a zero extension.

Burg (1967, 1968) has suggested a radically different approach to the computation of a power spectrum. We present here a brief and somewhat heuristic treatment of this technique, the maximum entropy method (MEM). A detailed treatment is presented by Smylie *et al.* (1973). Andersen (1974) discusses computational details. Other pertinent references are Ulrych (1972), Ulrych *et al.* (1973), and Edward and Fitelson (1973).

Entropy may be considered to be a measure of disorder in a system (Brillouin 1956). Alternately, entropy is a measure of our ignorance about the actual structure of a system. As elegantly pointed out by Abels (1972), this relationship between entropy and uncertainty or ignorance led Jaynes (1963, 1968) to the formation of an important principle. The basic premise of this principle is that the probability distribution which describes the available information but is maximally noncommittal with regard to the unavailable information is the one with maximum entropy. We can see that the application of this concept eliminates the unreasonable constraints demanded by conventional spectral analysis. Beginning with the relationship between the entropy H and the spectral density $S(f)$ of a stationary Gaussian process

$$H = \frac{1}{4f_N}\int_{-f_N}^{f_N} \log S(f)df\,,$$

where f_N is the Nyquist frequency, the MEM estimate $P_E(f)$ is obtained by maximizing H with respect to the unknown autocorrelation coefficients. The constraints in the maximization are that $P_E(f)$ must be consistent with the known autocorrelation coefficients. The resulting expression for the estimate $P_E(f)$ is

$$P_E(f) = \frac{P_M}{f_N}\left|1 + \sum_{j=1}^{M-1} \gamma_j \exp(-i2\pi fj\Delta t)\right|^{-2} \tag{A1}$$

In equation (A1), P_M is a constant and the γ_j are prediction error coefficients determined from the data.

The parameters in equation (A1) are more readily visualized if we make use of the correspondence between MEM spectral analysis and the autoregressive (AR) representation of a stochastic process which was recently established by Van den Bos (1972). Van den Bos has shown that the extrapolation of the autocovariance function of a process x_t past the known lags using the principle of maximum entropy is equivalent to the least-squares fitting of an AR model to that process. In other words, it is equivalent to the least-squares estimation of the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$ in the AR model:

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \cdots + \alpha_p x_{t-p} + a_t\,. \tag{A2}$$

The quantity a_t is known as the innovation of the process and is a white-noise series with zero mean and variance σ_a^2. It may be easily shown that the power spectrum of the process in equation (A2) is given by

$$P(f) = 2\sigma_a^2\left|1 - \sum_{j=1}^{p} \alpha_j \exp(-i2\pi jf)\right|^{-2} \tag{A3}$$

for unit sampling. A comparison of equations (A3) and (A2) identifies the coefficients γ_j with α_j. Since an equivalent interpretation of the AR process is that of linearly predicting x_t from a number of its past values, we see that the α_j's are the coefficients of a prediction filter. Rewriting equation (A2) as

$$a_t = x_t - \alpha_1 x_{t-1} - \alpha_2 x_{t-2} - \cdots - \alpha_p x_{t-p} \tag{A4}$$

identifies $1 - \alpha_1 - \alpha_2 - \cdots - \alpha_p$ with a prediction error filter and a_t with the prediction error with variance σ_a^2.

The calculation of the prediction error coefficients and of the prediction error variance will not be presented here, but details are given in the cited references.

Of major importance in MEM spectral analysis is the choice of M in equation (A1) or equivalently of p in equation (A2). In other words, we must choose the order of the AR process which is to represent the data. In this regard we have found that the estimating procedure proposed by Akaike (1969*a*, *b*, 1970) gives excellent results. The Akaike criterion, which is called the final prediction error, or FPE, has been investigated by Gersch and Sharpe (1973) and by Fryer, Odegard, and Sutton (1974), who are the first to have applied it in the published literature specifically to MEM analysis.

The FPE is defined as the mean square prediction error. If for a given AR process $\hat{x}_t$ is the estimated prediction of x_t, then

$$\mathrm{FPE} = E[(x_t - \hat{x}_t)^2]\,,$$

where E is the expectation operator.

Akaike shows that the FPE is composed of the sum of two contributions, one which decreases as the assumed value of p (or M) increases whereas the second increases. It is natural therefore to adopt that value of M which minimizes the FPE. Akaike (1969*b*, 1970) develops expressions for an efficient estimate of this minimum criterion which we will call $(\mathrm{FPE})_M$:

$$(\mathrm{FPE})_M = \left\{\frac{N + (M+1)}{N - (M-1)}\right\} S_M^2\,, \tag{A5}$$

where N is the number of data points and S_M^2 is an estimate of the prediction-error variance. As pointed out earlier, the computation of S_M^2 is detailed in the cited references.

As an example of MEM spectral analysis, we consider a fourth-order AR process, a realization of which is shown in figure 11. The parameters which we have used are

$$\alpha_1 = 2.7607\,, \quad \alpha_2 = -3.8106\,, \quad \alpha_3 = 2.6535\,, \quad \alpha_4 = -0.9238\,.$$

Figure 12 shows the true power spectrum for this process. In Figure 13 we show a plot of the logarithm of the normalized (FPE) which is the ratio $(\mathrm{FPE})_M/(\mathrm{FPE})_0$ computed from a 40-point realization of the AR process. The minimum value occurs at $M = 4$, and figure 14 shows the MEM power spectrum for the 40-point realization computed using this value. For the purpose of comparison, figure 15 shows the unsmoothed periodogram of the 40-point realization. The striking improvement in the spectral resolution using MEM is evident in this illustration.

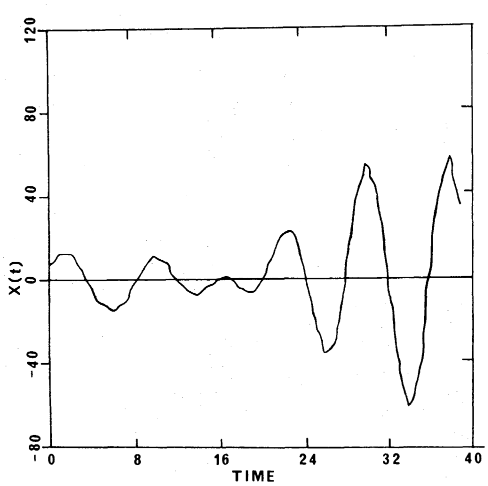

FIG. 11.—A realization $x(t)$, of a fourth-order AR process.

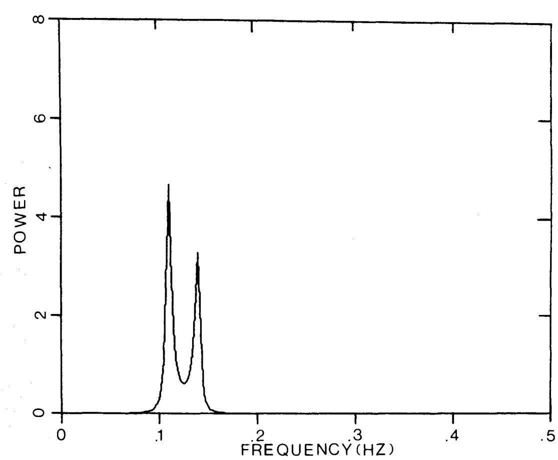

FIG. 12.—True power spectrum for the fourth-order process illustrated in fig. A1.

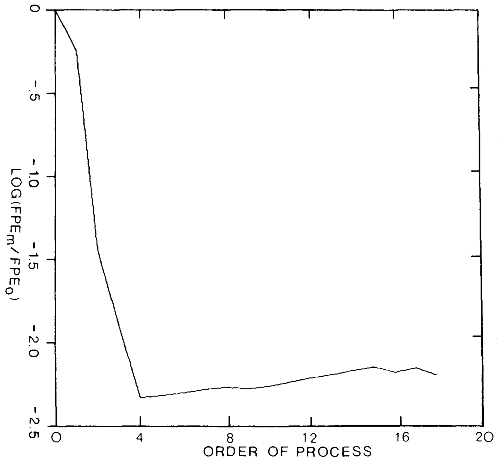

FIG. 13.—Variation of the logarithm of the ratio $(FPE)_M/(FPE)_0$ with assumed order of AR process.

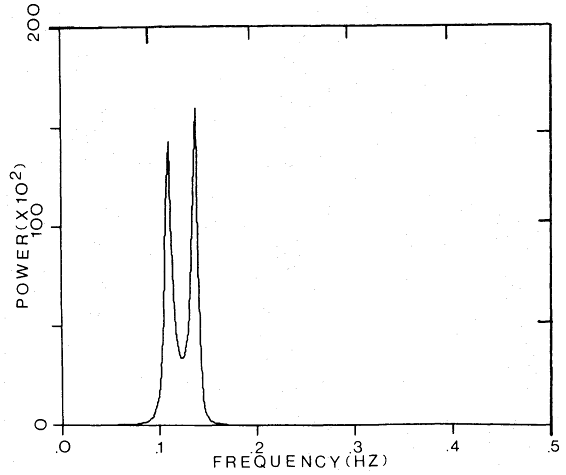

FIG. 14.—MEM power spectrum of the realization of fig. 11 using $M = 4$.

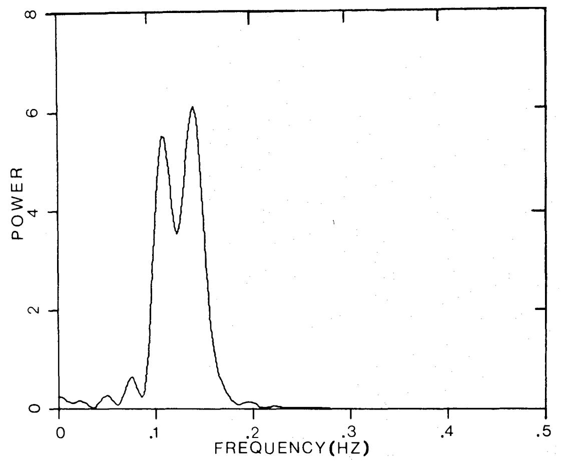

FIG. 15.—Unsmoothed periodogram of the realization of fig. 11

It will be noticed that the MEM power spectrum differs somewhat from the true spectrum. In view of the fact that the MEM estimate has been computed from one short realization of the process, this is hardly surprising.

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